In this paper, we examine the brief history of the ring of four almonds regarding Mesopotamian mathematics, and present reasons why the Omar Khayyam’s triangle, a special right triangle in a ring of four almonds, was essential for artisans due to its unique pattern. We presume that the ring of four almonds originated from a point symmetry figure given two concentric squares used in the proto-Sumerian Jemdet Nasr period (approximately 3000 B.C.) and a square halfway between two given concentric squares used during the time of the Old Akkadian period (2340–2200 B.C.) and the Old Babylonian age (2000–1600 B.C.). Artisans tried to create a new intricate pattern as almonds and 6-pointed stars by subdividing right triangles in the pattern of the popular altered Old Akkadian square band at the time. Therefore, artisans needed the Omar Khayyam’s triangle, whose hypotenuse equals the sum of the short side and the perpendicular to the hypotenuse. We presume that artisans asked mathematicians how to construct the Omar Khayyam’s triangle at a meeting between artisans and mathematicians in Isfahan. The construction of Omar Khayyam’s triangle requires solving an irreducible cubic polynomial. Omar Khayyam was the first to classify equations of integer polynomials of degree up to three and then proceeded to solve all types of cubic equations by means of intersections of conic sections. Omar Khayyam’s triangle gave practical meaning to the type of cubic equation \( x^3 + bx = cx^2 + a \). The work of Omar Khayyam was completed by Descartes in the 17th century.

**Keywords:** Islamic Art Design, Cubic Equation, Ornamental Pattern, Altered Old Akkadian Square Band, Omar Khayyam, Omar Khayyam’s Triangle; 이슬람 예술 디자인, 3차방정식, 장식패턴, 변형된 고 아카디안 사각미, 오마르 하이얌, 오마르 하이얌의 삼각형.

**MSC:** 01A17, 01A30, 97–03
1 Introduction

The untitled treatise by Omar Khayyam (1044–1123), a well-known Persian poet, philosopher, scientist, and mathematician of the 11th century, dealt with a geometrical question raised by artisans at a meeting attended by artisans and geometers in Isfahan [1, p. 323; 24, p. 172]. Such meetings were a widespread phenomenon in the Islamic world [25, p. 171].

In his treatise, Omar Khayyam stated [1, p. 323]:

We want to divide one-fourth of AB of the circle ABCD by a point R into two parts such that if RH is drawn perpendicular to the diameter BD, the ratio of AE to RH is the same as EH is to HB. The point E is the center of the circle and AE is half of its diameter.

![Figure 1. Division of a quadrant of a circle](image)

The point R is related to the construction of a right triangle, in which the hypotenuse is equal to the sum of the short side and the perpendicular to the hypotenuse. The right triangle is called Omar Khayyam’s triangle [22, p. 61]. We do not think that Omar wrote the artisans’ question in his untitled treatise without transforming it mathematically. To accurately predict the artisans’ questions, we first refer to the design called the ring of four almonds, also called the ring of four rhombuses or the whirling kites (see Figure 3e). We propose that the artisans’ question is related to the four entangled acrobats (UE 3, 393), a point symmetry figure used in the proto-Sumerian Jemdet Nasr period in Mesopotamia, around the beginning of the third millennium BCE [14, p. 79, p. 167]. The point symmetry figure is based on a chain of concentric rectangles as shown in Figure 2.
2 Ring of Four Almonds and Its Origin

Figure 3b is the geometric demonstration of the equipartitioned trapezoid equation in VAT 8512 [16, p. 106] and the Old Akkadian hand tablet IM 58045 using two concentric squares [13, p. 409–10]. Therefore, we have named Figure 3a the Old Akkadian square band formed between the two given concentric squares. Figures 3c and 3d are related to the find and proof of the Babylonian diagonal rule found in IM 86118, TMS 5, and BM 13901 [13, p. 206]. It is presumed that the ring of four almonds in Figure 3e derived from the Old Babylonian square halfway (Figure 3c) by inverting four right triangles located at each corner. Figure 4 displays a two-square tiling regarding a ring of four almonds. The almond is one of the common components in ornamental geometrical patterns throughout the Muslim world. Note that Figures 2 and 3c have exactly four rotation symmetries and no mirror reflection symmetries. Figures 3f and 4 illustrate two-square tilings classified as p4 according to the seventeen wall paper groups [20, p. 108].

