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ON STRONG METRIC DIMENSION OF ZERO-DIVISOR GRAPHS OF RINGS

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Abstract. In this paper, we study the strong metric dimension of zero-divisor graph $\Gamma(R)$ associated to a ring R. This is done by transforming the problem into a more well-known problem of finding the vertex cover number $\alpha(G)$ of a strong resolving graph G_{sr}. We find the strong metric dimension of zero-divisor graphs of the ring $\mathbb{Z}_n$ of integers modulo n and the ring of Gaussian integers $\mathbb{Z}_n[i]$ modulo n. We obtain the bounds for strong metric dimension of zero-divisor graphs and we also discuss the strong metric dimension of the Cartesian product of graphs.

1. Introduction

Let $G(V,E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The set of vertices adjacent to a vertex $v \in V(G)$ is the *neighborhood* of v and is denoted by $N(v)$. Further $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, or more simply we write $d(v)$ means the cardinality of $N(v)$. If the two vertices u and v are adjacent, we denote it by u *adj* v. A graph is *regular* if each of its vertex has the same degree. A *path* between two vertices $x_1, x_n \in V(G)$ is an ordered sequence of distinct vertices $x_1, x_2, \ldots, x_n$ of G such that $x_{i-1}x_i$ is an edge for

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$2 \le i \le n$. A closed path is a cycle. In G, the *distance* between two vertices x and y, denoted by $d(x,y)$, is the length of the shortest $x-y$ path in G. If there is no such path, we define $d(x,y)=\infty$. We say that G is *connected* if there exists a path between every pair of vertices in G. A graph that contains no cycles is called a *tree*. A *cut vertex* of a connected graph is a vertex whose removal results in a graph having two or more connected components. The *diameter* of a graph G is $diam(G) = sup\{d(x,y) \mid x \text{ and } y \text{ are distinct vertices of } G\}$. A clique is a maximal complete subgraph and the cardinality of its vertex set, denoted by $\omega(G)$, is called the *clique number* of G. In a graph G, a set $S \subset V(G)$ is an *independent set* if the subgraph induced by S is totally disconnected. We denote the complete graph on n vertices by K_n and the complete bipartite graph on m and n vertices by $K_{m,n}$. We will sometimes call a $K_{1,t}$ a star graph. A vertex u of G is *maximally distant* from v if for every vertex $w \in N(u), d(u,v) \ge d(v,w)$. If v is also maximally distant from u, then we say that u and v are *mutually maximally distant* and denote this by $uMMDv$. Also *boundary* of $G(V,E)$ is defined as $\partial(G) = \{u \in V : \text{there exists } v \in V \text{ with } uMMDv\}$. A set T of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of T. The *vertex cover number* of G, denoted by $\alpha(G)$, is the cardinality of smallest vertex cover of G. For basic definitions, we refer the reader to any standard graph theory book, such as [17, 26].

The idea of associating a graph to a ring is due to Beck [11], in which the author is primarily concerned with colorings. In [5], Anderson and Livingston defined the zero-divisor graph of a commutative ring R, denoted $\Gamma(R)$, to be the graph whose vertices are the nonzero zero-divisors of R, and in which x and y are connected by an edge if $xy=0$. Since then, there have been many papers written on the subject of zero-divisor graphs and and their variants (of which there are many). The interrelation between the ring-theoretic structure of R and the graph-theoretic structure of $\Gamma(R)$ has brought out interesting results from the perspective of both algebra and graph theory (cf. [2, 4–6], for example). Zero-divisor graphs were initially defined for commutative rings and later the concept of zero-divisor graphs was generalized to non-commutative rings by Redmond [21] and to the modules (see for example [10]). The concept widened the scope of this research area and many other graphs have been defined like total graphs, co-maximal graphs, unit graphs, Jacobson graphs, ideal based zero-divisor graphs, zero-divisor graphs of equivalence classes (cf. [1, 3, 7, 9, 22, 25]). For basic definitions, we refer the reader to [8, 15].

Throughout, unless otherwise stated, R denotes a finite commutative ring with $1 \neq 0$, the set of all non-zero zero-divisors of R is denoted by $Z^*(R) = Z(R) \setminus \{0\}$. A finite field on q number of elements is denoted by $\mathbb{F}_q$ and the ring of integers modulo n is denoted by Z_n. A ring R is a *local ring* if and only if R has a unique maximal ideal. An element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. A ring R is a *reduced ring* if it contains no non-zero nilpotent element. An *annihilator* of an element x of a ring R is defined as $ann(x) = \{r \in R \mid rx = 0\}$.

