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SOME REMARKS ON BOUNDED COHOMOLOGY GROUP OF PRODUCT OF GROUPS

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Abstract. In this paper, for discrete groups G and K, we show that the bounded cohomology group of $G \times K$ is isomorphic to the cohomology group of the complex of the projective tensor product $B^*(G)\widehat{\otimes}B^*(K)$, where $B^*(G)$ and $B^*(K)$ are the complexes of bounded cochains with real coefficients $\mathbb{R}$ of G and K, respectively.

1. Introduction

Throughout this paper, G and K denote discrete groups. Also, we consider Banach spaces over the field of real numbers $\mathbb{R}$.

Bounded cohomology of G with real coefficients, denoted by $\widehat{H}^*(G)$, is cohomology of the bounded cochain complex $B^*(G)$, which are Banach spaces. Then we can define bounded cohomology $\widehat{H}^*(G \times K)$ of the product of groups $G \times K$ as the cohomology of the bounded cochain complex $B^*(G \times K)$. In [4], under some conditions required for a category of Banach spaces, it is shown that there is a spectral sequence with

$$E_2^{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^p \left(\widehat{H}^s(G), \widehat{H}^t(K)\right)$$

and it converges to $H^n\left(B^*(G)\widehat{\otimes}B^*(K)\right)$. So, it seems natural to consider the relationship between cochian complexes $B^*(G)\widehat{\otimes}B^*(K)$ and $B^*(G \times K)$, and compare $H^n\left(B^*(G)\widehat{\otimes}B^*(K)\right)$ with $\widehat{H}^n(G \times K)$.

As we deal with bounded cohomology of a group, we first briefly review its definition developed by Ivanov. In [2], he defined it by using the method of relative homological algebra, which we will modify, and cultivated the theory of bounded cohomology.

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Definition 1.1. *Let V be a Banach space equipped with a norm $\|\cdot\|$. We say V is a bounded left G-module if there is a left action of G on V such that $\|g \cdot v\| \leq \|v\|$ for all $g \in G$ and $v \in V$.*

Similarly, we can define a bounded right G-module. For simplicity, a bounded left G-module will be called a G-module.

Notice that $\mathbb{R}$ is considered as a G-module with the trivial G-action.

Let M be a Banach space equipped with a norm $\|\cdot\|_M$ and $B(G, M)$ be the set of all bounded functions on G, that is,

$$B(G, M) = \{f : G \to M \mid \|f\| = \sup_{a \in G} \|f(a)\|_M < \infty\}.$$

Then $B(G, M)$ is a Banach space with the norm $\|\cdot\|$. Similarly, for every $n \geq 1$, let G^n be the n-product of G so that $G^n = \underbrace{G \times \cdots \times G}_{n}$. We denote by $B^n(G, M)$ the set of all bounded functions on G^n, that is,

$$B^n(G, M) = \{f : G^n \to M \mid \|f\| < \infty\},$$

where $\|f\| = \sup\{\|f(a_1, \cdots, a_n)\|_M \mid (a_1, \cdots, a_n) \in G^n\}$.

In case that $M = \mathbb{R}$, we denote $B^n(G, \mathbb{R})$ by $B^n(G)$.

Remark 1.2. **(1):** *For a Banach space M, every $B^n(G, M)$ for $n > 0$ is a G-module with the action defined by*

$$(x \cdot f)(g_1, g_2, \cdots, g_{n-1}, g_n) = f(g_1, g_2, \cdots, g_{n-1}, g_n x) \text{ for } x \in G.$$

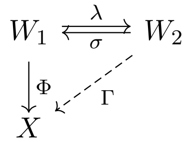
(2): *$B^{n+1}(G)$ is isomorphic with $B(G, B^n(G))$, where $B^n(G)$ is considered simply as a Banach space.*

Definition 1.3. *Let W_1 and W_2 be G-modules. A bounded linear operator $\lambda : W_1 \to W_2$ is called a G-morphism if λ commutes with the action of G. Furthermore, we say an injective G-morphism $\lambda : W_1 \to W_2$ is strongly injective if there is a bounded linear operator $\sigma : W_2 \to W_1$ such that $\sigma \circ \lambda = id$ and $\|\sigma\| \leq 1$.*

Definition 1.4. *Let X be a G-module. We say X is relatively injective if for any strongly injective G-morphism $\lambda : W_1 \to W_2$ of G-modules and any given G-morphism $\Phi : W_1 \to X$, there is a G-morphism $\Gamma : W_2 \to X$ such that $\Gamma \circ \lambda = \Phi$ and $\|\Gamma\| \leq \|\Phi\|$.*

Notice that Definition 1.4 is illustrated by the following commutative diagram

(1.4.1)



$\sigma \circ \lambda = id$

Proposition 1.5. *For any group G and a Banach space M, the G-module $B(G,M)$ is relatively injective. In particular, every G-module $B^n(G)$ for $n > 0$ is relatively injective.*

Proof. This is Lemma (3.2.2) in [2]. We review the idea for our own use and refer to [2] in detail.

We consider the following diagram in (1.4.1) by setting $X = B(G,M)$:

$$\begin{array}{ccc} W_1 & \underset{\sigma}{\overset{\lambda}{\rightleftarrows}} & W_2 \\ \downarrow{\scriptstyle \Phi} & \swarrow{\scriptstyle \Gamma} & \\ B(G,M) & & \end{array}$$

where λ is a strongly injective G-morphism and σ is a bounded linear operator satisfying $\sigma \circ \lambda = id$ and $\|\sigma\| \le 1$. For $w \in W_2$ and $x \in G$, we define

$$\Gamma(w)(x) = \Phi\left(\sigma(x \cdot w)\right)(e_G),$$

where e_G is the identity of G.

Then, by a standard calculation, we can show Γ is a G-morphism such that $\Gamma \circ \lambda = \Phi$ and $\|\Gamma\| \le \|\Phi\|$. Hence $B(G,M)$ is a relatively injective G-module. Then, since $B(G) = B(G,\mathbb{R})$, a G-module $B(G)$ is relatively injective. For $n \ge 1$, as $B^{n+1}(G)$ is isomorphic to $B\left(G, B^n(G)\right)$ from Remark 1.2, a G-module $B^n(G)$ for every $n > 0$ is also relatively injective. □

Definition 1.6. *For a G-module V, a G-resolution of V is a sequence of G-modules and G-morphisms of the form*

$$0 \to V \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots \tag{1.6.1}$$

such that it is exact as a sequence of vector spaces over $\mathbb{R}$. We say this G-resolution (1.6.1) of V is strong if it is provided with a contracting homotopy $t_0 : V_0 \to V$ and $t_n : V_n \to V_{n-1}$ for $n > 0$ satisfying the condition $\|t_n\| \le 1$ for all $n \ge 0$, that is, a sequence $\{t_n\}$ of linear operators

$$V \xleftarrow{t_0} V_0 \xleftarrow{t_1} V_1 \xleftarrow{t_2} V_2 \xleftarrow{t_3} \cdots$$

such that $d_{n-1} \circ t_n + t_{n+1} \circ d_n = id$ for $n \ge 0$ and $t_0 \circ d_{-1} = id$.

Also, we say this G-resolution (1.6.1) of V is relatively injective if every V_n is a relatively injective G-module.

