MULTIPlicITY RESULTS OF positive solutions for
SINGULAR GENERALIZED LApLAcian systems

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ABSTRACT. We study the homogeneous Dirichlet boundary value problem of generalized Laplacian systems with a singular weight which may not be in $L^1$. Using the well-known fixed point theorem on cones, we obtain the multiplicity results of positive solutions under two different asymptotic behaviors of the nonlinearities at 0 and $\infty$. Furthermore, a global result of positive solutions for one special case with respect to a parameter is also obtained.

1. Introduction

In this paper, we study the following nonlinear differential system

\[ (P_{\lambda}) \left\{ \begin{array}{l}
-\Phi(u')' = \lambda h(t) \cdot f(u), \quad t \in (0, 1), \\
u(0) = 0 = u(1),
\end{array} \right. \]

where $\Phi(u') = (\varphi(u'_1), \ldots, \varphi(u'_N))$ with $\varphi : \mathbb{R} \to \mathbb{R}$ an odd increasing homeomorphism, $\lambda > 0$ a parameter, $h(t) = (h_1(t), \ldots, h_N(t))$ with $h_i : (0, 1) \to \mathbb{R}_+$ continuous, $h_i \neq 0$ on any subinterval in $(0, 1)$ and $f(u) = (f_1(u), \ldots, f_N(u))$ with $f^i : \mathbb{R}_+^N \to \mathbb{R}_+$, here we denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_+^N = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ and $x \cdot y = (x_1y_1, x_2y_2, \ldots, x_Ny_N)$ the Hadamard product of $x$ and $y$ in $\mathbb{R}^N$. Thus problem $(P_{\lambda})$ can be rewritten as

\[ \left\{ \begin{array}{l}
-\varphi(u'_1)' = \lambda h_1(t)f_1^1(u), \\
\hspace{1cm} \vdots \\
-\varphi(u'_N)' = \lambda h_N(t)f_N^N(u), \\
u_i(0) = 0 = u_i(1), \quad i = 1, \ldots, N.
\end{array} \right. \]

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The generalized Laplacian problems like \( P_\lambda \) appear in various applications which describe reaction-diffusion systems, nonlinear elasticity, glaciology, population biology, combustion theory, and non-Newtonian fluids (see [8, 10, 11, 16]). They also have received growing attention in connection with positive radial solutions of elliptic problems in both annular and exterior domains (see [9, 21] and the references therein).

In recent years, existence and multiplicity of positive solutions of these problems have been extensively studied under various assumptions on the weight functions and nonlinearities (see [1–6], [9], [12], [14, 16–23]). For example, Wang [20] obtained the criteria of determining the number of positive solutions of problem \( P_\lambda \) with respect to the parameter \( \lambda \) when each \( h_i : [0, 1] \to \mathbb{R}_+ \) is continuous and \( \varphi \) satisfies that there exist two increasing homeomorphisms \( \psi_1 \) and \( \psi_2 \) of \((0, \infty)\) onto \((0, \infty)\) such that

\[
\psi_1(\sigma)\varphi(x) \leq \varphi(\sigma x) \leq \psi_2(\sigma)\varphi(x) \quad \text{for} \quad \sigma, \ x > 0.
\]

In this paper, we give assumptions on \( \varphi, h \) and \( f \) as follows.

(A) There exist an increasing homeomorphism \( \psi \) of \((0, \infty)\) onto \((0, \infty)\) and a function \( \gamma \) of \((0, \infty)\) into \((0, \infty)\) such that

\[
\psi(\sigma) \leq \frac{\varphi(\sigma x)}{\varphi(x)} \leq \gamma(\sigma) \quad \text{for all} \quad \sigma > 0, \ x \in \mathbb{R}/\{0\}.
\]

(H) \( h_i : (0, 1) \to \mathbb{R}_+ \) is a continuous function satisfying

\[
\int_0^{\frac{1}{2}} \psi^{-1}(\int_0^{\frac{1}{2}} h_i(\tau) d\tau) d\tau + \int_{\frac{1}{2}}^{1} \psi^{-1}(\int_{\frac{1}{2}}^{s} h_i(\tau) d\tau) ds < \infty,
\]

for \( i = 1, \ldots, N \).

(F1) \( f^i : \mathbb{R}_+^N \to \mathbb{R}_+ \) is continuous for \( i = 1, \ldots, N \).

(F2) \( f^i(u) > 0 \) for \( u \in \mathbb{R}_+^N \) with \( \|u\| > 0 \), \( i = 1, \ldots, N \).

(F3) \( f^i(u_1, \ldots, u_N) \leq f^i(v_1, \ldots, v_N) \), whenever \( u_i = v_i, u_j \leq v_j, i \neq j \).

Note that \( \varphi \) covers the case of \( p \)-Laplace operator, namely \( \varphi(x) = \varphi_p(x) := |x|^{p-2}x, \ x \in \mathbb{R}, \ p > 1 \). Clearly, \( \varphi_p \) satisfies condition (A) with \( \varphi_p \equiv \psi \equiv \gamma \).

Specially, conditions (A), (H) on \( \varphi \) and \( h_i \) were introduced first by Xu and Lee [22] and more general than the ones given by Wang [20]. For convenience, we introduce a new class of weight functions. For a bijection \( \iota : \mathbb{R} \to [0, 1] \), define \( \mathcal{H}_\iota \) a subset of \( C((0, 1), \mathbb{R}_+) \) given by

\[
\mathcal{H}_\iota = \left\{ g \in C((0, 1), \mathbb{R}_+): \int_0^{\frac{1}{2}} \iota^{-1}(\int_0^{\frac{1}{2}} g(\tau) d\tau) d\tau + \int_{\frac{1}{2}}^{1} \iota^{-1}(\int_{\frac{1}{2}}^{s} g(\tau) d\tau) ds < \infty \right\}.
\]

By the notation, condition (H) means \( h_i \in \mathcal{H}_\iota \).

Now we introduce some notations for the statement of the main theorem. Denote

\[
f_0 := \sum_{i=1}^{N} f_0^i, \ f_\infty := \sum_{i=1}^{N} f_\infty^i,
\]
where
\[ f_0' := \lim_{\|u\| \to 0} \frac{f_i(u)}{\varphi(\|u\|)}, \quad f_\infty' := \lim_{\|u\| \to \infty} \frac{f_i(u)}{\varphi(\|u\|)} \]
for \( u \in \mathbb{R}_+^N \) and \( i = 1, \ldots, N \). For simplicity, we denote \( \|u\| = \sum_{i=1}^N |u_i| \) for \( u \in \mathbb{R}_+^N \) in this paper.

When \( N = 1, \varphi = \varphi_p \), Agarwal-Lù-O’Regan [1] and Sánchez [18] proved the multiplicity of positive solutions of problem \((P_\lambda)\) for \( \lambda \) belonging to some open interval if either \( f_0 = f_\infty = 0 \) or \( f_0 = f_\infty = \infty \). Later, Wang [20] extended the multiplicity results in [1, 18] to \( \varphi \)-Laplacian system with each \( h_i \in C[0,1] \).

Recently, Xu and Lee [23] derived some explicit intervals for the multiplicity results in [1, 18] to \( \varphi \)-Laplacian system \((P_\lambda)\) for the cases \( f_0 = f_\infty = 0 \) and \( f_0 = f_\infty = \infty \). Further, under the monotone-type assumption \((F_2)\), we firstly obtain a global result of positive solutions for problem \((P_\lambda)\) with respect to \( \lambda \) for the case \( f_0 = f_\infty = \infty \). More precisely, main results can be stated as follows.

**Theorem 1.1.** Assume that (A), (H), \((F_1)\), and \((F_2)\) hold.

1. If \( f_0 = f_\infty = 0 \), then there exist \( \lambda > \lambda \) > 0 such that \((P_\lambda)\) has at least two positive solutions for \( \lambda > \lambda \), and no positive solution for \( \lambda \in (0, \lambda) \), where \( \lambda, \lambda \) are given by (3.2) and (3.9), respectively.

2. If \( f_0 = f_\infty = \infty \), then there exist \( \lambda > \lambda \) > 0 such that \((P_\lambda)\) has at least two positive solutions for \( \lambda \in (0, \lambda) \), and no positive solution for \( \lambda > \lambda \), where \( \lambda, \lambda \) are given by (3.11) and (3.21), respectively.

**Theorem 1.2.** Assume that (A), (H), \((F_1)\), \((F_2)\) and \((F_3)\) hold. If \( f_0 = f_\infty = \infty \), then there exist \( \lambda^* \geq \lambda_+ > 0 \) such that \((P_\lambda)\) has at least two positive solutions for \( \lambda \in (0, \lambda_+) \), one positive solution for \( \lambda \in [\lambda_+, \lambda^*] \), and no positive solution for \( \lambda > \lambda^* \), where \( \lambda^*, \lambda_+ \) are given by (3.28) and (3.29), respectively.

**Remark 1.3.** If \( f^{i_0}(0) > 0 \) for some \( i_0 \in \{1, \ldots, N\} \), then we can get \( \lambda_+ = \lambda^* \) in Theorem 1.2. The proof can be easily completed by the similar arguments in [15].