Many features regarded as traditionally Islamic go back to a time before Muhammad. From the historical point of view of mathematics, we call the ring of four almonds the altered Old Akkadian square band [26, p. 309]. This name reflects the Old Akkadian (VAT 8521) and the Old Babylonian mathematics (IM 87118). On the other hand, Özdural [22, p. 55, p. 57] proposed the ring of four almonds (Figure 3e) as the ornamental pattern of Abu’l-Wafa’s proof of the Pythagorean theorem by joining four congruent triangles for each side of the rectangle. Also, Cromwell and Beltrami [9, p. 86] suggested that the ring of four almonds is influenced by the Chinese regarding Figure 5 (Chinese Text Project) [6].

Zhao Shuang’s commentary on the Chou Pei Suan Chig (China, 250 B.C.) con-
Many features regarded as traditionally Islamic go back to a time before Muhammad. From the historical point of view of mathematics, we call the ring of four almonds the altered Old Akkadian square band \cite{26, p.309}. This name reflects the Old Akkadian (VAT 8521) and the Old Babylonian mathematics (IM 87118). On the other hand, Özdural \cite{22, p.55, p.57} proposed the ring of four almonds (Figure 3 e) as the

tains the agreement and diagrams in support of the Pythagorean Theorem. However, the original diagrams have been lost. Figure 5 displays what is believed to be “the Zhao’s hypotenuse diagram” \cite[p. 204–5]{18}.

The length of the green highlighted square in Figure 7a equals the short edge of the almond. Therefore, the right triangle corresponding to half of the almond
ornamental pattern of Abu ‘l-Wafa’s proof of the Pythagorean theorem by joining four congruent triangles for each side of the rectangle. Also, Cromwell and Beltrami suggested that the ring of four almonds is influenced by the Chinese regarding octagon, decagon, and dodecagon. According to El-Said and Parman, we propose that it is very easy for artisans to form the square at the center of a ring of four almonds; the right triangle corresponding to half of the almond features a ratio of 1:2. This proportion is commonly found within the historical record [4, p. 171]. We propose that it is very easy for artisans to form the square at the center of a 3 × 3 square as in Figure 7a, to divide the ring of rectangles as shown in Figure 7b, to divide each rectangle diagonally as in Figure 7c, and to invert four yellow highlighted right triangles located at each corner as shown in Figure 7d.

Figure 5. An estimating picture of the Zhao’s hypotenuse diagram

Figure 6. Altered Old Akkadian square band with 1:2 almond pattern, Tillya Kari Madrasah, Samarkand (Photo: J. Park).

features a ratio of 1 : 2. This proportion is commonly found within the historical record [4, p. 171]. We propose that it is very easy for artisans to form the square at the center of a 3 × 3 square as in Figure 7a, to divide the ring of rectangles as shown in Figure 7b, to divide each rectangle diagonally as in Figure 7c, and to invert four yellow highlighted right triangles located at each corner as shown in Figure 7d.

Figure 7. Four-step construction of an altered Old Akkadian square band with 1 : 2 almond pattern

Figure 4 illustrates a design based upon the two-square tiling with an altered Old Akkadian square band, which is prominent in the brickwork façade of the western
tomb tower at Kharaqan, Iran (1093). This is considered one of the prominent works of Seljuk. The tomb towers at Kharaqan feature more than thirty types of brickwork with very beautiful arch building façades and outstanding Kufic brick lines [10, p. 385]. We can use a two-square tiling to interpret the underlying geometric structure of decorative brickworks of the Qazvin Kharaqan towers, as displayed in Figures 8a and 8b (cf. [7, p. 762-63]).

Figure 8. a The design of the Kharaqan Tomb Towers featuring two-square tiling and ring of four almonds; b The design of the Kharaqan Tomb Towers utilizing two-square tiling.

Most rings of four almonds are based on constructive polygons including the heptagon, octagon, decagon, and dodecagon. According to El-Said and Parman [12, p. 29], the basic figure of the brickwork of the western tomb tower (see Figure 9) is generated using a regular octagon, and so we obtain $\angle ABC = 22.5^\circ$. Thus the right triangle $ABC$ in the almond has the approximate ratio of $AC : AB = 1 : \cot(22.5^\circ) \approx 1 : 2.4$. However, Bonner [4, p. 171-72] proposed that this ratio is $AC : AB = 1 : 2$.