Remark 1.7. *For G-modules U and V, let*

$$0 \to U \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots$$

be a strong resolution of U, and

$$0 \to V \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be a complex of relatively injective G-modules. Then, as in the ordinary case of the Comparison Theorem [3], it is easy to check that any G-morphism $\rho : U \to V$ can be extended to a G-morphism of complexes, that is, there are G-morphisms $\gamma_n : U_n \to V_n$ for $n \geq 0$ such that $\gamma_{n+1} \circ \partial_n = \partial'_n \circ \gamma_n$ and $\gamma_0 \circ \partial_{-1} = \partial'_{-1} \circ \rho$. Also, any two such extensions are chain homotopic.

Theorem 1.8. *The sequence of G-modules*

$$0 \to \mathbb{R} \xrightarrow{d_{-1}} B(G) \xrightarrow{d_0} B^2(G) \xrightarrow{d_1} B^3(G) \xrightarrow{d_2} \cdots \tag{1.8.1}$$

is a strong and relatively injective G-resolution of the trivial G-module $\mathbb{R}$, where boundary operators d_* are defined by the formulas: for $n \geq 0$

$$\begin{aligned} &d_n(f)(g_0, g_1, \ldots, g_{n+1}) \\ &= (-1)^{n+1} f(g_1, \ldots, g_{n+1}) + \sum_{i=0}^{n} (-1)^{n-i} f(g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \end{aligned} \tag{1.8.2}$$

and $d_{-1}(r)(g) = r$ for $r \in \mathbb{R}$ and every $g \in G$.

Proof. It follows from (3.4) in [2]. In fact, the operators

$$t_0 : B(G) \to \mathbb{R} \quad \text{and} \quad t_n : B^{n+1}(G) \to B^n(G) \text{ for } n > 0$$

defined by the formulas

$$t_0(f) = f(e_G) \quad \text{and} \quad t_n(f)(g_1, \cdots, g_n) = f(g_1, \cdots, g_n, e_G)$$

provide a contracting homotopy for the sequence (1.8.1). □

Definition 1.9. *The strong and relatively injective G-resolution of the trivial G-module $\mathbb{R}$ in (1.8.1) is called the standard G-resolution.*

Let V be a G-module. The space of all elements of V invariant under the action of G is denoted by V^G. Thus

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}.$$

Observe that, as V^G is a closed subspace of a Banach space V, it is also a Banach space.

Let $0 \to \mathbb{R} \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots$ be a G-resolution of $\mathbb{R}$. It is easy to check that its induced sequence

$$0 \to V_0^G \xrightarrow{d_0} V_1^G \xrightarrow{d_1} V_2^G \xrightarrow{d_2} \cdots$$

obtained by taking the spaces of G-invariant elements is a complex.

Definition 1.10. *Let* $0 \to \mathbb{R} \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots$ *be a strong and relatively injective G-resolution of the trivial G-module $\mathbb{R}$. The nth cohomology of its induced complex*

$$0 \to V_0^G \xrightarrow{d_0} V_1^G \xrightarrow{d_1} V_2^G \xrightarrow{d_2} \cdots \tag{1.10.1}$$

is called the n-th bounded cohomology of G. We denote it by $\widehat{H}^n(G)$.

Proposition 1.11. *The bounded cohomology groups $\widehat{H}^*(G)$ of G depend only on G.*

Proof. Let

$$0 \to \mathbb{R} \to V_0 \to V_1 \to V_2 \to \cdots \text{ and } 0 \to \mathbb{R} \to U_0 \to U_1 \to U_2 \to \cdots$$

be two strong and relatively injective G-resolutions of $\mathbb{R}$. From Remark 1.7, there are G- morphisms $\lambda_n : V_n \to U_n$ and $\gamma_n : U_n \to V_n$ extending the identity map on $\mathbb{R}$. Notice that both $\lambda_* \circ \gamma_*$ and $\gamma_* \circ \lambda_*$ are chain homotopic to the identities. Then, as explained in [2], the morphisms λ_* and γ_* respectively induce maps of complexes $\lambda_n^G : V_n^G \to U_n^G$ and $\gamma_n^G : U_n^G \to V_n^G$. Notice that the homotopy between $\lambda_* \circ \gamma_*$ and id defines a homotopy between $\lambda_*^G \circ \gamma_*^G$ and id. Similarly, $\gamma_*^G \circ \lambda_*^G$ is chain homotopic to the identity. This shows that λ_*^G is an isomorphism. Finally, as any two extensions $V_* \to U_*$ are chain homotopic, this isomorphism λ_*^G is uniquely determined. Hence, the cohomology groups of the complexes

$$0 \to V_0^G \to V_1^G \to V_2^G \to \cdots \quad \text{and} \quad 0 \to U_0^G \to U_1^G \to U_2^G \to \cdots$$

are canonically isomorphic. □

Notice that the group $\widehat{H}^n(G)$ has a vector space structure over $\mathbb{R}$. On the other hand, as $\widehat{H}^n(G)$ is a quotient space of a normed space, it has the natural seminorm induced by the norm on G-module in the resolution used for its computation. Thus the seminorm on $\widehat{H}^*(G)$ depends on the choice of a resolution.

Definition 1.12. *The canonical seminorm on $\widehat{H}^n(G)$ is defined as the infimum of seminorms arising from all strong and relatively injective G-resolutions of $\mathbb{R}$.*

Again, in [2], it is proved that the canonical seminorm on $\widehat{H}^n(G)$ can be achieved by the standard G-resolution. So it seems reasonable to use the standard G-resolution to compute $\widehat{H}^*(G)$.

Notice that, for $f \in B^n(G)$ and $n \geq 1$

$f \in (B^n(G))^G$
$\Leftrightarrow x \cdot f = f$ for $x \in G$
$\Leftrightarrow (x \cdot f)(g_1, \cdots, g_n) = f(g_1, \cdots, g_n)$ for $x \in G$ and $(g_1, \cdots, g_n) \in G^n$
$\Leftrightarrow f(g_1, \cdots, g_{n-1}, g_n x) = f(g_1, \cdots, g_{n-1}, g_n)$ for $x \in G$ and $(g_1, \cdots, g_n) \in G^n$.

Thus $f \in B^n(G)$ is an element of $(B^n(G))^G$ if and only if f is not affected by the last component of G^n.

Remark 1.13. **(1):** *For every $n > 0$, $\left(B^{n+1}(G)\right)^G$ and $B^n(G)$ are isomorphic as Banach spaces. In particular, $(B(G))^G$ is isomorphic to $\mathbb{R}$.*

(2): *The standard G-resolution (1.8.1) induces a complex*

$$0 \to B(G)^G \xrightarrow{d_0=0} \left(B^2(G)\right)^G \xrightarrow{d_1} \left(B^3(G)\right)^G \xrightarrow{d_2} \left(B^4(G)\right)^G \xrightarrow{d_3} \cdots$$

and this can be written as

$$0 \to \mathbb{R} \xrightarrow{d_0=0} B(G) \xrightarrow{d_1} B^2(G) \xrightarrow{d_2} B^3(G) \xrightarrow{d_3} \cdots . \tag{1.13.1}$$

The boundary operator d_* in the complex (1.13.1) can be redefined by

$$\begin{aligned} d_n(f)(g_1, g_2, \ldots, g_{n+1}) = {} & f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + f(g_1, \ldots, g_n). \end{aligned}$$

The sequence (1.13.1) is a complex of Banach spaces and its cohomology is $\widehat{H}^*(G)$ equipped with the canonical seminorm.