**Remark 1.4.** Quasi-monotone condition \((F_3)\) is redundant in one dimensional case so that Theorem 1.2 is valid for scalar \( \varphi \)-Laplacian problem without any monotonicity condition on \( f \).

**Remark 1.5.** Under the same assumptions in Theorem 1.2, we expect a similar result for the case \( f_0 = f_\infty = 0 \), but the analysis can not follow in a similar way.

As a benefit of a constructive technique used in this paper, we note that \( \lambda, \lambda \) appeared in Theorem 1.1 can be computed explicitly (see examples in Section 4). For the proofs, we employ a newly developed solution operator introduced by Xu and Lee [22] and then we apply the fixed point theorem on cones for our main results.
Our paper is organized as follows. In Section 2, we establish a solution operator for problem \((P_{\lambda})\) and introduce some preliminary facts. In Section 3, we prove the main theorems and in Section 4, we give some examples.

2. Preliminaries

Main condition of weight function \(h_i\) in problem \((P_{\lambda})\) is of \(H_{\psi}\)-class which includes singular functions specially on the boundary, i.e., \(h_i\) may not be integrable near the boundary, \(t = 0\) and/or \(t = 1\). In this case, solutions need not be in \(C^{1}[0,1]\). So by a solution to problem \((P_{\lambda})\), we understand a function \(u \in C_{0}((0,1], \mathbb{R}^{N}) \cap C^{1}((0,1), \mathbb{R}^{N})\) with \(\Phi(u')\) absolutely continuous which satisfies problem \((P_{\lambda})\).

Basic tool for proving our main results is the following well-known fixed point theorem ([7, 13]).

**Theorem 2.1.** Let \(E\) be a Banach space and let \(K\) be a cone in \(E\). Assume that \(\Omega_1\) and \(\Omega_2\) are open subsets of \(E\) with \(0 \in \Omega_1\), \(\overline{\Omega_1} \subset \Omega_2\). Assume that

\[ T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K \]

is completely continuous such that either

\[ \|Tu\| \leq \|u\| \text{ for } u \in K \cap \partial \Omega_1 \text{ and } \|Tu\| \geq \|u\| \text{ for } u \in K \cap \partial \Omega_2, \]
\[ \|Tu\| \geq \|u\| \text{ for } u \in K \cap \partial \Omega_1 \text{ and } \|Tu\| \leq \|u\| \text{ for } u \in K \cap \partial \Omega_2. \]

Then \(T\) has a fixed point in \(K \cap (\overline{\Omega_2} \setminus \Omega_1)\).

To set up the solution operator for \((P_{\lambda})\), let us define \(E\) the Banach space \(C_{0}[0,1] \times \cdots \times C_{0}[0,1]\) with norm \(\|u\|_\infty = \sum_{i=1}^N \|u_i\|_\infty\) and define a cone \(K\) by taking \(K = \{u \in E \mid u_i\text{ is concave on }[0,1], i = 1, \ldots, N\}\).

Let us consider a simple scalar problem of the form

\begin{align*}
(W) & \quad -\varphi(w')' = g(t), \quad t \in (0, 1), \\
(D) & \quad w(0) = w(1) = 0,
\end{align*}

where \(\varphi\) satisfies \((A)\) and \(g \in H_{\varphi}\). Note from condition \((A)\) that \(H_{\psi} \subset H_{\varphi}\) (see Remark 2.3). Let \(w\) be a solution of \((W)+(D)\). Then integrating both sides of \((W)\) on the interval \([s, \frac{1}{2}]\) for \(s \in (0, \frac{1}{2}]\) and \([\frac{1}{2}, s]\) for \(s \in [\frac{1}{2}, 1)\), respectively, we find that \((W)+(D)\) is equivalent to

\begin{equation}
\begin{cases}
\begin{align*}
&w'(s) = \varphi^{-1} \left( a + \int_s^\frac{1}{2} g(\tau)d\tau \right), \quad w(0) = 0, \quad s \in (0, \frac{1}{2}], \\
&w'(s) = \varphi^{-1} \left( a - \int_\frac{1}{2}^s g(\tau)d\tau \right), \quad w(1) = 0, \quad s \in [\frac{1}{2}, 1),
\end{align*}
\end{cases}
\end{equation}

where \(a = \varphi(w'(\frac{1}{2}))\). Showing the fact \(\varphi^{-1} \left( a + \int_s^\frac{1}{2} g(\tau)d\tau \right) \in L^1(0, \frac{1}{2}]\) is not obvious since \(g\) can not be in \(L^1(0, \frac{1}{2}]\). One may refer to Xu and Lee [22] for the proof. Now we may integrate both sides of \((2.1)\) on the interval \([0, \frac{1}{2}]\) for \(t \in [0, \frac{1}{2}]\) and on the interval \([\frac{1}{2}, 1]\) for \(t \in [\frac{1}{2}, 1]\), respectively. And we get

\[
w(t) = \begin{cases}
\begin{align*}
&\int_0^t \varphi^{-1} \left( a + \int_s^\frac{1}{2} g(\tau)d\tau \right) ds, \quad t \in [0, \frac{1}{2}], \\
&\int_\frac{1}{2}^t \varphi^{-1} \left( a - \int_\frac{1}{2}^s g(\tau)d\tau \right) ds, \quad t \in [\frac{1}{2}, 1].
\end{align*}
\end{cases}
\]
To check \(w(\frac{1}{2}^-) = w(\frac{1}{2}^+)\), define for \(a \in \mathbb{R}\),

\[
(2.2) \quad G(a) = \int_0^\frac{1}{2} \varphi^{-1}\left(a + \int_s^\frac{1}{2} g(\tau)d\tau\right)ds - \int_0^1 \varphi^{-1}\left(-a + \int_s^1 g(\tau)d\tau\right)ds.
\]

Then the function \(G : \mathbb{R} \to \mathbb{R}\) is well-defined and has a unique zero \(a = a(g)\) in \(\mathbb{R}\) (See Xu and Lee [22] for the proof). This implies \(w(\frac{1}{2}^-) = w(\frac{1}{2}^+)\). Consequently, if \(\varphi\) satisfies \((A)\) and \(g \in \mathcal{H}_\varphi\), then the solution \(w\) of \((W)+(D)\) can be represented by

\[
(2.3) \quad w(t) = \begin{cases} \int_0^t \varphi^{-1}\left(a(g) + \int_s^\frac{1}{2} g(\tau)d\tau\right)ds, & t \in [0, \frac{1}{2}], \\ \int_0^1 \varphi^{-1}\left(-a(g) + \int_s^1 g(\tau)d\tau\right)ds, & t \in [\frac{1}{2}, 1], \end{cases}
\]

where \(a(g) \in \mathbb{R}\) uniquely satisfies

\[
\int_0^\frac{1}{2} \varphi^{-1}\left(a(g) + \int_s^\frac{1}{2} g(\tau)d\tau\right)ds = \int_0^1 \varphi^{-1}\left(-a(g) + \int_s^1 g(\tau)d\tau\right)ds.
\]

Replacing \(g(t)\) with \(\lambda h_i(t)f^i(u(t))\) in \((W)+(D)\), we may define

\[
T_\lambda(u) = (T^1_\lambda(u), \ldots, T^N_\lambda(u))
\]

for \(\lambda > 0\), \(u \in K\) and for \(i = 1, \ldots, N\), given by

\[
T^i_\lambda(u)(t) = \begin{cases} \int_0^t \varphi^{-1}\left(a^i(\lambda h_i f^i(u)) + \int_s^\frac{1}{2} \lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds, & t \in [0, \frac{1}{2}], \\ \int_0^1 \varphi^{-1}\left(-a^i(\lambda h_i f^i(u)) + \int_s^\frac{1}{2} \lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds, & t \in [\frac{1}{2}, 1], \end{cases}
\]

where \(a^i(\lambda h_i f^i(u)) \in \mathbb{R}\) uniquely satisfies

\[
\int_0^\frac{1}{2} \varphi^{-1}\left(a^i(\lambda h_i f^i(u)) + \int_s^\frac{1}{2} \lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds = \int_\frac{1}{2}^1 \varphi^{-1}\left(-a^i(\lambda h_i f^i(u)) + \int_s^\frac{1}{2} \lambda h_i(\tau)f^i(u(\tau))d\tau\right)ds.
\]

One may show that \(T_\lambda : K \to K\) is completely continuous (See Lemma 11 in Xu and Lee [22] for details). Thus we see that \(u\) is a positive solution of \((P_\lambda)\) if and only if

\[
u = T_\lambda(u) \text{ on } K.
\]

We finally give some remarks and lemma for later use.