Figure 9. Basic construction of the design from the Kharaqan Tomb Towers for Fig. 8a

Bulatov [3, p. 14] proposed that the design of the fortress of Deu-kala (Khorezm, XII century) featured the ring of four right triangles. Figure 10a displays Bulatov’s
analysis of the restored fortress, Deu-kala. His construction is based on the ring of four right triangles with ratio 1 : 2.

As a consequence, Bulatov based his analysis on the regular octagon and the square as displayed in Figure 10b. On the other hand, Critchlow [8, p. 72] introduced a ring of four almonds based on the dodecagon and the square as in Figure 11. Note that each right triangle in the Critchlow’s ring of four almonds features a ratio of $1 : \sqrt{3}$. According to his doctrine, the coincidence of twelve and four suggests a zodiacal symbolism controlling or embracing the square which can be taken to symbolize the four seasons, the four elements, and the four qualities of heat and cold, moist and dry.

Therefore, the chain of rectangles can be applied to dividing into a ring of four right triangles and four almonds. Moreover, the chain of rectangles generates a two-
square tiling classified as p4 in the wall paper groups (see Figures 4, 8a, and 13). According to James, the ancient theory of "the four qualities and four elements" was expressed by a diagram formed in Egypt by a chain of rectangles as in Figure 12a. James described this as [17, p. 80–1]:

The corners of the outer squares carried the names of elements: fire, water, earth and air, while the corner of the inner squares, being at the mid points of the sides of the outer square, carried the four fundamental qualities: the hot, the dry, the cold and the wet.

Bernal [2, p. 38] suggested that James makes the plausible case for Greek philosophy having borrowed massively from Egypt. Also, BM 15285 shows that the Old Babylonian mathematics dealt with outer and inner rectangles in metric algebra frequently [14, p. 127].

![Figure 12. a The four qualities and four elements; b BM 15285 Col. iii ##12](image)

The Pythagorean Theorem is derived from the two-square tiling as displayed in Figures 8) and 13.

![Figure 13. Two-Square Tiling, Hierapolis, Turkey (Photo: J. Park).](image)
Consider centers \( A, A', B \) and \( B' \) in each small square rotating around the central large square in Figure 14. It is intuitively true that \( AB = CD = EF \) and \( EF^2 = EG^2 + GF^2 \). Figure 14 was first presented by H. E. Dudeney in 1917 [11, p. 95].

3 Artisans Question In A Conversation

Artisans utilize various interlocking patterns from the four triangles in the square as shown in Figure 15. A specific ratio of a right triangle will construct an almond in which one side of the almond coincides with one side of the inside square (see Figure 15b).

Moreover, subdivisions of the almond, as in Figure 16, generate various interlocking patterns of polygons and 6-pointed stars by a two-square tiling as displayed in Figure 17.

It is presumed that artisans asked mathematicians how to construct an interlocking figure with \( AB = AD \) as shown in Figure 15b.

In the next section, we summarize the main theories of Omar Khayyam’s untitled treaties related to the artisans’ question. He showed that the construction of
It is presumed that artisans asked mathematicians how to construct an interlocking figure with $AB \% AD$ as shown in Figure 15b. In the next section, we summarize the main theories of Omar Khayyam’s untitled treaties related to the artisans’ question. He showed that the construction of the Omar Khayyam’s triangle is equivalent to the solution of cubics solved by the intersection of a hyperbola and a semicircle.

4 Omar Khayyam’s Triangle and Cubic Equation

To construct a ring of Omar Khayyam’s triangles, Omar Khayyam suggested using point R on a circle AB with center E such that $AE : RH = EH : HB$ [1, p. 324] (see Figure 18 below).

Omar Khayyam’s Theorem.

Let R be a point on the circle AB such that $AE : RH = EH : HB$. Then we obtain the following:

1. $RI = IB = RK$,
2. $RE + RH = ET$,
3. $ER + EH = RT$,
(4) Let $EH = 10$ and $RH = x$. Then we obtain $x^3 + 200x = 20x^2 + 2000$.

Proof. (1): Choose point $J$ on $RH$ such that $IJ$ and $RH$ are perpendicular. Then triangles $ERH$ and $RIJ$ are similar, and so $ER : RI = RI : IJ = EH : HB = EH : IJ$.

This results in $RI = EH$ and hence, $RI = IB = RK$.

(2) and (3): These follow from [1, p. 326].