In Section 2, we study tensor products of Banach spaces and form a $G \times K$-resolution from the standard G- and K-resolutions. In Section 3, by using the method of relative homological algebra as in [2], we prove the cohomology groups $\widehat{H}^*(G \times K)$ and $H^*(B^*(G) \widehat{\otimes} B^*(K))$ are isomorphic.

2. The tensor product of resolutions

We review a tensor product of Banach spaces.

For Banach spaces X with a norm $\|\cdot\|_X$ and Y with a norm $\|\cdot\|_Y$, we consider their algebraic tensor product $X\otimes Y$ and define the **projective tensor norm** $\|\cdot\|_\pi$ on $X\otimes Y$ as follows: for $\omega\in X\otimes Y$,

$$\|\omega\|_\pi = \inf\left\{\sum_{i=1}^{n}\|x_i\|_X\|y_i\|_Y,\ \text{where } \omega=\sum_{i=1}^{n}x_i\otimes y_i\right\},$$

where the infimum is taken over all representations of $\omega\in X\otimes Y$.

Unless the spaces X and Y are finite dimensional, $X\otimes Y$ endowed with projective tensor norm is not complete and so not a Banach space.

Definition 2.1. *The projective tensor product of Banach spaces X and Y is defined as the completion of $X\otimes Y$ with respect to the projective tensor norm $\|\cdot\|_\pi$. It is denoted by $X\widehat{\otimes}Y$.*

Proposition 2.2. *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Then $\|x\otimes y\|_\pi=\|x\|_X\|y\|_Y$ for every $x\in X$ and $y\in Y$.*

Proof. This is Proposition 2.1. in [6]. □

In [6], it is shown that $\omega\in X\widehat{\otimes}Y$ if and only if, for every $\epsilon>0$, there exist $x_k\in X$ and $y_k\in Y$ such that

$$\omega=\sum_{k=1}^{\infty}x_k\otimes y_k \ \text{ and }\ \|\omega\|_\pi\le\sum_{k=1}^{\infty}\|x_k\|_X\|y_k\|_Y\le\|\omega\|_\pi+\epsilon$$

and also $\sum_{k=1}^{\infty}\|x_k\|_X\|y_k\|_Y<\infty$. Thus

$$\|\omega\|_\pi=\inf\left\{\sum_{k=1}^{\infty}\|x_k\|_X\|y_k\|_Y \mid \sum_{k=1}^{\infty}\|x_k\|_X\|y_k\|_Y<\infty, \omega=\sum_{k=1}^{\infty}x_k\otimes y_k\right\},$$

where the infimum is taken over all representations of $\omega\in X\widehat{\otimes}Y$. For more properties of projective tensor products, we refer to [6].

Remark 2.3. *In the category of Banach spaces and bounded linear morphisms, there exists finite (co)product. However, infinite (co)products do not exist as explained in* [1].

Remark 2.4. *From Remark 1.13, there is a complex of Banach spaces*

$$0\to\mathbb{R}\xrightarrow{\partial_0=0}B(G)\xrightarrow{\partial_1}B^2(G)\xrightarrow{\partial_2}B^3(G)\xrightarrow{\partial_3}\cdots. \tag{2.4.1}$$

Similarly, for K, there is a complex of Banach spaces

$$0\to\mathbb{R}\xrightarrow{\delta_0=0}B(K)\xrightarrow{\delta_1}B^2(K)\xrightarrow{\delta_2}B^3(K)\xrightarrow{\delta_3}\cdots. \tag{2.4.2}$$

As in the ordinary case, for $n \geq 1$ we let

$$\left(B^*(G)\widehat{\otimes}B^*(K)\right)_n = \bigoplus_{p+q=n} B^p(G)\widehat{\otimes}B^q(K).$$

From Remark 2.3, every $\left(B^*(G)\widehat{\otimes}B^*(K)\right)_n$ for $n \geq 1$ is a Banach space. Then we have a complex of Banach spaces

$$(2.4.3)\quad 0 \longrightarrow \mathbb{R} \overset{d_0'=0}{\longrightarrow} \left(B^*(G)\widehat{\otimes}B^*(K)\right)_1 \overset{d_1'}{\longrightarrow} \left(B^*(G)\widehat{\otimes}B^*(K)\right)_2 \overset{d_2'}{\longrightarrow} \cdots,$$

where the boundary operators d_n' are defined on the monomial tensor $\alpha \otimes \beta \in B^p(G)\widehat{\otimes}B^q(K)$ by the formula

$$d_n'(\alpha \otimes \beta) = \partial_p\alpha \otimes \beta + (-1)^p \alpha \otimes \delta_q\beta.$$

As we know it, the nth cohomology group of the complex (2.4.3) is denoted by $H^n\left(B^*(G)\widehat{\otimes}B^*(K)\right)$.

Let U and V be Banach spaces. Recall that we have a G-module $B(G,U)$ and a K-module $B(K,V)$. We consider their projective tensor product $B(G,U)\widehat{\otimes}B(K,V)$. A monomial tensor $\alpha \otimes \beta \in B(G,U)\widehat{\otimes}B(K,V)$ can be considered as a function

$$\alpha \otimes \beta : G \times K \to U \otimes V$$

defined by $(\alpha\otimes\beta)(x,y) = \alpha(x)\otimes\beta(y)$. Notice that $G\times K$ acts diagonally on each monomial tensor $\alpha \otimes \beta \in B(G,U)\widehat{\otimes}B(K,V)$ by

$$(x,y)\cdot(\alpha\otimes\beta) = (x\cdot\alpha)\otimes(y\cdot\beta).$$

Proposition 2.5. *Let U and V be Banach spaces. Then the projective tensor product $B(G,U)\widehat{\otimes}B(K,V)$ is a $G\times K$-module with respect to the diagonal action. Furthermore, $B^p(G)\widehat{\otimes}B^q(K)$ is a $G\times K$-module for positive integers p and q.*

Proof. Recall that $B(G,U)\widehat{\otimes}B(K,V)$ is a Banach space.

Let $(x,y) \in G\times K$ and $\omega \in B(G,U)\widehat{\otimes}B(K,V)$ be represented by

$$\omega = \sum_{k=1}^{\infty} \alpha_k \otimes \beta_k, \quad \text{where } \alpha_k \in B(G,U) \text{ and } \beta_k \in B(K,V) \text{ for each } k.$$

By extending linearly the diagonal $G\times K$ action on each monomial tensor, it is clear that $G\times K$ acts on $B(G,U)\widehat{\otimes}B(K,V)$ by

$$(2.5.1)\qquad (x,y)\cdot\omega = (x,y)\cdot\left(\sum_{k=1}^{\infty}\alpha_i\otimes\beta_i\right) = \sum_{k=1}^{\infty}(x\cdot\alpha_k)\otimes(y\cdot\beta_k).$$

Notice that

$$\|(x,y)\cdot\omega\|_\pi \le \|\sum_{k=1}^{\infty}(x\cdot\alpha_k)\otimes(y\cdot\beta_k)\| \le \sum_{k=1}^{\infty}\|x\cdot\alpha_k\|\|y\cdot\beta_k\| \le \sum_{k=1}^{\infty}\|\alpha_k\|\|\beta_k\|.$$

Since the projective tensor norm of ω is defined as the infimum arising from all norms of its representations, we have $\|(x,y)\cdot\omega\|_\pi \le \|\omega\|_\pi$. Thus this diagonal action is bounded. Hence $B(G,U)\widehat{\otimes}B(K,V)$ is a $(G\times K)$-module.