**Remark 2.2.** From condition \((A)\), we get

\[
\sigma x \leq \varphi^{-1}[\gamma(\sigma)\varphi(x)],
\]

and

\[
\varphi^{-1}[\sigma \varphi(x)] \leq \psi^{-1}(\sigma)x
\]

for \(\sigma\) and \(x > 0\).
Remark 2.3. Let $h \in L^1_{\text{loc}}((0,1),\mathbb{R}_+).$ Then for any fixed $s \in (0, \frac{1}{2}),$ we know $\int_s^2 h(\tau) \, d\tau < \infty.$ Applying $\sigma = \int_s^2 h(\tau) \, d\tau$ and $x = \varphi^{-1}(1)$ in Remark 2.2, we get
\[
\varphi^{-1}\left(\int_s^2 h(\tau) \, d\tau\right) \leq \varphi^{-1}(1)\varphi^{-1}\left(\int_s^2 h(\tau) \, d\tau\right).
\]
This implies $\mathcal{H}_\psi \subseteq \mathcal{H}_\varphi.$

Proposition 2.4. ([20]) Let $w \in C_0[0,1] \cap C^1(0,1)$ satisfy $\varphi(w')' \leq 0$ on $(0,1).$
Then $w$ is concave on $[0,1]$ and $\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} w(t) \geq \frac{1}{4} \|w\|_{\infty},$ where $\|w\|_{\infty}$ is the supremum norm of $w.$

3. Proofs of main results

In this section, we need to give some lemmas which will play a crucial role in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. Assume that (A), (H), (F1), and (F2) hold. If $f_0 = f_\infty = 0,$ then there exists $\lambda > 0$ such that $(P_\lambda)$ has at least two positive solutions for $\lambda > \lambda.$

Proof. For any $r > 0,$ define
\[
\hat{m}_r = \min\{f^i(x) \mid x \in \mathbb{R}_+^N, \frac{r}{4} \leq \|x\| \leq r, \ i = 1, \ldots, N\}.
\]
We see that $\hat{m}_r > 0,$ by (F2). For $K_r \equiv \{u \in K \mid \|u\|_{\infty} < r\},$ let $u \in \partial K_r,$ then by Proposition 2.4, for $t \in \left[\frac{1}{4}, \frac{3}{4}\right],$
\[
r = \|u\|_{\infty} \geq \|u(t)\| = \sum_{i=1}^N u_i(t) \geq \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \sum_{i=1}^N u_i(t) \geq \frac{1}{4} \|u\|_{\infty} = \frac{r}{4},
\]
and
\[
f^i(u(t)) \geq \hat{m}_r \text{ for } i = 1, \ldots, N.
\]
For simplicity, denote $a^i_{\lambda,u} = \lambda f^i(u).$ Then for $u \in \partial K_r,$ we get
\[
2T_\lambda^2(u)\left(\frac{1}{2}\right) = \int_0^1 \varphi^{-1}\left(\int_s^2 \lambda h_i(\tau)f^i(u(\tau)) \, d\tau\right) \, ds
\]
\[
+ \int_0^1 \varphi^{-1}\left(-a^i_{\lambda,u} + \int_s^2 \lambda h_i(\tau)f^i(u(\tau)) \, d\tau\right) \, ds.
\]
If $a^i_{\lambda,u} \geq 0,$ then
\[
\int_0^1 \varphi^{-1}\left(a^i_{\lambda,u} + \int_s^2 \lambda h_i(\tau)f^i(u(\tau)) \, d\tau\right) \, ds
\]
\[
\geq \int_0^1 \varphi^{-1}\left(\int_s^2 \lambda h_i(\tau)f^i(u(\tau)) \, d\tau\right) \, ds.
\]
and by the definition of $a_{\lambda,u}$,

$$
\int_0^1 \varphi^{-1} \left( -a_{\lambda,u} + \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds \\
= \int_0^1 \varphi^{-1} \left( a_{\lambda,u} + \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds \geq 0.
$$

Thus,

$$
2T^i_{\lambda}(u) \left( \frac{1}{2} \right) \geq \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds.
$$

If $a_{\lambda,u} < 0$, then $-a_{\lambda,u} > 0$ and

$$
\int_0^1 \varphi^{-1} \left( -a_{\lambda,u} + \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds \\
\geq \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds,
$$

and by the same argument, we get

$$
2T^i_{\lambda}(u) \left( \frac{1}{2} \right) \geq \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds.
$$

Thus, we obtain

$$
2T^i_{\lambda}(u) \left( \frac{1}{2} \right) \\
\geq \min \left\{ \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds, \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds \right\}.
$$

By using (3.1), we get

$$
2\|T^i_{\lambda}(u)\|_\infty \\
\geq 2T^i_{\lambda}(u) \left( \frac{1}{2} \right) \\
\geq \min \left\{ \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds, \int_0^1 \varphi^{-1} \left( \int_{s}^{s} \lambda h_i(\tau) f'(u(\tau)) d\tau \right) ds \right\}.
$$
\[
\begin{align*}
&= \frac{1}{4} \varphi^{-1}\left(\lambda \tilde{m}_r \min \left\{ \int_{\frac{1}{2}}^{1} h_i(\tau) d\tau, \int_{\frac{1}{2}}^{1} h_i(\tau) d\tau \right\} \right) \\
&\geq \frac{1}{4} \varphi^{-1}(\lambda \tilde{m}_r \Gamma),
\end{align*}
\]

where \( \Gamma \triangleq \min \{ \min \{ \int_{\frac{1}{2}}^{1} h_i(\tau) d\tau, \int_{\frac{1}{2}}^{1} h_i(\tau) d\tau \} | i = 1, \ldots, N \} \). Define
\[
p(r) = \frac{\varphi(8r)}{m_r \Gamma},
\]
then \( p : (0, \infty) \to (0, \infty) \) is continuous. Since \( f_0 = f_\infty = 0 \), we get
\[
\lim_{r \to 0} p(r) = \lim_{r \to \infty} p(r) = \infty.
\]
Thus, there exists \( r_3 \in (0, \infty) \) such that
\[
(3.2) \quad p(r_3) = \inf \{ p(r) | r > 0 \} \leq \bar{\lambda}.
\]
Then for any \( \lambda > \bar{\lambda} \), there exist \( r_1, r_2 > 0 \) such that \( 0 < r_1 < r_3 < r_2 < \infty \) with \( p(r_1) = p(r_2) = \lambda \). Therefore, if \( u \in \partial K_{r_1} \), then for any \( \lambda > \bar{\lambda} \),
\[
2 \| T_{\lambda}^r(u) \|_\infty \geq 2T_{\lambda}^r(u)(\frac{1}{2}) \geq \frac{1}{4} \varphi^{-1}(\varphi(8r_1) \tilde{m}_r \Gamma) = 2r_1 = 2\| u \|_\infty,
\]
and thus
\[
(3.3) \quad \| T_{\lambda}^r(u) \|_\infty \geq \| T_{\lambda}^r(u) \|_\infty \geq \| u \|_\infty \text{ for } u \in \partial K_{r_1}, \lambda > \bar{\lambda}.
\]
Similarly,
\[
(3.4) \quad \| T_{\lambda}^r(u) \|_\infty \geq \| T_{\lambda}^r(u) \|_\infty \geq \| u \|_\infty \text{ for } u \in \partial K_{r_2}, \lambda > \bar{\lambda}.
\]
Let \( f_0 = f_\infty = 0 \), then \( f_0^i = f_\infty^i = 0 \), \( i = 1, \ldots, N \). For \( \lambda > \bar{\lambda} \), we can choose \( \epsilon(= \epsilon(\lambda)) > 0 \) sufficiently small so that
\[
\psi^{-1}(\lambda \epsilon) \Upsilon \leq \frac{1}{N},
\]
where
\[
\Upsilon \triangleq \max \left\{ \max \left\{ \int_{0}^{r_3} \psi^{-1}\left( \int_{s}^{1} h_i(\tau) d\tau \right) ds, \int_{0}^{r_3} \psi^{-1}\left( \int_{s}^{1} h_i(\tau) d\tau \right) ds \right\} | i = 1, \ldots, N \right\},
\]
Since \( f_0^i = 0 \), there exists \( r_3^i(= r_3^i(\epsilon)) > 0 \) such that for \( x \in \mathbb{R}_+^N \) with \( \| x \| \leq r_3^i \),
\[
f_i(x) \leq \epsilon \varphi(\| x \|) \text{ for } i = 1, \ldots, N.
\]
Take \( 0 < r_3 < \min \{ r_1, \min \{ r_3^i | i = 1, \ldots, N \} \} \). Then for \( u \in \partial K_{r_3} \), we get
\[
(3.5) \quad f_i(u(t)) \leq \epsilon \varphi(\| u(t) \|) \leq \epsilon \varphi(r_3) \text{ for } i = 1, \ldots, N.
\]
Since \( f_\infty^i = 0 \), we define a function \( \tilde{f}^i(t) : \mathbb{R}_+ \to \mathbb{R}_+ \) by
\[
\tilde{f}^i(t) = \max \{ f^i(x) | x \in \mathbb{R}_+^N, \| x \| \leq t \}.
\]
By Lemma 2.8 in Wang [20], we have

\[
\hat{f}_i^\infty = \lim_{t \to \infty} \frac{\hat{f}(t)}{\varphi(t)} = f_i^\infty = 0.
\]