(4): Since $\triangle ERT$ and $\triangle EHR$ are similar, we obtain $ET : ER = ER : EH$, and so $ER^2 = x^2 + 100$. From (2), we obtain $x^2/10 + 10 = \sqrt{x^2 + 100} + x$. Therefore, $(x^2/10 - x + 10)^2 = x^2 + 100$ or $x^3 + 200x = 20x^2 + 2000$, or in the literal translation of his words: a cubic root plus two hundred roots are equal to twenty root squares plus two thousand numbers [1, p. 327–28].

Note that if $y := RT/ER = x/10$, then $10y$ satisfies $x^3 + 200x = 20x^2 + 2000$. Thus, we obtain $y^3 + 2y = 2y^2 + 2$. If $f = y^3 - 2y^2 + 2y - 2 \in \mathbb{Z}[y]$, then the Eisenstein Criterion with $p = 2$ shows that $f$ is an irreducible polynomial in $\mathbb{Q}[y]$ and its discriminant is:

$$D(f(y)) = D(f(y + \frac{2}{3})) = D(y^2 + \frac{2}{3}y - \frac{34}{27})$$
$$= -4(\frac{2}{3})^2 - 27(\frac{34}{27})^2$$
$$= -44 < 0$$

Therefore, it has precisely one real root of $y \approx 1.543689013$ according to the computation of Maple 13, and its Galois group is $S_3$ [29, p. 216]. Omar Khayyam solved the cubic equation $x^3 + 200x = 20x^2 + 2000$ using the intersection of a hyperbola and a semicircle whose modern notations are $xy = 100\sqrt{2}$ and $(x + 15)^2 + (y + 10\sqrt{2})^2 = 5^2$, respectively [1, p. 332]. After solving the cubic equation geometrically, he proposed the approximations of $\angle RET \approx 57^\circ$ and $\sin(\angle RET) \approx 50/60$ without performing calculations [1, p. 336–37]. Geometrically, but not arithmetically, it is possible that if $RE = 60$, then he could determine that $RH = 50.3$ (see Figure 18 above),
2 \times 50.3 = 100.6 \equiv \text{crd}114^\circ, \text{and } \sin(\angle RET) \equiv 50/60 + 18/60^2 \equiv 50/60 \text{ from the chord table of Almagest} \ [30, \text{p. 59}]. \text{Therefore, the ratio } ER : RT \text{ is approximately } 1 : \tan(57^\circ) \equiv 1.54. \text{ The construction of Omar Khayyam’s triangle was not easy for artisans, so it is assumed that Omar Khayyam advised artisans to use a simple instrument such as a ruler, i.e. the T-ruler as in MS Persan 169, Fol. 191r} \ [23, \text{p. 197}].

There is no record in Egypt of the solution of cubic equations, but among the Babylonians there are many instances such as \( x^3 + x^2 = a \) \ [5, \text{p. 30–1}]. Merzbach and Boyer \ [5, \text{p. 86} \] proposed that Menaechmus (4th century B.C.) knew the duplication problem could be solved by the point of intersection of the hyperbola \( xy = a^2 \) and the parabola \( y^2 = (a/2)x \). Cubic equations were first studied systematically by Islamic mathematicians including Omar Khayyam. Omar Khayyam wrote the \textit{Algebra (Treatise on Demonstrations of Problems of al-jabr and al-mmuqabala)} \ (c. 1097) that went beyond the scope of al-Khwarizmi to include cubics. He believed that a general cubic could only be solved geometrically using conic sections. Thus, he provided only geometric solutions. In \textit{Algebra}, he was the first to classify equations of integer polynomials of degree up to three (see Table 1 below) and then proceeded to solve all types of cubic equations by intersections of conic sections \ [28]. \text{However, these revolutionary solutions were of little practical importance. In addition, Omar Khayyam himself solved the equation } x^3 + 2a^2 + 10x = 20, \text{ for which Fibonacci} \text{ would later detail a long solution} \ [27, \text{p. 379–80}]. \text{In Algebra, Omar Khayyam set forth a rule for finding the fourth, fifth, sixth, and higher powers of a binomial, but such a work is not extant. It is presumed that he was referring to the Pascal triangle arrangement} \ [5, \text{p. 219}].