The second statement follows from that $B^p(G)\widehat{\otimes}B^q(K)$ is isomorphic to

$$B\left(G, B^{p-1}(G)\right)\widehat{\otimes}B\left(K, B^{q-1}(K)\right).$$

□

Now we consider the space of $G\times K$-invariant elements in $B^p(G)\widehat{\otimes}B^q(K)$.

Remark 2.6. *From a diagonal action of $G\times K$ on $B^p(G)\widehat{\otimes}B^q(K)$ defined in (2.5.1), observe that G acts only on $B^p(G)$ and K only on $B^q(K)$. Hence it is easy to see that*

$$\left(B^p(G)\widehat{\otimes}B^q(K)\right)^{G\times K} = B^p(G)^G\widehat{\otimes}B^q(K)^K = B^{p-1}(G)\widehat{\otimes}B^{q-1}(K).$$

Now we construct a strong resolution from the tensor product of two strong resolutions.

Theorem 2.7. *For a G-module A and a K-module B, let*

$$0 \longrightarrow A \underset{s_0}{\overset{\partial_{-1}}{\rightleftarrows}} X_0 \underset{s_1}{\overset{\partial_0}{\rightleftarrows}} X_1 \underset{s_2}{\overset{\partial_1}{\rightleftarrows}} X_2 \underset{s_3}{\overset{\partial_2}{\rightleftarrows}} X_3 \underset{s_4}{\overset{\partial_3}{\rightleftarrows}} X_4 \underset{s_5}{\overset{\partial_4}{\rightleftarrows}} \cdots$$

and

$$0 \longrightarrow B \underset{t_0}{\overset{\delta_{-1}}{\rightleftarrows}} Y_0 \underset{t_1}{\overset{\delta_0}{\rightleftarrows}} Y_1 \underset{t_2}{\overset{\delta_1}{\rightleftarrows}} Y_2 \underset{t_3}{\overset{\delta_2}{\rightleftarrows}} Y_3 \underset{t_4}{\overset{\delta_3}{\rightleftarrows}} Y_4 \underset{t_5}{\overset{\delta_4}{\rightleftarrows}} \cdots$$

be the strong G- and K-resolutions of A and B, respectively, satisfying the conditions $\|\partial_{-1}\circ s_0\|\le 1$ and $\|\delta_{-1}\circ t_0\|\le 1$. For $n\ge 0$, let

$$\left(\mathbf{X}\widehat{\otimes}\mathbf{Y}\right)_n = \bigoplus_{p+q=n} X_p\widehat{\otimes}Y_q.$$

Then the sequence

$$(2.7.1)\qquad 0\to A\widehat{\otimes}B \xrightarrow{d_{-1}} \left(\mathbf{X}\widehat{\otimes}\mathbf{Y}\right)_0 \xrightarrow{d_0} \left(\mathbf{X}\widehat{\otimes}\mathbf{Y}\right)_1 \xrightarrow{d_1} \left(\mathbf{X}\widehat{\otimes}\mathbf{Y}\right)_2 \xrightarrow{d_2} \cdots$$

is a strong $G \times K$-resolution of the $G \times K$-module $A \widehat{\otimes} B$, where the boundary operators d_* are defined as follows: $d_{-1} = \partial_{-1} \otimes \delta_{-1}$ and for $n \geq 0$

$$d_n = \sum_{p+q=n} \partial_p \otimes id_{Y_q} + (-1)^p id_{X_p} \otimes \delta_q.$$

Proof. It is easy to see that $A \widehat{\otimes} B$ and $\left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_n$ are $G \times K$-modules with a diagonal action. It is clear that the sequence (2.7.1) is a complex of $G \times K$-modules. So it is enough to construct the contracting homotopy, that is, a sequence of linear operators $k_0 : X_0 \widehat{\otimes} Y_0 \to A \widehat{\otimes} B$ and for each $n \geq 0$

$$k_{n+1} : \left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_{n+1} \to \left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_n$$

such that

$$k_0 \circ d_{-1} = \mathrm{id}, \quad d_{n-1} \circ k_n + k_{n+1} \circ d_n = id \quad \text{and} \quad \|k_n\| \leq 1.$$

Consider the sequence of Banach spaces

$$0 \longrightarrow A \widehat{\otimes} B \underset{k_0}{\overset{d_{-1}}{\rightleftarrows}} \left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_0 \underset{k_1}{\overset{d_0}{\rightleftarrows}} \left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_1 \underset{k_2}{\overset{d_1}{\rightleftarrows}} \left(\mathbf{X} \widehat{\otimes} \mathbf{Y}\right)_2 \underset{k_3}{\overset{d_2}{\rightleftarrows}} \cdots .$$

We define the operator k_0 as $k_0 = s_0 \otimes t_0$. It is clear that k_0 is linear and $\|k_0\| \leq \|s_0\| \|t_0\| \leq 1$. Also, for $n = p + q \geq 1$ and $x_p \otimes y_q \in X_p \widehat{\otimes} Y_q$, we define the operator k_n as follows:

$$\begin{aligned}
k_n\left(x_n \otimes y_0\right) &= \frac{1}{2}\left(s_n(x_n) \otimes y_0 + s_n(x_n) \otimes \delta_{-1} t_0(y_0)\right) && \text{for } p=n \text{ and } q=0 \\
k_n\left(x_0 \otimes y_n\right) &= \frac{1}{2}\left(x_0 \otimes t_n(y_n) + \partial_{-1} s_0(x_0) \otimes t_n(y_n)\right) && \text{for } p=0 \text{ and } q=n \\
k_n\left(x_p \otimes y_q\right) &= \frac{1}{2}\left(s_p(x_p) \otimes y_q + (-1)^p x_p \otimes t_q(y_q)\right) && \text{for } p, q \geq 1 .
\end{aligned}$$

It is clear that every k_n is linear. We show $\|k_n\| \leq 1$. It is enough to see that $\|k_n(x_p \otimes y_q)\|_\pi \leq \|x_p \otimes y_q\|_\pi$ for each monomial tensor $x_p \otimes y_q \in$

$X_p \widehat{\otimes} Y_q$. Notice that

$$\begin{aligned}
&\|k_n\left(x_n \otimes y_0\right)\|_\pi \\
&= \frac{1}{2}\|s_n(x_n) \otimes y_0 + s_n(x_n) \otimes \delta_{-1} t_0(y_0)\|_\pi \\
&\leq \frac{1}{2}\|s_n(x_n) \otimes y_0\|_\pi + \frac{1}{2}\|s_n(x_n) \otimes \delta_{-1} t_0(y_0)\|_\pi \\
&= \frac{1}{2}\|s_n(x_n)\|\|y_0\| + \frac{1}{2}\|s_n(x_n)\|\|\delta_{-1} t_0(y_0)\| && \text{by Proposition 2.2} \\
&\leq \frac{1}{2}\|x_n\|\|y_0\| + \frac{1}{2}\|x_n\|\|\delta_{-1} t_0\|\|y_0\| \\
&\leq \frac{1}{2}\|x_n\|\|y_0\| + \frac{1}{2}\|x_n\|\|y_0\| && \|\delta_{-1} t_0\| \leq 1 \\
&= \frac{1}{2}\|x_n \otimes y_0\|_\pi + \frac{1}{2}\|x_n \otimes y_0\|_\pi && \text{by Proposition 2.2} \\
&= \|x_n \otimes y_0\|_\pi
\end{aligned}$$

Similarly, we can show $\|k_n\left(x_0 \otimes y_n\right)\|_\pi \leq \|x_0 \otimes y_n\|_\pi$. Also,

$$\begin{aligned}
\|k_n\left(x_p \otimes y_q\right)\|_\pi &= \frac{1}{2}\|s_p(x_p) \otimes y_q + (-1)^p x_p \otimes t_q(y_q)\|_\pi \\
&\leq \frac{1}{2}\|s_p(x_p) \otimes y_q\|_\pi + \frac{1}{2}\|x_p \otimes t_q(y_q)\|_\pi \\
&= \frac{1}{2}\|s_p(x_p)\|\|y_q\| + \frac{1}{2}\|x_p\|\|t_q(y_q)\| \\
&\leq \frac{1}{2}\|x_p\|\|y_q\| + \frac{1}{2}\|x_p\|\|y_q\| = \|x_p\|\|y_q\| \\
&= \|x_p \otimes y_q\|_\pi.
\end{aligned}$$

Hence k_n is a linear operator such that $\|k_n\| \leq 1$ for each $n \geq 0$.