Since \( \hat{f}_i^\infty = 0 \), then for \( \epsilon \) given above, there exists \( r_4' = r_4'(\epsilon) > 0 \) such that for \( t \in \mathbb{R}_+ \) with \( t \geq r_4' \),

\[
\hat{f}(t) \leq \epsilon \varphi(t) \quad \text{for} \quad i = 1, \ldots, N.
\]

Take \( r_4 > \max \{ r_2, \max \{ r_i' \mid i = 1, \ldots, N \} \} \). Then for \( u \in \partial K_{r_4} \), we get

\[
(3.6) \quad f^j(u(t)) \leq \hat{f}(r_4) \leq \epsilon \varphi(r_4) \quad \text{for} \quad i = 1, \ldots, N.
\]

Since \( T_\lambda(u) \in K \) for \( u \in \partial K_{r_j} (j = 3, 4) \), there exists a unique \( \sigma_i \in (0, 1) \) such that \( T_\lambda(u)(\sigma_i) = \max_{t \in [0, 1]} T_\lambda(u)(t) \) and \( T_\lambda(u)'(\sigma_i) = 0 \). We first consider the case \( \sigma_i \in (0, \frac{1}{2}] \).

\[
0 = T_\lambda(u)'(\sigma_i) = \varphi^{-1} \left( a_{\lambda,u}^j + \int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \right).
\]

Since \( \varphi \) is an odd homeomorphism, \( a_{\lambda,u}^j = -\int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \). Applying (3.5), (3.6) and Remark 2.2 with \( \sigma = \lambda \epsilon, x = \varphi^{-1} \left( \varphi(r_j) \int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) d\tau \right) \) and then \( \sigma = \int_{\sigma_i}^{\frac{1}{2}} h_i(\tau) d\tau, x = r_j \) consecutively, we obtain

\[
\| T_\lambda(u) \|_\infty = T_\lambda(u)(\sigma_i)
\]

\[
= \int_0^{\sigma_i} \varphi^{-1} \left( a_{\lambda,u}^j + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \right) ds
\]

\[
= \int_0^{\sigma_i} \varphi^{-1} \left( -\int_{\sigma_i}^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau + \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \right) ds
\]

\[
= \int_0^{\sigma_i} \varphi^{-1} \left( \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \right) ds
\]

\[
\leq \int_0^{\frac{1}{2}} \varphi^{-1} \left( \int_s^{\frac{1}{2}} \lambda h_i(\tau) f^j(u(\tau)) d\tau \right) ds
\]

\[
\leq \int_0^{\frac{1}{2}} \varphi^{-1} \left( \lambda \varphi(r_j) \int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds
\]

\[
\leq \psi^{-1}(\lambda \epsilon) \int_0^{\frac{1}{2}} \varphi^{-1} \left( \varphi(r_j) \int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds
\]

\[
\leq \psi^{-1}(\lambda \epsilon) \int_0^{\frac{1}{2}} \psi^{-1} \left( \int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds \]
Similarly for the case $\sigma_i \in [\frac{1}{2}, 1)$, we get
\[
\|T^s_\lambda(u)\|_\infty \leq \psi^{-1}(\lambda e) \left[ \int_{\frac{1}{2}}^1 \psi^{-1} \left( \int_{\frac{1}{2}}^s h_\lambda(\tau) \, d\tau \right) \, ds \right] r_j.
\]
Combining the above two inequalities and using the choice of $\epsilon$, we get
\[
\|T^s_\lambda(u)\|_\infty \leq \psi^{-1}(\lambda e) \frac{r_j}{N}
\]
for $i = 1, \ldots, N$, $j = 3, 4$, and thus
\[
(3.7) \quad \|T^s_\lambda(u)\|_\infty = \sum_{i=1}^N \|T^s_\lambda(u)\|_\infty \leq \|u\|_\infty \text{ for } u \in \partial K_r (j = 3, 4).
\]
Combining (3.3), (3.4) and (3.7), we conclude that problem $(P_\lambda)$ has at least two positive solutions $u_1, u_2$ with $r_3 \leq \|u_1\| \leq r_1 < r_2 \leq \|u_2\| \leq r_4$ for $\lambda > \lambda$.

**Lemma 3.2.** Assume that $(A)$, $(H)$, and $(F_1)$ hold. If $f_0 = f_\infty = 0$, then there exists $\Delta \in (0, \lambda)$ such that $(P_\lambda)$ has no positive solution for $\lambda \in (0, \Delta)$.

**Proof.** Since $f_0 = f_\infty = 0 < \infty$, then $f_0^i < \infty$ and $f_\infty^i < \infty$, $i = 1, \ldots, N$. Thus, for any $i = 1, \ldots, N$, there exist positive numbers $\beta_1^i, \beta_2^i, R_1^i, R_2^i$ such that $R_1^i < R_2^i$, $\beta_1^i > f_0^i$, $\beta_2^i > f_\infty^i$, $f^i(\mathbf{x}) \leq \beta_1^i \varphi(\|\mathbf{x}\|)$ for $\mathbf{x} \in \mathbb{R}_+^N$, $\|\mathbf{x}\| \leq R_1^i$, and $f^i(\mathbf{x}) \leq \beta_2^i \varphi(\|\mathbf{x}\|)$ for $\mathbf{x} \in \mathbb{R}_+^N$, $\|\mathbf{x}\| \geq R_2^i$. Let
\[
\beta^i = \max\{\beta_1^i, \beta_2^i, \max\{\frac{f^i(\mathbf{x})}{\varphi(\|\mathbf{x}\|)} \mid \mathbf{x} \in \mathbb{R}_+^N, R_1^i \leq \|\mathbf{x}\| \leq R_2^i\}\},
\]
and
\[
\beta = \max\{\max\{\beta^i \mid i = 1, \ldots, N\}, \inf\{\beta \mid \beta > 0, \frac{\psi(\frac{1}{\lambda \beta})}{\beta} < \lambda\}\}.
\]
Thus, we have
\[
(3.8) \quad f^i(\mathbf{x}) \leq \beta \varphi(\|\mathbf{x}\|) \text{ for } \mathbf{x} \in \mathbb{R}_+^N, i = 1, \ldots, N.
\]
Assume that $\mathbf{v}(t)$ is a positive solution of $(P_\lambda)$. We prove that if $(P_\lambda)$ has a positive solution, then $\lambda \geq \Delta$, where
\[
\Delta := \frac{\psi(\frac{1}{\lambda \beta})}{\beta}.
\]
Indeed, on the contrary, suppose that $(P_\lambda)$ has a positive solution $\mathbf{v}$ for $0 < \lambda < \Delta$. Since $\mathbf{v}(t) = T^s_\lambda(\mathbf{v})(t)$ for $t \in [0, 1]$, applying the same argument in the proof of Lemma 3.1 with aid of (3.8) and Remark 2.2 with $\sigma = \lambda \beta$,
\[ x = \varphi^{-1}\left(\varphi(\|v\|_{\infty}) f^\frac{1}{2} h_i(\tau) d\tau \right) \] and \( \sigma = \int h_i(\tau) d\tau, \) \( x = \|v\|_{\infty} \) consecutively, we get for \( 0 < \lambda < \lambda_0 \),

\[ \|v\|_{\infty} = \|T_\lambda(v)\|_{\infty} = \sum_{i=1}^{N} \|T_\lambda^i(v)\|_{\infty} \leq N \cdot \psi^{-1}(\lambda \beta) \|v\|_{\infty} < \|v\|_{\infty}, \]

which is a contradiction. \( \square \)

**Lemma 3.3.** Assume that (A), (H), (F_1), and (F_2) hold. If \( f_0 = f_\infty = \infty \), then there exists \( \lambda > 0 \) such that \( (P_\lambda) \) has at least two positive solutions for \( \lambda \in (0, \lambda_0) \).

**Proof.** For any \( r > 0 \), define

\[ \hat{M}_r = \max \{ f^i(x) \mid x \in \mathbb{R}^N_r, \|x\| \leq r, \ i = 1, \ldots, N \}. \]

By (F_2), then \( \hat{M}_r > 0 \). Let \( u \in \partial K_r \), then for \( t \in [0, 1] \),

\[ \|u(t)\| \leq \|u\|_{\infty} = r, \]

and

\[ \|T_\lambda^i(u)\|_{\infty} = \hat{M}_r \text{ for } i = 1, \ldots, N. \]

Since \( T_\lambda(u) \in K \) for \( u \in \partial K_r \), there exists a unique \( \sigma_i \in (0, 1) \) such that \( T_\lambda^i(u)(\sigma_i) = \max_{t \in [0, 1]} T_\lambda^i(u)(t) \) and \( T_\lambda^i(u)'(\sigma_i) = 0 \). We also consider two cases \( \sigma_i \in (0, \frac{1}{2}] \) and \( \sigma_i \in [\frac{1}{2}, 1) \) with the similar argument in the proof of Lemma 3.1 with aid of (3.10), we get

\[ \|T_\lambda^i(u)\|_{\infty} \leq \varphi^{-1}(\lambda \hat{M}_r) \beta \text{ for } i = 1, \ldots, N. \]

Define

\[ q(r) = \frac{\varphi(\frac{r}{\hat{M}_r})}{\hat{M}_r}, \]

then \( q : (0, \infty) \to (0, \infty) \) is continuous clearly. Since \( f_0 = f_\infty = \infty \), we get

\[ \lim_{r \to 0} q(r) = \lim_{r \to \infty} q(r) = 0. \]