<table>
<thead>
<tr>
<th>simple equation</th>
<th>binomial</th>
<th>compound equation</th>
<th>tetranomial</th>
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<td>( a=x )</td>
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<td>( x^3 + bx = a )</td>
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<td>2.</td>
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<td>3.</td>
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<td>4.</td>
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<td>( x^3 + cx^2 + bx = a )</td>
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<td>6.</td>
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<td>12.</td>
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<td>25.</td>
<td>( x^3 + a = cx^2 + bx )</td>
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Table 1. Classification of equations up to degree three by Omar Khayyam

Omar Khayyam was not the first to solve cubic equations by means of the intersection of conic sections. According to Omar Khayyam, Abu Jafar Khazin was the first to show that a cubic equation of the form \( x^3 + a = cx^2 \) could be solved by means of conic sections. It is equivalent to the following problem: given two straight lines \( AB, AC \) and an area \( D \), to divide \( AB \) at \( M \) so that \( AM : AC = D : AB^2 \) \ [15, \text{p. 71}].
Also, Aboo Nassre ibn Aragh solved the equation \( x^3 + ax^2 = b \) with conic sections [1, p. 331]. According to his untitled treatise, mathematicians including al-Bujani, al-Quhi, and al-Saghani debated the solution of \( x + y = 10; x^2 + y^2 + y/x = 70 \) or \( 2x^3 + 29x + 10 = 20x^2 \) at the royal court of Adud al-Dawala. Eventually, Abu al-Jud solved this equation [1, p. 331], and Rashed [27, p. 379] posited that Abu al-Jud solved the type of cubic equation of \( x^3 + bx = cx^2 + a \).

The solution of cubic equations using intersecting conics is the greatest accomplishment of Arab mathematicians in algebra. The geometric interpretation based on the conic sections of Omar Khayyam led to the achievements of René Descartes (1596–1650), the pioneer of analytical geometry, which later interpreted \( x^2 \) and \( x^3 \) as line segments, rather than as geometric squares and cubes. This permitted him to abandon the principal of homogeneity [5, p. 312].

However, around 1300, the decorative pattern design of the Omar Khayyam’s triangle was described in MS Persan 169 (Fol. 191r, Fol. 189v, and Fol. 188r) [21, p. 350–56], but it cannot be found in actual works of art. We assume that it was not only difficult to locate the point in the process of making the decorative pattern design, but the decorative pattern resulting from the use of the Omar Khayyam’s triangle did not attract the attention of artisans at the time. However, Omar Khayyam’s triangle was applied to the geometric scheme of the North Dome Chamber of the Friday Mosque of Isfahan [24, p. 711].

5 Conclusion

In order to reflect mathematics developed in Mesopotamia, we call the ring of four almonds as the altered Old Akkadian square band. We claim that the ring of four almonds was the result of a point symmetry figure between two given concentric squares used in the proto-Sumerian Jemdet Nasr period (approximately 3000 B.C.) and a square halfway problem in area between two given concentric squares by the time of Old Akkadian period (2340–2200 B.C.) and the Old Babylonian age (2000–1600 B.C.). Also, this can be used to create a new pattern regarding a two-square tiling classified as p4 in the seventeen wall paper groups.

Artisans designed almonds and 6-pointed stars by dividing rotating right triangles in altered Old Akkadian square bands that were prevalent at the time. For this design, artisans needed a ratio of a right triangle such as the Omar Khayyam’s triangle. Thus, it is assumed that artisans consulted mathematicians regarding a method for constructing Omar Khayyam’s triangle. Omar Khayyam’s triangle was described in MS Persan 169, Fol. 191r, Fol. 189v, and Fol. 188r. We have inferred reasons the Omar Khayyam’s triangle was important for artisans, and examined the algebraic meaning of the Omar Khayyam’s triangle.
Omar Khayyam was the first to consider all types of cubic equations that possess a positive root [11, p. 246]. He classified cubic equations of integer polynomials and claimed that the solution of cubic equations can be obtained by means of intersections of conic sections. He only found positive solutions using geometric methods, but it was not until the 17th century that Descartes established a relation between geometry and algebra. Descartes’ methods are similar to those of Omar Khayyam, but Descartes realized that certain intersection points represented negative roots and imaginary roots [18, p. 483]. The cubic equation $x^3 + 200x = 20x^2 + 2000$ was created by Omar Khayyam’s triangle. Therefore, another contribution of Omar Khayyam’s triangle is to give practical meaning to the type of cubic equation of $x^3 + bx = cx^2 + a$. Following Özdural [22, p. 59], in the history of Islamic art and architecture, Omar’s triangle is evidence of Omar Khayyam’s involvements in the ornamental arts.

Acknowledgements. We would like to thank Mariya Glazirina (Deputy Principal of the School Named After Al Khorezmy) for her helpful advice on the Figure 10.

References