It remains to show that k_* is a contracting homotopy. By linearity, we only check it for a monomial tensor. First, we show $k_0 \circ d_{-1} = id$. For $a \otimes b \in A \widehat{\otimes} B$, we have

$$\begin{aligned}
(k_0 \circ d_{-1})(a \otimes b) &= (s_0 \otimes t_0) \circ (\partial_{-1} \otimes \delta_{-1})\,(a \otimes b) \\
&= (s_0 \circ \partial_{-1}) \otimes (t_0 \circ \delta_{-1})\,(a \otimes b) \\
&= (s_0 \circ \partial_{-1})\,(a) \otimes (t_0 \circ \delta_{-1})\,(b) = a \otimes b.
\end{aligned}$$

Thus $k_0 \circ d_{-1} = id$.

To show $d_{-1}\circ k_0 + k_1\circ d_0 = \mathrm{id}_{X_0\widehat{\otimes}Y_0}$, let $x_0\otimes y_0 \in X_0\widehat{\otimes}Y_0$. Then

$$\begin{aligned}
&(d_{-1}\circ k_0)\,(x_0\otimes y_0) = d_{-1}\,(s_0\otimes t_0)\,(x_0\otimes y_0)\\
&= (\partial_{-1}\otimes\delta_{-1})\,(s_0x_0\otimes t_0y_0)\\
&= (\partial_{-1}\circ s_0)\,(x_0)\otimes(\delta_{-1}\circ t_0)\,(y_0)\\
&\text{and}\\
&(k_1\circ d_0)\,(x_0\otimes y_0) = k_1\,(d_0(x_0\otimes y_0))\\
&= k_1\,(\partial_0x_0\otimes y_0 + x_0\otimes\delta_0y_0)\\
&= k_1\,(\partial_0x_0\otimes y_0) + k_1\,(x_0\otimes\delta_0y_0)\\
&= \frac{1}{2}\left[(s_1\circ\partial_0)x_0\otimes y_0 + (s_1\circ\partial_0)x_0\otimes(\delta_{-1}\circ t_0)y_0\right]\\
&+ \frac{1}{2}\left[x_0\otimes(t_1\circ\delta_0)y_0 + (\partial_{-1}\circ s_0)x_0\otimes(t_1\circ\delta_0)y_0\right].
\end{aligned}$$

Since $s_1\circ\partial_0 + \partial_{-1}\circ s_0 = id$ and $\delta_{-1}\circ t_0 + t_1\circ\delta_0 = id$, we have

$$\begin{aligned}
&(d_{-1}\circ k_0 + k_1\circ d_0)\,(x_0\otimes y_0)\\
&= (d_{-1}\circ k_0)\,(x_0\otimes y_0) + k_1\,(d_0(x_0\otimes y_0))\\
&= (\partial_{-1}\circ s_0)\,x_0\otimes(\delta_{-1}\circ t_0)\,y_0\\
&+ \frac{1}{2}\left[(s_1\circ\partial_0)x_0\otimes y_0 + (s_1\circ\partial_0)x_0\otimes(\delta_{-1}\circ t_0)y_0\right]\\
&+ \frac{1}{2}\left[x_0\otimes(t_1\circ\delta_0)y_0 + (\partial_{-1}\circ s_0)x_0\otimes(t_1\circ\delta_0)y_0\right]\\
&= \frac{1}{2}\left[(\partial_{-1}\circ s_0)\,x_0\otimes(\delta_{-1}\circ t_0)\,y_0 + (\partial_{-1}\circ s_0)x_0\otimes(t_1\circ\delta_0)y_0\right]\\
&+ \frac{1}{2}\left[(\partial_{-1}\circ s_0)\,x_0\otimes(\delta_{-1}\circ t_0)\,y_0 + (s_1\circ\partial_0)x_0\otimes(\delta_{-1}\circ t_0)y_0\right]\\
&+ \frac{1}{2}\left[(s_1\circ\partial_0)x_0\otimes y_0 + x_0\otimes(t_1\circ\delta_0)y_0\right]\\
&= \frac{1}{2}\left[(\partial_{-1}\circ s_0)\,x_0\otimes y_0 + x_0\otimes(\delta_{-1}\circ t_0)\,y_0\right]\\
&+ \frac{1}{2}\left[(s_1\circ\partial_0)x_0\otimes y_0 + x_0\otimes(t_1\circ\delta_0)y_0\right]\\
&= x_0\otimes y_0.
\end{aligned}$$

This shows $d_{-1}\circ k_0 + k_1\circ d_0 = id$.

Now, let $n \geq 1$. For $x_n \otimes y_0 \in X_n \widehat{\otimes} Y_0$, we have

$$
\begin{aligned}
&(d_{n-1} \circ k_n)(x_n \otimes y_0) = \frac{1}{2} d_{n-1} [s_n x_n \otimes y_0 + s_n x_n \otimes (\delta_{-1} \circ t_0) y_0] \\
&= \frac{1}{2} \left[(\partial_{n-1} \circ s_n) x_n \otimes y_0 + (-1)^{n-1} s_n x_n \otimes \delta_0 y_0 \right] \\
&+ \frac{1}{2} \begin{bmatrix} (\partial_{n-1} \circ s_n) x_n \otimes (\delta_{-1} \circ t_0) y_0 \\ + (-1)^{n-1} s_n x_n \otimes (\delta_0 \circ \delta_{-1} \circ t_0) y_0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} (\partial_{n-1} \circ s_n) x_n \otimes y_0 + (-1)^{n-1} s_n x_n \otimes \delta_0 y_0 \\ + (\partial_{n-1} \circ s_n) x_n \otimes (\delta_{-1} \circ t_0) y_0 \end{bmatrix}
\end{aligned}
$$

and also

$$
\begin{aligned}
&(k_{n+1} \circ d_n)(x_n \otimes y_0) = k_{n+1} (\partial_n x_n \otimes y_0 + (-1)^n x_n \otimes \delta_0 y_0) \\
&= \frac{1}{2} [(s_{n+1} \circ \partial_n) x_n \otimes y_0 + (s_{n+1} \circ \partial_n) x_n \otimes (\delta_{-1} \circ t_0) y_0] \\
&+ (-1)^n \frac{1}{2} [s_n x_n \otimes \delta_0 y_0 + (-1)^n x_n \otimes (t_1 \circ \delta_0) y_0].
\end{aligned}
$$

