INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5) FUNCTION

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Abstract. A necessary and sufficient condition in terms of lower cut sets is given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that whose $F_\sigma$-kernel of sets are $F_\sigma$-sets.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [17]. He investigated the sets that can be represented as union of closed sets and called them $V$–sets. Complements of $V$–sets, i.e., sets that are intersection of open sets are called $\Lambda$–sets [17].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$–continuous [22] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$–continuity. However, for unknown concepts the reader may refer to [4, 10].

In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 21].

Results of Katětov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are
used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that whose $F_\sigma$—kernel of sets are $F_\sigma$—sets.

A real-valued function $f$ defined on a topological space $X$ is called *contra-Baire-1* (Baire-.5) if the preimage of every open subset of $\mathbb{R}$ is a $G_\delta$—set in $X$ [23].

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all $x$ in $X$.

The following definitions are modifications of conditions considered in [15].

A property $P$ defined relative to a real-valued function on a topological space is a $B - .5$—property provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and any Baire-.5 function also has property $P$. If $P_1$ and $P_2$ are $B - .5$—properties, the following terminology is used:

(i) A space $X$ has the weak $B - .5$—insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a Baire-.5 function $h$ such that $g \leq h \leq f$.

(ii) A space $X$ has the $B - .5$—insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a Baire-.5 function $h$ such that $g < h < f$.

In this paper, for a topological space that whose $F_\sigma$—kernel of sets are $F_\sigma$—sets, is given a sufficient condition for the weak $B - .5$—insertion property. Also for a space with the weak $B - .5$—insertion property, we give a necessary and sufficient condition for the space to have the $B - .5$—insertion property. Several insertion theorems are obtained as corollaries of these results.

### 2. The Main Result and Applications

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^A$ and $A^V$ as follows:

$A^A = \cap \{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup \{F : F \subseteq A, F^c \in (X, \tau)\}$.

In [6, 16, 20], $A^A$ is called the *kernel* of $A$.

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$G_\delta(A) = \cup \{O : O \subseteq A, O \text{ is } G_\delta \text{ set}\}$ and

$F_\sigma(A) = \cap \{F : F \supseteq A, F \text{ is } F_\sigma \text{ set}\}$.
The following first two definitions are modifications of conditions considered in [13, 14].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \) then \( \bar{\rho} \) is defined as follows: \( x \bar{\rho} y \) if and only if \( y \rho v \) implies \( x \rho v \) and \( u \rho x \) implies \( u \rho y \) for any \( u \) and \( v \) in \( S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a strong binary relation in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

1) If \( A_i \rho B_j \) for any \( i \in \{1, \ldots, m\} \) and for any \( j \in \{1, \ldots, n\} \), then there exists a set \( C \) in \( P(X) \) such that \( A_i \rho C \) and \( C \rho B_j \) for any \( i \in \{1, \ldots, m\} \) and any \( j \in \{1, \ldots, n\} \).

2) If \( A \subseteq B \), then \( A \bar{\rho} B \).

3) If \( A \rho B \), then \( F_\sigma(A) \subseteq B \) and \( A \subseteq G_\delta(B) \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f, \ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f, \ell) \) is a lower indefinite cut set in the domain of \( f \) at the level \( \ell \).

We now give the following main results:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), that \( F_\sigma- \text{kernel sets} \) in \( X \) are \( F_\sigma- \text{sets} \), with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \), then there exists a Baire-.5 function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** [19, Theorem 2.1].

**Definition 2.5.** A real-valued function \( f \) defined on a space \( X \) is called contra-upper semi-Baire-.5 (resp. contra-lower semi-Baire-.5) if \( f^{-1}(-\infty, t) \) (resp. \( f^{-1}(t, +\infty) \)) is a \( G_\delta \)-set for any real number \( t \).
The abbreviations \textit{usc}, \textit{lsc}, \textit{cusB}.5 and \textit{clsB}.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

\textbf{Remark 1} ([13, 14]). A space $X$ has the weak $c$–insertion property for (\textit{usc}, \textit{lsc}) if and only if $X$ is normal.

Before stating the consequences, we suppose that $X$ is a topological space that whose $F_\sigma$–kernel of sets are $F_\sigma$–sets.

\textbf{Corollary 2.1.} For each pair of disjoint $F_\sigma$–sets $F_1, F_2$, there are two $G_\delta$–sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if $X$ has the weak $B - .5$–insertion property for $(\textit{cusB} - .5, \textit{clsB} - .5)$.

\textit{Proof.} [19, Corollary 2.1].