Thus, there exists \( r^* \in (0, \infty) \) such that

\[ q(r^*) = \sup \{ q(r) \mid r > 0 \} \triangleq \lambda. \]

Then for any \( \lambda \in (0, \lambda_0) \), there exist \( r_1, r_2 > 0 \) such that \( 0 < r_1 < r^* < r_2 < \infty \) with \( q(r_1) = q(r_2) = \lambda \). Therefore, if \( u \in \partial K_{r_1} \), then for \( \lambda \in (0, \lambda_0) \),

\[ \|T_\lambda^i(u)\|_{\infty} \leq \varphi^{-1}\left(\frac{\varphi(\frac{r_1}{\hat{M}_{r_1}})}{\hat{M}_{r_1}}\right) \beta = \frac{r_1}{N} \text{ for } i = 1, \ldots, N, \]

and thus

\[ \|T_\lambda(u)\|_{\infty} \leq \sum_{i=1}^{N} \|T_\lambda^i(u)\|_{\infty} \leq \|u\|_{\infty} \text{ for } u \in \partial K_{r_1}, \lambda \in (0, \lambda_0). \]
Similarly,
\[
\|T_\lambda(u)\|_\infty = \sum_{i=1}^N \|T^{(i)}_\lambda(u)\|_\infty \leq \|u\|_\infty \text{ for } u \in \partial K_{r_2}, \lambda \in (0, \lambda_o).
\]
Let \( f_0 = f_\infty = \infty, \) then \( f_0^{j_0} = f_\infty^{j_0} = \infty, \) where
\[
f_0^{j_0} := \max\{f_0^i \mid i = 1, \ldots, N\}, \quad f_\infty^{j_0} := \max\{f_\infty^i \mid i = 1, \ldots, N\}
\]
for some \( i_0, j_0 \in \{1, \ldots, N\}. \) For \( \lambda \in (0, \lambda_o), \) we can take \( M = \frac{\gamma(32)}{M} > 0. \) Since \( f_0^{j_0} = \infty, \) there exists \( r_M > 0 \) such that for \( x \in \mathbb{R}_+^N \) with \( \|x\| \leq r_M, \) we have
\[
f^{j_0}(x) \geq M\varphi(\|x\|).
\]
If \( u \in K \) with \( \|u\|_\infty \leq r_M, \) then by Proposition 2.4, for \( t \in \left[\frac{1}{4}, \frac{3}{4}\right], \)
\[
\|u(t)\| \leq \|u\|_\infty \leq r_M,
\]
and
\[
f^{j_0}(u(t)) \geq M\varphi(\|u(t)\|) \geq M\varphi\left(\frac{1}{4}\|u\|_\infty\right).
\]
Take \( 0 < r_3 < \min\{r_1, r_M\}. \) Then for \( u \in \partial K_{r_3}, \) we get
\[
f^{j_0}(u(t)) \geq M\varphi(\|u(t)\|) \geq M\varphi\left(\frac{1}{4}\|u\|_\infty\right).
\]
Since \( f_\infty^{j_0} = \infty, \) for \( M \) given above, there exists \( R_M > 0 \) such that for \( x \in \mathbb{R}_+^N \) with \( \|x\| \geq R_M, \) we have
\[
f^{j_0}(x) \geq M\varphi(\|x\|).
\]
If \( u \in K \) with \( \|u\|_\infty \geq 4R_M, \) then by Proposition 2.4, for \( t \in \left[\frac{1}{4}, \frac{3}{4}\right], \)
\[
\|u(t)\| = \sum_{i=1}^N u_i(t) \geq \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \sum_{i=1}^N u_i(t) \geq \frac{1}{4}\|u\|_\infty \geq R_M,
\]
and
\[
f^{j_0}(u(t)) \geq M\varphi(\|u(t)\|) \geq M\varphi\left(\frac{1}{4}\|u\|_\infty\right).
\]
Take \( r_4 > \max\{r_2, 4R_M\}. \) Then for \( u \in \partial K_{r_4}, \) we get
\[
f^{j_0}(u(t)) \geq M\varphi(\|u(t)\|) \geq M\varphi\left(\frac{1}{4}\|u\|_\infty\right).
\]
We also consider two cases \( a^{i_0}_{M, u} \geq 0 \) and \( a^{i_0}_{M, u} < 0 \) \( (i = i_0, j_0). \) Applying the same argument in the proof of Lemma 3.1 with aids of (3.15), (3.17) and by the definition of \( M, \) we get
\[
2\|T^i_\lambda(u)\|_\infty \geq 2T^{j_0}_\lambda(u)(\frac{1}{2}) = \frac{1}{4}\varphi^{-1}\left(\lambda M\varphi\left(\frac{1}{4}\|u\|_\infty\right)\right)
\geq \frac{1}{4}\varphi^{-1}\left(\gamma(32)\varphi\left(\frac{1}{4}\|u\|_\infty\right)\right).
\]
Applying Remark 2.2 with $\sigma = 32$ and $x = \frac{3}{4}||u||_\infty$, we get

$$2||T_\lambda^j(u)||\infty \geq \frac{1}{4} \times 32 \times \frac{1}{4}||u||\infty = 2||u||\infty.$$ 

Thus, for $i = i_0, j_0$, we have

$$(3.18) \quad ||T_\lambda(u)||\infty \geq ||T_\lambda^j(u)||\infty \geq ||u||\infty \text{ for } u \in \partial K_{r_j} (j = 3, 4).$$

Combining (3.12), (3.13) and (3.18), we conclude that problem $(P_\lambda)$ has at least two positive solutions $u_1, u_2$ with $r_3 \leq ||u_1||\infty \leq r_2 \leq ||u_2||\infty \leq r_4$ for $\lambda \in (0, \bar{\lambda})$.

**Lemma 3.4.** Assume that $(A)$, $(H)$, and $(F_1)$ hold. If $f_0 = f_\infty = \infty$, then there exists $\lambda \in (\bar{\lambda}, \infty)$ (here $\bar{\lambda}$ is given in Lemma 3.3) such that $(P_\lambda)$ has no positive solution for $\lambda > \bar{\lambda}$.

**Proof.** Since $f_0 = f_\infty = \infty$, we can easily get $f_0^1 > 0$ and $f_\infty^1 > 0$. Thus, there exist positive numbers $\eta_1, \eta_2, r'_1$ and $r'_2$ such that $r'_1 < r'_2, 0 < \eta_1 < f_0^1, 0 < \eta_2 < f_\infty^1$, 

$$f^1(x) \geq \eta_1 \varphi(||x||) \text{ for } x \in \mathbb{R}^N_+, ||x|| \leq r'_1,$$

and 

$$f^1(x) \geq \eta_2 \varphi(||x||) \text{ for } x \in \mathbb{R}^N_+, ||x|| \geq r'_2.$$ 

Let 

$$\eta_3 = \min\{\eta_1, \eta_2, \min\{\frac{f^1(x)}{\varphi(||x||)} \mid x \in \mathbb{R}^N_+, \frac{r'_1}{4} \leq ||x|| \leq r'_2\}\},$$

$$\sup\{\eta \mid \eta > 0, \frac{\gamma(32)}{\eta^1} > \bar{\lambda}\} > 0.$$ 

Then, we have

$$(3.19) \quad f^1(x) \geq \eta_3 \varphi(||x||) \text{ for } x \in \mathbb{R}^N_+, ||x|| \leq r'_1,$$

and 

$$(3.20) \quad f^1(x) \geq \eta_3 \varphi(||x||) \text{ for } x \in \mathbb{R}^N_+, ||x|| \geq \frac{r'_1}{4}.$$ 

Assume that $v$ is a positive solution of $(P_\lambda)$, we prove that if $(P_\lambda)$ has a positive solution, then $\lambda \leq \bar{\lambda}$, where

$$(3.21) \quad \bar{\lambda} := \frac{\gamma(32)}{\eta_3^1}.$$ 

Indeed, on the contrary, suppose that $(P_\lambda)$ has a positive solution $v$ for $\lambda > \bar{\lambda}$. If $||v||\infty \leq r'_1$, then by (3.19) and Proposition 2.4, we get for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$,

$$(3.22) \quad f^1(v(t)) \geq \eta_3 \varphi(||v(t)||) \geq \eta_3 \varphi\left(\frac{1}{4}||v||\infty\right).$$
On the other hand, if $\|v\|_\infty > r'_1$, then by Proposition 2.4 and (3.20),

$$\|v(t)\| = \sum_{i=1}^{N} v_i(t) \geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \sum_{i=1}^{N} v_i(t) \geq \frac{1}{4}\|v\|_\infty > \frac{r'_1}{4},$$

and

$$(3.23) \quad f^{j_0}(v(t)) \geq \eta \varphi(\|v(t)\|) \geq \eta \varphi(\frac{1}{4}\|v\|_\infty)$$

for $t \in [\frac{1}{4}, \frac{3}{4}]$. Since $v(t) = T_\lambda(v)(t)$ for $t \in [0, 1]$, applying the same argument in the proof of Lemma 3.1 with aids of (3.22), (3.23) and Remark 2.2 with $\sigma = 32$, $x = \frac{1}{4}\|v\|_\infty$, then for $\lambda > \bar{\lambda}$,

$$\|v\|_\infty = \|T_\lambda(v)\|_\infty \geq \frac{1}{8}\varphi^{-1}\left(\lambda\eta\varphi(\frac{1}{4}\|v\|_\infty)\Gamma\right)$$

$$> \frac{1}{8}\varphi^{-1}\left(\gamma(32)\varphi(\frac{1}{4}\|v\|_\infty)\right)$$

$$\geq \frac{1}{8} \times 32 \times \frac{1}{4}\|v\|_\infty = \|v\|_\infty,$$

which is a contradiction. \hfill \Box

Proof of Theorem 1.1. Theorem 1.1(1) follows from Lemma 3.1 and Lemma 3.2. Theorem 1.1(2) follows from Lemma 3.3 and Lemma 3.4. \hfill \Box

Lemma 3.5. Assume that (A), (H), (F_1), (F_3), and $f_0 = \infty$ hold. If $(P_\lambda)$ has a positive solution at $\lambda = \hat{\lambda}$, then $(P_\lambda)$ has at least one positive solution for $\lambda \in (0, \hat{\lambda})$.