Then

$$
\begin{aligned}
&(d_{n-1} \circ k_n + k_{n+1} \circ d_n)(x_n \otimes y_0) \\
&= \frac{1}{2} \begin{bmatrix} (\partial_{n-1} \circ s_n) x_n \otimes y_0 + (-1)^{n-1} s_n x_n \otimes \delta_0 y_0 \\ + (\partial_{n-1} \circ s_n) x_n \otimes (\delta_{-1} \circ t_0) y_0 \end{bmatrix} \\
&+ \frac{1}{2} [(s_{n+1} \circ \partial_n) x_n \otimes y_0 + (s_{n+1} \circ \partial_n) x_n \otimes (\delta_{-1} \circ t_0) y_0] \\
&+ (-1)^n \frac{1}{2} [s_n x_n \otimes \delta_0 y_0 + (-1)^n x_n \otimes (t_1 \circ \delta_0) y_0] \\
&= \frac{1}{2} [(\partial_{n-1} \circ s_n) x_n \otimes y_0 + (s_{n+1} \circ \partial_n) x_n \otimes y_0] \\
&+ \frac{1}{2} \begin{bmatrix} (\partial_{n-1} \circ s_n) x_n \otimes (\delta_{-1} \circ t_0) y_0 \\ + (s_{n+1} \circ \partial_n) x_n \otimes (\delta_{-1} \circ t_0) y_0 + x_n \otimes (t_1 \circ \delta_0) y_0 \end{bmatrix} \\
&= \frac{1}{2} [x_n \otimes y_0 + x_n \otimes (\delta_{-1} \circ t_0) y_0 + x_n \otimes (t_1 \circ \delta_0) y_0] \\
&= \frac{1}{2} [x_n \otimes y_0 + x_n \otimes y_0] = x_n \otimes y_0.
\end{aligned}
$$

This shows $(d_{n-1} \circ k_n + k_{n+1} \circ d_n)(x_n \otimes y_0) = x_n \otimes y_0$. Similarly, we can show that, for $x_0 \otimes y_n \in X_0 \widehat{\otimes} Y_n$,

$$
(d_{n-1} \circ k_n + k_{n+1} \circ d_n)(x_0 \otimes y_n) = x_0 \otimes y_n.
$$

Now we consider $x_p \otimes y_q \in X_p \widehat{\otimes} Y_q$ for $p, q \geq 1$ and $p+q=n$. Then

$$\begin{aligned}
&(d_{n-1} \circ k_n)(x_p \otimes y_q) = \frac{1}{2} d_{n-1}\left[s_p x_p \otimes y_q + (-1)^p x_p \otimes t_q y_q\right] \\
&= \frac{1}{2}\left[(\partial_{p-1} \circ s_p) x_p \otimes y_q + (-1)^{p-1} s_p x_p \otimes \delta_q y_q\right] \\
&+ \frac{1}{2}\left[(-1)^p \partial_p x_p \otimes t_q y_q + (-1)^{2p} x_p \otimes (\delta_{q-1} \circ t_q) y_q\right]
\end{aligned}$$

and also

$$\begin{aligned}
&(k_{n+1} \circ d_n)(x_p \otimes y_q) \\
&= k_{n+1}\left(\partial_p x_p \otimes y_q + (-1)^p x_p \otimes \delta_q y_q\right) \\
&= \frac{1}{2}\left[(s_{p+1} \circ \partial_p) x_p \otimes y_q + (-1)^{p+1} \partial_p x_p \otimes t_q y_q\right] \\
&+ \frac{1}{2}\left[(-1)^p s_p x_p \otimes \delta_q y_q + (-1)^{2p} x_p \otimes (t_{q+1} \circ \delta_q) y_q\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&(d_{n-1} \circ k_n + k_{n+1} \circ d_n)(x_p \otimes y_q) \\
&= \frac{1}{2}\left[(\partial_{p-1} \circ s_p) x_p \otimes y_q + (-1)^{p-1} s_p x_p \otimes \delta_q y_q\right] \\
&+ \frac{1}{2}\left[(-1)^p \partial_p x_p \otimes t_q y_q + (-1)^{2p} x_p \otimes (\delta_{q-1} \circ t_q) y_q\right] \\
&+ \frac{1}{2}\left[(s_{p+1} \circ \partial_p) x_p \otimes y_q + (-1)^{p+1} \partial_p x_p \otimes t_q y_q\right] \\
&+ \frac{1}{2}\left[(-1)^p s_p x_p \otimes \delta_q y_q + (-1)^{2p} x_p \otimes (t_{q+1} \circ \delta_q) y_q\right] \\
&= \frac{1}{2}\left[(\partial_{p-1} \circ s_p) x_p \otimes y_q + (s_{p+1} \circ \partial_p) x_p \otimes y_q\right] \\
&+ \frac{1}{2}\left[x_p \otimes (\delta_{q-1} \circ t_q) y_q + x_p \otimes (t_{q+1} \circ \delta_q) y_q q\right] \\
&= x_p + y_q.
\end{aligned}$$

Thus $(d_{n-1} \circ k_n + k_{n+1} \circ d_n)(x_p \otimes y_q) = x_p \otimes y_q$.

By linear properties of d_* and k_*, we can conclude

$$d_{n-1} \circ k_n + k_{n+1} \circ d_n = id.$$

□

From now on, we denote the projective tensor products formed from the standard G- and K-resolutions by $\left(\mathbf{B}(G) \widehat{\otimes} \mathbf{B}(K)\right)_*$. Thus, for every

$n \geq 0$
$$\left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_n = \bigoplus_{\substack{p+q=n+2\\1\leq p\leq n+1}} B^p(G)\widehat{\otimes}B^q(K).$$

Corollary 2.8. *The sequence (2.8.1) below*

$$0 \longrightarrow \mathbb{R} \xrightarrow{d_{-1}} \left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_0 \xrightarrow{d_0} \left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_1 \xrightarrow{d_1} \cdots$$

is a strong resolution of the trivial $G \times K$-module $\mathbb{R}$.

Proof. Recall that $\mathbb{R}\widehat{\otimes}\mathbb{R} = \mathbb{R}$.
From Theorem 1.8, the standard G-resolution

$$0 \longrightarrow \mathbb{R} \underset{s_0}{\overset{\partial_{-1}}{\rightleftarrows}} B(G) \underset{s_1}{\overset{\partial_0}{\rightleftarrows}} B^2(G) \underset{s_2}{\overset{\partial_1}{\rightleftarrows}} B^3(G) \underset{s_3}{\overset{\partial_2}{\rightleftarrows}} \cdots$$

and the standard K-resolution

$$0 \longrightarrow \mathbb{R} \underset{t_0}{\overset{\delta_{-1}}{\rightleftarrows}} B(K) \underset{t_1}{\overset{\delta_0}{\rightleftarrows}} B^2(K)) \underset{t_2}{\overset{\delta_1}{\rightleftarrows}} B^3(K) \underset{t_3}{\overset{\delta_2}{\rightleftarrows}} \cdots$$

are strong, where the contracting homotopies $\{s_n\}$ and $\{t_n\}$ are defined as the same formula in Theorem 1.8. Notice that, for $\alpha \in B(G)$ and $g \in G$,

$$\| \left(\partial_{-1} \circ s_0\right)(\alpha)(g)\| = \|\partial_{-1}\left(\alpha(e_G)\right)(g)\| = \|\alpha(e_G)\| \leq \|\alpha\|.$$

Hence $\|\partial_{-1} \circ s_0\| \leq 1$. Similarly, we have $\|\delta_{-1} \circ t_0\| \leq 1$. Thus it follows from Theorem 2.7. □

3. Bounded cohomology of product of groups

Now we consider bounded cohomology groups of product of groups.