\textbf{Remark 2} ([24]). A space $X$ has the weak $c$–insertion property for (\textit{lsc}, \textit{usc}) if and only if $X$ is extremally disconnected.

\textbf{Corollary 2.2.} For every $G$ of $G_\delta$–set, $F_\sigma(G)$ is a $G_\delta$–set if and only if $X$ has the weak $B - .5$–insertion property for $(\textit{clsB} - .5, \textit{cusB} - .5)$.

\textit{Proof.} [19, Corollary 2.2].

\textbf{Theorem 2.2.} Let $P_1$ and $P_2$ be $B - .5$–property and $X$ be a space that satisfies the weak $B - .5$–insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the $B - .5$–insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of $X$ with empty intersection and such that for each $n$, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by Baire-.5 functions.

\textit{Proof.} [18, Theorem 2.1].

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

\textbf{Lemma 2.1.} The following conditions on the space $X$ are equivalent:

(i) For every $G$ of $G_\delta$–set we have $F_\sigma(G)$ is a $G_\delta$–set.

(ii) For each pair of disjoint $G_\delta$–sets as $G_1$ and $G_2$ we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$. 

\begin{itemize}
\item \textbf{Proof.} [19, Corollary 2.2].
\end{itemize}
The proof of Lemma 2.1 is a direct consequence of the definition $F_\sigma-$kernel of sets.

**Lemma 2.2.** The following conditions on the space $X$ are equivalent:

(i) Every two disjoint $F_\sigma-$sets of $X$ can be separated by $G_\delta-$sets of $X$.

(ii) If $F$ is a $F_\sigma-$set of $X$ which is contained in a $G_\delta-$set $G$, then there exists a $G_\delta-$set $H$ such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

*Proof.* [19, Lemma 2.2].

**Lemma 2.3.** Suppose that $X$ is a topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If $F_1$ and $F_2$ be two disjoint $F_\sigma-$sets of $X$, then there exists a Baire-.5 function $h : X \to [0, 1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.

*Proof.* [19, Lemma 2.3].

**Lemma 2.4.** Suppose that $X$ is the topological space such that every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets. The following conditions are equivalent:

(i) Every countable covering of $G_\delta-$sets of $X$ has a refinement consisting of $G_\delta-$sets such that, for every $x \in X$, there exists a $G_\delta-$set containing $x$ such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection there exists a decreasing sequence $\{G_n\}$ of $G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

*Proof.* (i) $\Rightarrow$ (ii). Suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets. By hypothesis (i) and Lemma 2.2, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every $V_n$ is a $G_\delta-$set and $F_\sigma(V_n) \subseteq F_n^c$. By setting $F_n = (F_\sigma(V_n))^c$, we obtain a decreasing sequence of $G_\delta-$sets with the required properties.

(ii) $\Rightarrow$ (i). Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets, we set for $n \in \mathbb{N}, F_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of $G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets $W_n$ of $X$ in the following manner:

$W_1$ is a $G_\delta-$set of $X$ such that $G_1^c \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$. 

$W_2$ is a $G_\delta$–set of $X$ such that $F_\sigma(W_1) \cup G_2^c \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 2.2, $W_n$ exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for $X$, $\{W_n : n \in \mathbb{N}\}$ is a covering for $X$ consisting of $G_\delta$–sets. Moreover, we have

(i) $F_\sigma(W_n) \subseteq W_{n+1}$

(ii) $G_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$.

Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $G_\delta$–sets and covers $X$. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Finally, consider the following sets:

$S_1 \cap H_1$, \quad $S_1 \cap H_2$

$S_2 \cap H_1$, \quad $S_2 \cap H_2$, \quad $S_2 \cap H_3$

$S_3 \cap H_1$, \quad $S_3 \cap H_2$, \quad $S_3 \cap H_3$, \quad $S_3 \cap H_4$

and continue ad infinitum. These sets are $G_\delta$–sets, cover $X$ and refine $\{H_n : n \in \mathbb{N}\}$.

In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $G_\delta$–set containing $x$ that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i + 1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $G_\delta$–sets, and for every point in $X$ we can find a $G_\delta$–set containing the point that intersects only finitely many elements of that refinement. \hfill $\Box$

**Remark 3** ([12, 13]). A space $X$ has the $c$–insertion property for $(usc, lsc)$ if and only if $X$ is normal and countably paracompact.