Proof. Let $\hat{u}$ be a positive solution of $(P_\lambda)$ at $\lambda = \hat{\lambda}$ and let $\lambda \in (0, \hat{\lambda})$ be fixed. Consider the following modified problem

$$(P_\lambda^*) \quad \begin{cases} -\Phi(u')' = \lambda h(t) \cdot f_\ast(u), & t \in (0, 1), \\ u(0) = 0 = u(1), \end{cases}$$

where $f_\ast = (f_1^\ast, \ldots, f_N^\ast)$ and each $f_i^\ast : \mathbb{R}_+^N \to \mathbb{R}_+$ is defined by $f_i^\ast(u_1, \ldots, u_N) = f^i(\gamma_1(u_1), \ldots, \gamma_N(u_N))$ with

$$\gamma_i(u_i) = \begin{cases} \hat{u}_i, & \text{if } u_i > \hat{u}_i, \\ u_i, & \text{if } 0 \leq u_i \leq \hat{u}_i. \end{cases}$$

First, we show that $(P_\lambda^*)$ has at least one positive solution. Define $T_\lambda^*$ the same as $T_\lambda$ replacing $f$ by $f_\ast$. Then $T_\lambda^* : K \to K$ is also completely continuous. By the fact that $f_\ast$ is bounded, there exists $R > 0$ such that $\|T_\lambda^*(u)\|_\infty \leq R$, for any $u \in K$, i.e.,

$$\|T_\lambda^*(u)\|_\infty \leq \|u\|_\infty \quad \text{for } u \in \partial K_R.$$
Let \( f_0 = \infty \), then \( f_0^\infty = \infty \). Applying the similar argument in Lemma 3.3 with \( 0 < r < \min\{\|\hat{u}\|_\infty, R\} \), we get
\[(3.25) \quad \|T^*_\lambda(u)\|_\infty \geq \|(T^*_\lambda)^{\ast}(u)\|_\infty \geq \|u\|_\infty \]
for \( u \in \partial K_r \). Combing (3.24) and (3.25), we conclude that (\( P^*_\lambda \)) has at least one solution \( u \) with \( r \leq \|u\|_\infty \leq R \), i.e., \( u \) is a positive solution.

Next, we show that if \( u \) is a solution of (\( P^*_\lambda \)), then \( 0 \leq u(t) \leq \hat{u}(t) \) for \( t \in [0, 1] \). If it is true, then (\( P^*_\lambda \)) and (\( P_\lambda \)) are equivalent and the proof is complete. Clearly, \( u(t) \geq 0 \) for \( t \in [0, 1] \). We also need show that \( u(t) \leq \hat{u}(t) \) for \( t \in [0, 1] \). If it is not true, then \( u_i(t) \leq \hat{u}_i(t) \) for some \( i \in \{1, \ldots, N\} \). By the boundary values of \( u_i \) and \( \hat{u}_i \), there exist \( T_1, T_2 \in (0, 1) \) such that
\[ u_i(t) - \hat{u}_i(t) > 0 \text{ on } (T_1, T_2) \text{ and } u_i(T_1) - \hat{u}_i(T_1) = u_i(T_2) - \hat{u}_i(T_2) = 0. \]
Thus, by (\( F_3 \)), we have for \( t \in (T_1, T_2) \),
\[-\varphi(u'_i(t))' = \lambda h_i(t)f_i^\ast(u_1, \ldots, u_i, \ldots, u_N) \]
\[= \lambda h_i(t)f_i(\gamma_1(u_1), \ldots, \hat{u}_i, \ldots, \gamma_i(u_N)) \]
\[\leq \hat{\lambda} h_i(t)f_i(\tilde{u}_i, \ldots, \hat{u}_i, \ldots, u_N) \]
\[= -\varphi(\hat{u}_i(t))', \]
i.e.,
\[(3.26) \quad \varphi(u'_i(t))' \geq \varphi(\hat{u}_i(t))'. \]
Since \( u_i - \hat{u}_i \in C_0[T_1, T_2] \), there exist \( t_0 \in (T_1, T_2) \) and \( 0 < \delta < T_2 - t_0 \) such that
\[ u_i(t_0) - \hat{u}_i(t_0) = \max_{t \in [t_0, t_0 + \delta]} \{u_i(t) - \hat{u}_i(t)\}, \]
and
\[ u'_i(t_0) - \hat{u}'_i(t_0) = 0, \quad u'_i(t) - \hat{u}'_i(t) < 0, \quad t \in (t_0, t_0 + \delta). \]
Integrating both sides of (3.26) from \( t_0 \) to \( t \in (t_0, t_0 + \delta) \), then we get
\[ \varphi(u'_i(t)) - \varphi(u'_i(t_0)) \geq \varphi(\hat{u}_i'(t)) - \varphi(\hat{u}_i'(t_0)). \]
Since \( \varphi \) is increasing, we have \( u'_i(t) \geq \hat{u}'_i(t), t \in (t_0, t_0 + \delta) \), which is a contradiction. \( \square \)

**Lemma 3.6.** Assume that (\( A \)), (\( H \)), (\( F_1 \)), and \( f_\infty = \infty \) hold. Let \( I \) be a compact interval of \((0, \infty) \). Then there exists a constant \( b_1 > 0 \) such that all possible positive solutions \( u \) of (\( P_\lambda \)) at \( \lambda \in I \) satisfy \( \|u\|_\infty < b_1 \).

**Proof.** Suppose on the contrary that there exists a sequence \( \{u_n\} \) of positive solutions of (\( P_\lambda \)) with \( \{\lambda_n\} \subset I = [\alpha, \beta] \subset (0, \infty) \) and \( \|u_n\|_\infty \to \infty \) as \( n \to \infty \). Take \( M = \frac{2\gamma_1(\alpha)}{\alpha} \). Let \( f_\infty = \infty \), then \( f_\infty^\alpha = \infty \). Since \( f_\infty^\beta = \infty \), for \( M \) given above, there exists \( R_M > 0 \) such that for \( x \in \mathbb{R}_+^N \) with \( \|x\| \geq R_M \), we have
\[ f_j^\alpha(x) \geq M \varphi(\|x\|). \]
From the assumption, we can get \( \|u_n\|_\infty \geq 4R_M \) for sufficiently large \( n \). Thus, by Proposition 2.4, we have
\[
\|u_n(t)\| = \sum_{i=1}^{N} u_i^n(t) \geq \min_{i \in [1, N]} \sum_{i=1}^{N} u_i^n(t) \geq \frac{1}{4} \|u_n\|_\infty \geq R_M,
\]
and
\[
(3.27) \quad f_i^{\text{in}}(u_n(t)) \geq M \varphi(\|u_n(t)\|) \geq M \varphi(\frac{1}{4} \|u_n\|_\infty)
\]
for \( t \in [\frac{1}{4}, \frac{3}{4}] \) and sufficiently large \( n \). Since \( u_n(t) = T_{\lambda_n}(u_n)(t) \) for \( t \in [0, 1] \), applying the same argument in Lemma 3.1 with aid of (3.27) and by the definition of \( M \) and Remark 2.2 with \( \sigma = 32 \), \( x = \frac{1}{4} \|u_n\|_\infty \), we get
\[
\|u_n\|_\infty = \|T_{\lambda_n}(u_n)\|_\infty \geq \frac{1}{8} \varphi^{-1}(\lambda_n M \varphi(\frac{1}{4} \|u_n\|_\infty) \Gamma)
\]
\[
\geq \frac{1}{8} \varphi^{-1}(\alpha M \varphi(\frac{1}{4} \|u_n\|_\infty))
\]
\[
\geq \frac{1}{8} \varphi^{-1}(2\gamma(32)\varphi(\frac{1}{4} \|u_n\|_\infty))
\]
\[
> \frac{1}{8} \varphi^{-1}(\gamma(32)\varphi(\frac{1}{4} \|u_n\|_\infty))
\]
\[
\geq \frac{1}{8} \times 32 \times \frac{1}{4} \|u_n\|_\infty = \|u_n\|_\infty
\]
for \( \lambda_n \in I \) with sufficiently large \( n \). This is a contradiction. \( \square \)

**Proof of Theorem 1.2.** Define
\[
(3.28) \quad \lambda^* := \sup\{\lambda \mid (P_\lambda) \text{ has at least one positive solution}\}.
\]
\[
(3.29) \quad \lambda_* := \sup\{\tilde{\lambda} \mid (P_\lambda) \text{ has at least two positive solutions for } \lambda \in (0, \tilde{\lambda})\}.
\]
By Lemma 3.3 and Lemma 3.4, \( \lambda_* \) and \( \lambda^* \) are both well-defined and \( 0 < \lambda_* \leq \lambda^* \leq \tilde{\lambda} \). By the definitions of \( \lambda_* \) and \( \lambda^* \), and Lemma 3.5, we get that \( (P_\lambda) \) has at least two positive solutions for \( \lambda \in (0, \lambda_0) \), one positive solution for \( \lambda \in [\lambda_0, \lambda^*] \), and no positive solution for \( \lambda > \lambda^* \).