Recall that the (external) direct product $G \times K$ is also a discrete group with the operation defined coordinatewise. Let M be a Banach space. Similar to $B(G, M)$, it is easy to see that the space $B(G \times K, M)$ of all bounded functions $f : G \times K \to M$ is a (bounded) $G \times K$-module with the action defined by

$$\left((x, y) \cdot f\right)(a, b) = f(ax, by) \quad \text{for } (x, y), (a, b) \in G \times K.$$

Again, $\mathbb{R}$ forms a bounded $G \times K$-module with the trivial $G \times K$-action. For each $n > 0$, we consider the Cartesian product $(G \times K)^n$. We denote by $B^n(G \times K)$ the set of all real -valued bounded functions $f : (G \times K)^n \to \mathbb{R}$, where

$$\|f\| = \sup\{\|f(z_1, \cdots, z_n)\| \mid (z_1, \cdots, z_n) \in (G \times K)^n\}$$

for $z_i = (x_i, y_i) \in G \times K$. It is clear that $B^n(G \times K)$ is a Banach space with the norm $\|\cdot\|$.

Observe that the Banach space $B^n(G\times K)$ have the similar properties of $B^n(G)$. We list some for our convenience.

Remark 3.1. *Let n be a positive integer.*

(1): *$B^{n+1}(G \times K)$ is isomorphic to $B\left(G \times K, B^n(G \times K)\right)$.*

(2): *$B^n(G\times K)$ is a bounded $G\times K$-module with the following action: for $(x, y) \in G \times K$ and $((a_1, b_1), \cdots, (a_{n-1}, b_{n-1}), (a_n, b_n)) \in (G \times K)^n$*

$$\begin{aligned} &\left((x, y) \cdot f\right)\left((a_1, b_1), \cdots, (a_{n-1}, b_{n-1}), (a_n, b_n)\right) \\ &= f\left((a_1, b_1), \cdots, (a_{n-1}, b_{n-1}), (a_n x, b_n y)\right). \end{aligned}$$

(3): *$\left(B^{n+1}(G \times K)\right)^{G\times K}$ is isomorphic to $B^n(G \times K)$ for every $n > 0$. In particular, $\left(B(G \times K)\right)^{G\times K}$ is isomorphic to $\mathbb{R}$.*

Corollary 3.2. *Every $G \times K$-module $B^n(G \times K)$ for $n > 0$ is relatively injective.*

Proof. Let M be a Banach space. As $G \times K$ is a discrete group, a $G\times K$-module $B(G\times K, M)$ is a relatively injective by Proposition 1.5. Since $B^n(G \times K)$ is isomorphic to $B\left(G \times K, B^{n-1}(G \times K)\right)$ for every $n > 0$, $B^n(G \times K)$ is also relatively injective. □

Corollary 3.3. *The sequence*

$$0 \to \mathbb{R} \xrightarrow{\widetilde{d}_{-1}} B(G \times K) \xrightarrow{\widetilde{d}_0} B^2(G \times K) \xrightarrow{\widetilde{d}_1} B^3(G \times K) \xrightarrow{\widetilde{d}_2} \cdots \tag{3.3.1}$$

is a strong and relatively injective $G\times K$-resolution of the trivial $G\times K$-module $\mathbb{R}$, where the boundary operators $\widetilde{d}_*$ is defined by the same formula as in (1.8.2).

Proof. Let $z_i = (x_i, y_i) \in G \times K$ and $e = (e_G, e_K)$ be the identity of $G \times K$. Notice that the boundary operators $\widetilde{d}_*$ are defined by the same formulas in (1.8.2) as follows:

$$\begin{aligned} &\widetilde{d}_{-1}(r)(a, b) = r \\ &\widetilde{d}_n(f)(z_0, z_1, \ldots, z_n, z_{n+1}) \\ &= (-1)^{n+1} f(z_1, \ldots, z_{n+1}) + \sum_{i=0}^{n} (-1)^{n-i} f(z_0, \ldots, z_i z_{i+1}, \ldots, z_{n+1}). \end{aligned}$$

Also, we define linear operators $t_0 : B(G \times K) \to \mathbb{R}$ and $t_n : B^{n+1}(G \times K) \to B^n(G \times K)$ for $n > 0$ as follows:

$$t_0(f) = f(e_G, e_K) \quad \text{and} \quad t_n(f)(z_1, \cdots, z_n) = f(z_1, \cdots, z_n, e).$$

Then, it is easy to verify that the sequence (3.3.1) is a strong $G \times K$-resolution with contracting homotopy t_*. By Corollary 3.2, the sequence (3.3.1) is also relatively injective. □

Notice that the sequence (3.3.1) is the standard $G \times K$-resolution. Hence the nth cohomology group of its induced complex

(3.3.2)

$$0 \to (B(G\times K))^{G\times K} \xrightarrow{\widetilde{d}_0} \big(B^2(G\times K)\big)^{G\times K} \xrightarrow{\widetilde{d}_1} \big(B^3(G\times K)\big)^{G\times K} \xrightarrow{\widetilde{d}_2} \cdots$$

is $\widehat{H}^n(G \times K)$. Recall that the complex (3.3.2) is equal to

$$(3.3.3) \quad 0 \to \mathbb{R} \xrightarrow{\widetilde{d}_0} B(G \times K) \xrightarrow{\widetilde{d}_1} B^2(G \times K) \xrightarrow{\widetilde{d}_2} B^3(G \times K) \xrightarrow{\widetilde{d}_3} \cdots .$$

Now we construct another relatively injective $G \times K$-module.

Theorem 3.4. *Let U and V be Banach spaces. Then $B(G, U)\widehat{\otimes}B(K, V)$ is a relatively injective $G \times K$-module. Furthermore, every $G \times K$-module $\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_n$ for $n \geq 0$ is relatively injective.*

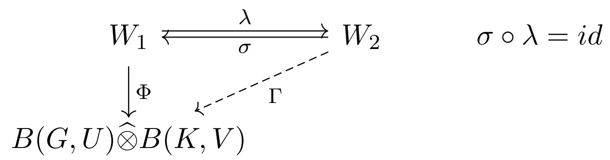
Proof. By Proposition 2.5, $B(G, U)\widehat{\otimes}B(K, V)$ is a $(G \times K)$-module.

Using a similar idea in Proposition 1.5, we show a $(G \times K)$-module $B(G, U)\widehat{\otimes}B(K, V)$ is relatively injective. Let $\lambda : W_1 \to W_2$ be any given strongly injective $G\times K$-morphism equipped with a bounded linear operator $\sigma : W_2 \to W_1$ such that $\sigma \circ \lambda = id$ and $\|\sigma\| \leq 1$. Also, let $\Phi : W_1 \to B(G, U)\widehat{\otimes}B(K, V)$ be any given $G \times K$-morphism. For $\omega \in W_1$, notice that $\Phi(\omega)$ is represented by $\Phi(\omega) = \sum_{k=1}^{\infty} \alpha_k \otimes \beta_k$ for $\alpha_k \in B(G, U)$ and $\beta_k \in B(K, V)$. In this case, for $x \in G$ and $y \in K$,

$$\Phi(\omega)(x, y) = \sum_{k=1}^{\infty} \alpha_k(x) \otimes \beta_k(y).$$

We consider the diagram illustrating the relatively injectivity

(3.4.1)



Let (e_G, e_K) be the identity of $G \times K$.