**Corollary 2.3.** $X$ has the $B - .5$–insertion property for $(cusB - .5, clsB - .5)$ if and only if every two disjoint $F_\sigma$–sets of $X$ can be separated by $G_\delta$–sets, and in addition, every countable covering of $G_\delta$–sets has a refinement that consists of $G_\delta$–sets such that, for every point of $X$ we can find a $G_\delta$–set containing that point such that, it intersects only a finite number of refining members.

**Proof.** Suppose that $F_1$ and $F_2$ are disjoint $F_\sigma$–sets. Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_2}$.

Since $F_2$ is a $F_\sigma$–set, and $F_1^c$ is a $G_\delta$–set, $g$ is $cusB - .5$. $f$ is $clsB - .5$ and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function $h$ such that,
$g < h < f$. Now by setting $G_1 = \{x \in X : h(x) < 1\}$ and $G_2 = \{x \in X : h(x) > 1\}$. We can say that $G_1$ and $G_2$ are disjoint $G_\delta-$sets that contain $F_1$ and $F_2$, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$, $f(x) = \frac{1}{n+1}$. Since $\bigcap_{n=0}^{\infty} F_n = \emptyset$ and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}$, $f$ is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then $\{x \in X : f(x) > r\} = X$ is a $G_\delta-$set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, we can find $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Now suppose that $k$ is the least natural number such that $\frac{1}{k+1} < r$. Hence by hypothesis there exists a Baire-.5 function $h$ on $X$ such that, $g < h < f$.

By setting $G_n = \{x \in X : h(x) < \frac{1}{n+1}\}$, we have $G_n$ is a $G_\delta-$set. But for every $x \in F_n$, we have $f(x) = \frac{1}{n+1}$ and since $g < h < f$, $0 < h(x) < \frac{1}{n+1}$, i.e., $x \in G_n$ therefore $F_n \subseteq G_n$ and since $h > 0$ it follows that $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Hence by Lemma 2.4, the conditions hold.

On the other hand, since every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets, by Corollary 2.1, $X$ has the weak $B-.5-$insertion property for $(\text{cus}B-.5, \text{cls}B-.5)$. Now suppose that $f$ and $g$ are real-valued functions on $X$ with $g < f$, such that, $g$ is $\text{cus}B-.5$ and $f$ is $\text{cls}B-.5$. For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$ 

Since $g$ is $\text{cus}B-.5$, and $f$ is $\text{cls}B-.5$, $f-g$ is $\text{cls}B-.5$. Hence $A(f-g, 3^{-n+1})$ is a $F_\sigma-$set of $X$. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $F_\sigma-$sets and furthermore since $0 < f-g$, it follows that

$$\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \emptyset.$$ 

Now by Lemma 2.4, there exists a decreasing sequence $\{D_n\}$ of $G_\delta-$sets such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 2.3, $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of $F_\sigma-$sets can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, there exists a Baire-.5 function $h$ defined on $X$ such that, $g < h < f$, i.e., $X$ has the $B-.5-$insertion property for $(\text{cus}B-.5, \text{cls}B-.5)$.

**Remark 4** ([14]). A space $X$ has the $c-$insertion property for $(lsc, usc)$ iff $X$ is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of
X with empty intersection there exists a decreasing sequence \( \{F_n\} \) of closed subsets of \( X \) with empty intersection such that \( G_n \subseteq F_n \) for each \( n \).

**Corollary 2.4.** For every \( G \) of \( \text{G}_\delta \)-set, \( F_\sigma(G) \) is a \( \text{G}_\delta \)-set and in addition for every decreasing sequence \( \{G_n\} \) of \( \text{G}_\delta \)-sets with empty intersection, there exists a decreasing sequence \( \{F_n\} \) of \( F_\sigma \)-sets with empty intersection such that for every \( n \in \mathbb{N}, G_n \subseteq F_n \) if and only if \( X \) has the \( B - .5 \)-insertion property for \( (\text{clsB} - .5, \text{cusB} - .5) \).

**Proof.** Since for every \( G \) of \( \text{G}_\delta \)-set, \( F_\sigma(G) \) is a \( \text{G}_\delta \)-set, by Corollary 2.2, \( X \) has the weak \( B - .5 \)-insertion property for \( (\text{clsB} - .5, \text{cusB} - .5) \). Now suppose that \( f \) and \( g \) are real-valued functions defined on \( X \) with \( g < f, g \) is \( \text{clsB} - .5 \), and \( f \) is \( \text{cusB} - .5 \). Set \( A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) < 3^{-n+1}\} \). Then since \( f - g \) is \( \text{cusB} - .5 \), \( \{A(f - g, 3^{-n+1})\} \) is a decreasing sequence of \( \text{G}_\delta \)-sets with empty intersection. By hypothesis, there exists a decreasing sequence \( \{D_n\} \) of \( F_\sigma \)-sets with empty intersection such that, for every \( n \in \mathbb{N}, A(f - g, 3^{-n+1}) \subseteq D_n \). Hence \( X \setminus D_n \) and \( A(f - g, 3^{-n+1}) \) are two disjoint \( \text{G}_\delta \)-sets and therefore by Lemma 2.1, we have