Finally, it is enough to show that \( (P_\lambda) \) has at least one positive solution at \( \lambda = \lambda^* \). By the definition of \( \lambda^* \) and Lemma 3.4, we can choose a sequence \( \{\lambda_n\} \) with \( \frac{\lambda_n}{2} \leq \lambda_n < \lambda^* \leq \lambda \) such that \( \lambda_n \to \lambda^* \) as \( n \to \infty \), and then by Lemma 3.6 with \( I = [\frac{\lambda_n}{2}, \lambda] \), there exists \( b_I > 0 \) such that the corresponding positive solutions \( u_n \) satisfying \( \|u_n\|_\infty < b_I \), i.e., \( \{u_n\} \) is bounded.

By the fact that \( T_{\lambda_n} \) is completely continuous, we get \( \{T_{\lambda_n}(u_n)\} \) is equicontinuous. This implies that \( \{u_n\} \) is equicontinuous, since \( u_n = T_{\lambda_n}(u_n) \). By the Ascoli-Arzela theorem, \( \{u_n\} \) is relatively compact. Hence, there exists a convergent subsequence \( \{u_n\} \), denoted again by \( \{u_n\} \) and \( u^* \in K \) such that \( u_n \to u^* \) as \( n \to \infty \). Since \( u_n = T_{\lambda_n}(u_n) \), by the Lebesgue Dominated Convergence Theorem, we can get \( u^* = T_{\lambda^*}(u^*) \), i.e., \( u^* \) is a solution of \( (P_{\lambda^*}) \).
Moreover, by $f_0 = \infty$ and applying the similar argument in Lemma 3.6, we see that $u^* \neq 0$. Therefore mainly due to condition $(F_2)$ and the Maximal Principle, it is not hard to see that $u^*$ is a positive solution of $(P_{\lambda^*})$. □

4. Applications

In this section, we give some examples applicable to our main results.

Example 4.1. Consider the following scalar $\varphi$-Laplacian problem

\[
(E_1) \quad \begin{cases}
\varphi(u')' + \lambda t^{-\frac{2}{3}} f(u) = 0, & t \in (0, 1), \\
u(0) = 0 = u(1),
\end{cases}
\]

where $\varphi(x) = |x|x + x$, $x \in \mathbb{R}$, and

\[f(u) = \begin{cases}
u^3, & \text{if } 0 \leq u < 1, \\
u, & \text{if } u \geq 1.
\end{cases}\]

We easily see that $\varphi$ is an odd increasing homeomorphism. Define functions $\psi$ and $\gamma$ given as

\[
\psi(\sigma) = \begin{cases}
\sigma^2, & \text{if } 0 < \sigma \leq 1, \\
\sigma, & \text{if } \sigma > 1,
\end{cases}
\]

and

\[
\gamma(\sigma) = \begin{cases}
1, & \text{if } 0 < \sigma \leq 1, \\
\sigma^2, & \text{if } \sigma > 1.
\end{cases}
\]

Then $\psi, \gamma : (0, \infty) \to (0, \infty)$ and $\psi$ is an increasing homeomorphism with

\[
\psi^{-1}(\sigma) = \begin{cases}
\sigma^{\frac{1}{2}}, & \text{if } 0 < \sigma \leq 1, \\
\sigma, & \text{if } \sigma > 1.
\end{cases}
\]

We may see that $(E_1)$ satisfies assumptions $(A)$, $(H)$, $(F_1)$ and $(F_2)$ (see Xu and Lee [22] for details). In addition,

\[f_0 = \lim_{\|u\| \to 0} \frac{f(u)}{\varphi(\|u\|)} = \lim_{\|u\| \to 0} \frac{u^3}{u^2 + u} = 0,
\]

\[f_\infty = \lim_{\|u\| \to \infty} \frac{f(u)}{\varphi(\|u\|)} = \lim_{\|u\| \to \infty} \frac{u}{u^2 + u} = 0.
\]

For any $r > 0$,

\[\hat{m}_r = \max \{f(x) \mid x \in \mathbb{R}_+, \frac{r}{4} \leq x \leq r\} = f(r),
\]

where

\[f(r) = \begin{cases}
r^3, & \text{if } 0 < r < 1, \\
r, & \text{if } r \geq 1.
\end{cases}\]

If $0 < r < 1$, then
\[ p(r) = \frac{\varphi(8r)}{m_r \Gamma} = \frac{(8r)^2 + 8r}{0.49r^3} \approx 64r + 8r, \]
and
\[ p'(r) = -\frac{31.36r - 7.84}{0.2401r^3} < 0. \]
If $r \geq 1$, then
\[ p(r) = \frac{\varphi(8r)}{m_r \Gamma} = \frac{(8r)^2 + 8r}{0.49r^2} = 64r + \frac{8r}{0.49}, \]
and
\[ p'(r) = \frac{64}{0.49} > 0. \]
Thus, we get
\[ \bar{\lambda} = \inf \{ p(r) \mid r > 0 \} = p(1) = \frac{64 \times 1 + 8}{0.49} \approx 146.94. \]
Since $f_0 = f_\infty = 0$, there exist $\beta_1 = 1 > f_0$, $\beta_2 = \frac{1}{10000} > f_\infty$, $R_1 = 1$, $R_2 = 10000$ such that
\[ f(x) \leq \varphi(x) \text{ for } 0 \leq x \leq 1, \]
and
\[ f(x) \leq \frac{1}{10000} \varphi(x) \text{ for } x \geq 10000. \]
Since for $x \geq 1$,
\[ \frac{f(x)}{\varphi(x)} = \frac{x}{x^2 + x} = \frac{1}{x + 1}, \]
we get
\[ \max \{ \frac{f(x)}{\varphi(x)} \mid x \in \mathbb{R}_+, 1 \leq x \leq 10000 \} = \frac{1}{2}. \]
From
\[ \psi\left(\frac{1}{\sqrt{\beta}}\right) < \bar{\lambda}, \]
we get
\[ \frac{(\frac{1}{\sqrt{\beta}})^2}{\beta} < 146.94, \]
i.e., $\beta > 0.0031$ and thus
\[ \inf \{ \beta \mid \beta > 0, \frac{\psi\left(\frac{1}{\sqrt{\beta}}\right)}{\beta} < \bar{\lambda} \} > 0.0031. \]
Therefore, we obtain
\[ \beta = \max\{ \beta_1, \beta_2, \max \{ \frac{f(x)}{\varphi(x)} \mid x \in \mathbb{R}_+, 1 \leq x \leq 10000 \}, \inf \{ \beta \mid \beta > 0, \frac{\psi\left(\frac{1}{\sqrt{\beta}}\right)}{\beta} < \bar{\lambda} \} \} = 1, \]
and
\[ \lambda = \frac{\psi(\frac{1}{N})}{\beta} = \left(\frac{1}{1 \times 0.46}\right)^2 = 0.46. \]

Consequently, by Theorem 1.1(1), we get the following Conclusion.

**Conclusion.** Problem \((E_1)\) has at least two positive solutions for \(\lambda > 146.94\), and no positive solution for \(\lambda \in (0, 0.46)\).