For $w \in W_2$, we define $\Gamma : W_2 \to B(G,U)\widehat{\otimes}B(K,V)$ by the formula

$$\Gamma(w)(x,y) = \Phi\left(\sigma\left((x,y)\cdot w\right)\right)(e_G, e_K) \qquad \text{for } x \in G \text{ and } y \in K.$$

First, we show that Γ is a $G \times K$-morphism. Let $w \in W_2$ and $(a,b) \in G \times K$. Then for $x \in G$ and $y \in K$, we have

$$\begin{aligned}\Gamma((a,b)\cdot w)(x,y) &= \Phi\left(\sigma\left((x,y)\cdot((a,b)\cdot w)\right)\right)(e_G,e_K)\\ &= \Phi\left(\sigma\left((xa,yb)\cdot w\right)\right)(e_G,e_K) = \Gamma(w)(xa,yb)\\ &= \left((a,b)\cdot\Gamma(w)\right)(x,y).\end{aligned}$$

Thus $\Gamma\left((a,b)\cdot w\right) = (a,b)\cdot\Gamma(w)$ and so Γ is a $G \times K$-morphism.

Secondly, we show that $\Gamma \circ \lambda = \Phi$. Let $\omega \in W_1$. For $x \in X$ and $y \in Y$, we have

$$\begin{aligned}(\Gamma\circ\lambda)(\omega)(x,y) &= \Gamma\left(\lambda(\omega)\right)(x,y) = \Phi\left(\sigma\left((x,y)\cdot\lambda(\omega)\right)\right)(e_G,e_K)\\ &= \Phi\left(\sigma\left(\lambda\left((x,y)\cdot\omega\right)\right)\right)(e_G,e_K) = \Phi\left((x,y)\cdot\omega\right)(e_G,e_K)\\ &= ((x,y)\cdot\Phi)(\omega)(e_G,e_K) = \Phi(\omega)(e_Gx,e_Ky) = \Phi(\omega)(x,y).\end{aligned}$$

Finally, for $w \in W_2$, notice that

$$\begin{aligned}\|\Gamma(w)(x,y)\| &= \|\Phi\left(\sigma\left((x,y)\cdot w\right)\right)(e_G,e_K)\\ &\le \|\Phi\|\|\sigma\|\|(x,y)\cdot w\| \le \|\Phi\|\|(x,y)\cdot w\| \le \|\Phi\|\|w\|.\end{aligned}$$

Thus $\|\Gamma\| \le \|\Phi\|$ and also Γ is bounded. Hence $B(G,U)\widehat{\otimes}B(K,V)$ is a relatively injective $G \times K$-module.

By setting that $U = \mathbb{R}$ and $V = \mathbb{R}$, $B(G)\widehat{\otimes}B(K)$ is a relatively injective $G \times K$-module. Also, for each $p > 0$ and $q > 0$, the $G \times K$-module $B^p(G)\widehat{\otimes}B^q(K)$ is isomorphic to $B\left(G, B^{p-1}(G)\right)\widehat{\otimes}B\left(K, B^{q-1}(K)\right)$ and so is relatively injective. Recall that

$$\left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_n = \bigoplus_{\substack{p+q=n+2\\ 1\le p\le n+1}} B^p(G)\widehat{\otimes}B^q(K).$$

By using projections π and injection ρ

$$\left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_n \xrightarrow{\pi} B^p(G)\widehat{\otimes}B^q(K) \xrightarrow{\rho} \left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_n,$$

it is easy to prove that its relatively injective property by the same method as the ordinary case shown in [5] that the direct product of injective modules is also injective. □

Remark 3.5. *From Remark 2.6, we have*

$$\left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)_0^{G\times K} = \left(B(G)\widehat{\otimes}B(K)\right)^{G\times K} = \mathbb{R}\widehat{\otimes}\mathbb{R} = \mathbb{R}$$

and for $n \geq 1$

$$\begin{aligned}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_n^{G\times K} &= \left(\bigoplus_{\substack{p+q=n+2\\1\leq p\leq n+1}} B^p(G)\widehat{\otimes}B^q(K)\right)^{G\times K}\\ &= \bigoplus_{\substack{p+q=n+2\\1\leq p\leq n+1}} B^p(G)^G\widehat{\otimes}B^q(K)^K\\ &= \bigoplus_{\substack{p+q=n\\0\leq p\leq n}} B^p(G)\widehat{\otimes}B^q(K)\\ &= \big(B^*(G)\widehat{\otimes}B^*(K)\big)_n .\end{aligned}$$

Theorem 3.6. *The cohomology groups* $H^*(B^*(G)\widehat{\otimes}B^*(K))$ *are isomorphic to the bounded cohomology groups* $\widehat{H}^*(G\times K)$ *of* $G\times K$, *that is, there is an isomorphism of groups*

$$H^*(B^*(G)\widehat{\otimes}B^*(K)) \cong \widehat{H}^*(G\times K).$$

Proof. Recall that $\widehat{H}^*(G\times K)$ can be computed by the complex induced from the standard $G\times K$-resolution

$$(3.3.1)\quad 0\to\mathbb{R}\xrightarrow{\widetilde{d}_{-1}} B(G\times K)\xrightarrow{\widetilde{d}_0} B^2(G\times K)\xrightarrow{\widetilde{d}_1} B^3(G\times K)\xrightarrow{\widetilde{d}_2}\cdots .$$

Recall that, by Corollary 2.8 and Theorem 3.4, the sequence (2.8.1) below

$$0\longrightarrow\mathbb{R}\xrightarrow{d_{-1}}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_0\xrightarrow{d_0}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_1\xrightarrow{d_1}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_2\xrightarrow{d_2}\cdots$$

is a strong and relatively injective $G\times K$-resolution of the trivial $G\times K$-module $\mathbb{R}$. It induces a complex

$$(3.6.1)\quad 0\longrightarrow\mathbb{R}\xrightarrow{d_0=0}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_1^{G\times K}\xrightarrow{d_1}\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)_2^{G\times K}\xrightarrow{d_2}\cdots .$$

and its nth cohomology is denoted by $H^n\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)$.

Since all cohomology groups of the complexes induced from strong and relatively injective $G\times K$-resolutions of the trivial $G\times K$-module $\mathbb{R}$ are canonically isomorphic by Proposition 1.11, the cohomology groups $H^*\big(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\big)$ and $\widehat{H}^*(G\times K)$ are isomorphic. On the other hand, from Remark 2.4, the nth cohomology of the complex (2.4.3)

$$(2.4.3)\qquad 0\longrightarrow\mathbb{R}\xrightarrow{d'_0=0}\big(B^*(G)\widehat{\otimes}B^*(K)\big)_1\xrightarrow{d'_1}\big(B^*(G)\widehat{\otimes}B^*(K)\big)_2\xrightarrow{d'_2}\cdots$$

is $H^n(B^*(G)\widehat{\otimes}B^*(K))$. Observe that the boundary operators d'_* in (2.4.3) are equal to d_* in (3.6.1), which are defined by the same boundary operators of the standard G- and K-resolutions. Also, by Remark 3.5, the complexes (3.6.1) and (2.4.3) are the same. So the cohomology groups $H^*(B^*(G)\widehat{\otimes}B^*(K))$ and $H^*\left(\mathbf{B}(G)\widehat{\otimes}\mathbf{B}(K)\right)$ are also the same. Hence the cohomology groups $H^*(B^*(G)\widehat{\otimes}B^*(K))$ and $\widehat{H}^*(G \times K)$ are isomorphic. $\square$

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