\[
F_\sigma(A(f - g, 3^{-n+1})) \cap F_\sigma((X \setminus D_n) = \emptyset
\]

and therefore by Lemma 2.3, \( X \setminus D_n \) and \( A(f - g, 3^{-n+1}) \) are completely separable by Baire-.5 functions. Therefore by Theorem 2.2, there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \), i.e., \( X \) has the \( B - .5 \)-insertion property for \( (\text{clsB} - .5, \text{cusB} - .5) \).

On the other hand, suppose that \( G_1 \) and \( G_2 \) are two disjoint \( \text{G}_\delta \)-sets. Since \( G_1 \cap G_2 = \emptyset \), we have \( G_2 \subseteq G_1^c \). We set \( f(x) = 2 \) for \( x \in G_1^c \), \( f(x) = \frac{1}{2} \) for \( x \notin G_1^c \) and \( g = \chi_{G_2} \).

Then since \( G_2 \) is a \( \text{G}_\delta \)-set and \( G_1^c \) is an \( F_\sigma \)-set, we conclude that \( g \) is \( \text{clsB} - .5 \) and \( f \) is \( \text{cusB} - .5 \) and furthermore \( g < f \). By hypothesis, there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \). Now we set \( F_1 = \{x \in X : h(x) \leq \frac{3}{4}\} \) and \( F_2 = \{x \in X : h(x) \geq 1\} \). Then \( F_1 \) and \( F_2 \) are two disjoint \( F_\sigma \)-sets containing \( G_1 \) and \( G_2 \), respectively. Hence \( F_\sigma(G_1) \subseteq F_1 \) and \( F_\sigma(G_2) \subseteq F_2 \) and consequently \( F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset \). By Lemma 2.1, for every \( G \) of \( \text{G}_\delta \)-set, the set \( F_\sigma(G) \) is a \( \text{G}_\delta \)-set.

Now suppose that \( \{G_n\} \) is a decreasing sequence of \( \text{G}_\delta \)-sets with empty intersection.
We set \( G_0 = X \) and \( f(x) = \frac{1}{n+1} \) for \( x \in G_n \setminus G_{n+1} \). Since \( \bigcap_{n=0}^{\infty} G_n = \emptyset \) and for every \( n \in \mathbb{N} \) there exists \( x \in G_n \setminus G_{n+1} \), \( f \) is well-defined. Furthermore, for every \( r \in \mathbb{R} \), if \( r \leq 0 \) then \( \{ x \in X : f(x) < r \} = \emptyset \) is a \( G_{\delta} \)-set and if \( r > 0 \) then by Archimedean property of \( \mathbb{R} \), there exists \( i \in \mathbb{N} \) such that \( \frac{1}{i+1} \leq r \). Suppose that \( k \) is the least natural number with this property. Hence \( \frac{1}{k} > r \). Now if \( \frac{1}{k+1} < r \) then \( \{ x \in X : f(x) < r \} = G_k \) is a \( G_{\delta} \)-set and if \( \frac{1}{k+1} = r \) then \( \{ x \in X : f(x) < r \} = G_{k+1} \) is a \( G_{\delta} \)-set. Hence \( f \) is a \( \text{cusB} - 0.5 \) on \( X \). By setting \( g = 0 \), we have conclude, \( g \) is \( \text{clsB} - 0.5 \) on \( X \) and in addition \( g < f \). By hypothesis there exists a Baire-5 function \( h \) on \( X \) such that, \( g < h < f \). Set \( F_n = \{ x \in X : h(x) \leq \frac{1}{n+1} \} \). This set is an \( F_{n} \)-set. But for every \( x \in G_n \), we have \( f(x) \leq \frac{1}{n+1} \) and since \( g < h < f \), \( h(x) < \frac{1}{n+1} \), which means that \( x \in F_n \) and consequently \( G_n \subseteq F_n \).

By definition of \( F_n \), \( \{ F_n \} \) is a decreasing sequence of \( F_{n} \)-sets and since \( h > 0 \), \( \bigcap_{n=1}^{\infty} F_n = \emptyset \). Thus the conditions hold.

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