**Example 4.2.** Consider the following \(\varphi\)-Laplacian system
\[
\begin{align*}
(E_2) & \quad \left\{ \begin{array}{ll}
\varphi(u')' + \lambda t^{\frac{N}{2}} f^1(u, v) = 0, \\
\varphi(v')' + \lambda t^{\frac{N}{2}} f^2(u, v) = 0, \\
u(0) = v(0) = u(1) = v(1) = 0,
\end{array} \right. \\
t \in (0, 1),
\end{align*}
\]
where \(\varphi(x) = x^\beta, x \in \mathbb{R}, f^1(u, v) = e^{-u(v+1)^{\frac{1}{2}}}, f^2(u, v) = (u + v + 2)^{\frac{1}{2}}\). Then \(\varphi\) is an odd increasing homeomorphism. By the homogeneity of \(\varphi\), taking \(\psi(\sigma) = \gamma(\sigma) \equiv \varphi(\sigma)\). We can easily check that \((E_2)\) satisfies assumptions \((A),(H),(F_1)\) and \((F_2)\) (see Xu and Lee [22] for details) and exactly obtain
\[ \Gamma = \min\{\min\{\int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau, \int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau\} | i = 1, 2\} = 0.4473. \]

In fact,
\[ \int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau = \int_{\frac{1}{2}}^{\frac{1}{2}} \tau^{-\frac{N}{2}}d\tau \]
\[ = -4\tau^{-\frac{1}{2}} \bigg|_{\frac{1}{2}}^{\frac{1}{2}} = -4[\left(\frac{1}{2}\right)^{-\frac{1}{2}} - \left(\frac{1}{2}\right)^{-\frac{1}{2}}] = -4\left(2^{\frac{1}{2}} - 4^{\frac{1}{2}}\right) \approx 0.9000, \]
\[ \int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau = \int_{\frac{1}{2}}^{\frac{1}{2}} \tau^{-\frac{N}{2}}d\tau \]
\[ = -4\tau^{\frac{1}{2}} \bigg|_{\frac{1}{2}}^{\frac{1}{2}} = -4[\left(\frac{3}{4}\right)^{-\frac{1}{2}} - \left(\frac{1}{2}\right)^{-\frac{1}{2}}] = -4\left(\frac{3}{4}^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) \approx 0.4585, \]
\[ \int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau = \int_{\frac{1}{2}}^{\frac{1}{2}} \tau^{-\frac{N}{2}}d\tau \]
\[ = -5\tau^{-\frac{1}{2}} \bigg|_{\frac{1}{2}}^{\frac{1}{2}} = -5[\left(\frac{1}{2}\right)^{-\frac{1}{2}} - \left(\frac{1}{2}\right)^{-\frac{1}{2}}] = -5\left(2^{\frac{1}{2}} - 4^{\frac{1}{2}}\right) \approx 0.8540, \]
\[ \int_{\frac{1}{2}}^{\frac{1}{2}} t_{i}(\tau)d\tau = \int_{\frac{1}{2}}^{\frac{1}{2}} \tau^{-\frac{N}{2}}d\tau \]
\[ = -5\tau^{\frac{1}{2}} \bigg|_{\frac{1}{2}}^{\frac{1}{2}} = -5[\left(\frac{3}{4}\right)^{-\frac{1}{2}} - \left(\frac{1}{2}\right)^{-\frac{1}{2}}] = -5\left(\frac{3}{4}^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) \approx 0.4473, \]
\[ \Upsilon = \max\{\max\{H^0_0, H^1_1\} | i = 1, 2\} = 53.8174. \]
In fact,
\[ H_0^1 = \int_0^1 \psi^{-1}(\int_s^1 h_1(\tau)d\tau)ds = \int_0^1 (\int_s^1 \tau^{-\frac{3}{2}}d\tau)^3 ds \approx 53.8174, \]
\[ H_1^1 = \int_0^1 \psi^{-1}(\int_s^1 h_1(\tau)d\tau)ds = \int_0^1 (\int_s^1 \tau^{-\frac{3}{2}}d\tau)^3 ds \approx 0.0690, \]
\[ H_0^2 = \int_0^1 \psi^{-1}(\int_s^1 h_2(\tau)d\tau)ds = \int_0^1 (\int_s^1 \tau^{-\frac{3}{2}}d\tau)^3 ds \approx 23.6831, \]
\[ H_1^2 = \int_0^1 \psi^{-1}(\int_s^1 h_2(\tau)d\tau)ds = \int_0^1 (\int_s^1 \tau^{-\frac{3}{2}}d\tau)^3 ds \approx 0.0648. \]

In addition,
\[ f_0^1 = \lim_{\|u,v\| \to 0} \frac{f^1(u,v)}{\varphi(\|u,v\|)} = \lim_{\|u,v\| \to 0} \frac{e^{-u}(v+1)^\frac{3}{2}}{(u+v)^\frac{3}{2}} = \lim_{\|u,v\| \to 0} \frac{(v+1)^\frac{3}{2}}{(u+v)^\frac{3}{2}} = \infty, \]
\[ 0 \leq f_\infty^1 = \lim_{\|u,v\| \to \infty} \frac{f^1(u,v)}{\varphi(\|u,v\|)} = \lim_{\|u,v\| \to \infty} \frac{e^{-u}(v+1)^\frac{3}{2}}{(u+v)^\frac{3}{2}} \leq \lim_{\|u,v\| \to \infty} \frac{(u+v+1)^\frac{3}{2}}{(u+v)^\frac{3}{2}} = \infty, \]
\[ f_0^2 = \lim_{\|u,v\| \to 0} \frac{f^2(u,v)}{\varphi(\|u,v\|)} = \lim_{\|u,v\| \to 0} \frac{(u+v+2)^\frac{3}{2}}{(u+v)^\frac{3}{2}} = \infty, \]
\[ f_\infty^2 = \lim_{\|u,v\| \to \infty} \frac{f^2(u,v)}{\varphi(\|u,v\|)} = \lim_{\|u,v\| \to \infty} \frac{(u+v+2)^\frac{3}{2}}{(u+v)^\frac{3}{2}} \geq \lim_{\|u,v\| \to \infty} (u+v)^\frac{3}{2} = \infty. \]

Thus,
\[ f_0 = f_0^1 + f_0^2 = \infty, \quad f_\infty = f_\infty^1 + f_\infty^2 = \infty. \]

For any \( r > 0 \),
\[ M_r = \max \{ f^i(x) \mid x \in \mathbb{R}_+^2, \|x\| \leq r, i = 1,2 \} = (r+2)^\frac{3}{2}. \]
Then we can easily get
\[ q(r) = \frac{\varphi\left(\sqrt[r]{r}\right)}{M_r} = \frac{\varphi\left(\sqrt[r]{r}\right)\varphi(r)}{M_r} = \frac{\left(\frac{1}{x^{3.8194}}\right)^{r^{0.5}}}{(r+2)^{0.5}} \] \[ = 0.2102r^{0.5}, \]
and
\[ q'(r) = \begin{cases} > 0, & \text{if } 0 < r < 4, \\ = 0, & \text{if } r = 4, \\ < 0, & \text{if } r > 4. \end{cases} \]
Thus, we get
\[ \bar{\lambda} = \sup\{ q(r) \mid r > 0 \} = q(4) \approx 0.1362. \]
Since \( f_0^2 = f_{\infty}^2 = \infty \), there exist \( \eta_1 = 1 < f_0^2, \eta_2 = 10 < f_\infty^2, r_1 = 1, r_2 = 10^6 \) such that
\[ f^2(x) \geq \varphi(||x||) \text{ for } x \in \mathbb{R}_1^2, ||x|| \leq 1, \]
and
\[ f^2(x) \geq 10\varphi(||x||) \text{ for } x \in \mathbb{R}_2^2, ||x|| \geq 10^6. \]
Since
\[ \frac{f^2(x)}{\varphi(||x||)} = \frac{(||x|| + 2)^{0.5}}{||x||^{0.5}}, \]
we get
\[ \min\left\{ \frac{f^2(x)}{\varphi(||x||)} \mid x \in \mathbb{R}_2^2, \frac{1}{4} \leq ||x|| \leq 10^6 \right\} = \frac{(4 + 2)^{0.5}}{4^{0.5}} = 1.5438. \]
From
\[ \gamma(32) \frac{\eta}{\eta} > \bar{\lambda}, \]
we get
\[ \frac{3.1748}{\eta \cdot 0.4473} > 0.1362, \]
i.e., \( \eta < 52.1123 \) and thus
\[ \sup\{ \eta \mid \eta > 0, \frac{\gamma(32)}{\eta} > \bar{\lambda} \} > 52.1123. \]
Therefore, we obtain
\[ \eta_3 = \min\{\eta_1, \eta_2, \min\left\{ \frac{f^2(x)}{\varphi(||x||)} \mid x \in \mathbb{R}_2^2, \frac{1}{4} \leq ||x|| \leq 10^6 \right\} \sup\{\eta \mid \eta > 0, \frac{\gamma(32)}{\eta} > \bar{\lambda} \} \} = 1, \]
and
\[ \bar{\lambda} = \frac{\gamma(32)}{\eta_3} = \frac{3.1748}{1 \times 0.4473} \approx 7.0977. \]
Consequently, by Theorem 1.1(2), we get the following conclusion.
Conclusion. Problem (E_2) has at least two positive solutions for \( \lambda \in (0, 0.1362) \), and no positive solution for \( \lambda > 7.0977 \).

Clearly, problem (E_2) also satisfies assumption (F_3). By Theorem 1.2, there must exist \( \lambda^* \geq \lambda_\ast > 0 \) such that problem (E_2) has at least two positive solutions for \( \lambda \in (0, \lambda_\ast) \), one positive solution for \( \lambda \in [\lambda_\ast, \lambda^*] \), and no positive solution for \( \lambda > \lambda^* \).

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