2. Metric dimension of some graphs

Harary and Melter [14] introduced the concept of *metric dimension* of graphs in the following way. Let G be a connected graph of order $n \geq 1$ and let $W = \{w_1, w_2, \dots, w_k\}$ be an (ordered) set of vertices. The metric vector of a vertex $v \in G$ relative to W is the vector $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. The set W is a *resolving set* of G if distinct vertices have distinct metric vectors and a minimum resolving set is called a *metric basis* for G and its cardinality, denoted by $dim(G)$, is called the metric dimension of G. This invariant has been further studied by a number of authors, including [12, 13, 18–20].

The *strong metric dimension* of a graph is defined as follows. In a connected graph G, for two distinct vertices u and v, the interval $I[u, v]$ is the collection of all vertices that belong to some shortest $u-v$ path. A vertex $w \in V(G)$ strongly resolves two vertices u and v if $v \in I[u, w]$ or $u \in I[v, w]$. In other words, two vertices u and v are strongly resolved by w if $d(w, u) = d(w, v) + d(v, u)$ or $d(w, v) = d(w, u) + d(u, v)$. A set W of vertices is a *strong resolving set* of G if every two distinct vertices of G are *strongly resolved* by some vertex of W and a minimum strong resolving set is called a *strong metric basis* and its cardinality is the *strong metric dimension* of G, denoted by $dim_s(G)$. If a vertex w strongly resolves u and v, it is easy to see that w also resolves these vertices. Hence every strong resolving set is a resolving set and $dim(G) \leq dim_s(G)$. In fact, $1 \leq dim_s(G) \leq n - 1$. Oellermann and Peters-Fransen [16] showed that the problem of finding the strong metric dimension of a graph G can be transformed into a more well-known problem of finding the vertex cover number $\alpha(G_{sr})$ of a *strong resolving graph* denoted by G_{sr} with vertex set $V(G_{sr}) = \partial(G)$ and $uv \in E(G_{sr})$ if and only if $uMMDv$ in G. We notice that every vertex of a strong resolving set is a boundary vertex.

EXAMPLE 2.1. For positive integers m and n,
(i) $(K_n)_{sr} = K_n$
(ii) $(K_{m,n})_{sr} = K_{m,n}$

THEOREM 2.2. [16] *For any connected graph G, $dim_s(G) = \alpha(G_{sr})$.*

Now, we discuss the strong metric dimension of some useful graphs. We start with the following lemma.

LEMMA 2.3. *For a connected graph G of order $n \geq 1$, $dim_s(G) = 1$ if and only if $G \cong P_n$, where P_n is the path on n vertices. Moreover, the only end vertices belong to the strong resolving set.*

Proof. Let $P_n := u = v_1 - v_2 - \cdots - v_n = v$ be a path. To show $dim_s(G) = 1$, by Theorem 2.2, it is enough to prove that $V((P_n)_{sr}) = \partial(P_n) = \{u, v\}$, that is, u and v are the only *MMD* vertices. First we show u and v are *MMD*. Clearly, $d(u, v) \geq d(v, w)$ for all $w \in N(u)$, implies u is maximally distant from v. Also, $d(v, u) \geq d(u, w)$ for all $w \in N(v)$, implies v is maximally distant from u. Therefore, by definition $uMMDv$. Now, we show that there is no any other pair of vertices which are MMD. Let $v_i, v_j \in V(G)$ for $1 < i, j \leq n - 1$. We consider the following three cases.
Case 1. If v_i and u are adjacent, then $d(u, v_i) > d(v_i, w)$ for all $w \in N(u)$. But $d(v_i, u) \ngeq d(u, w)$ for all $w \in N(v_i)$, therefore v_i is not MMD to u.
Case 2. If v_i and u are not adjacent, then $d(v_i, u) \geq$ or $\leq d(u, w)$ for all $w \in N(v_i)$ and $d(u, v_i) \geq d(v_i, w)$ for all $w \in N(u)$, therefore v_i is not MMD to u.
Case 3. Now, consider the vertices v_i and v_j, we observe that $d(v_i, v_j) \ngeq d(v_j, w)$, for all $w \in N(v_i)$ and $d(v_i, v_j) \ngeq d(v_i, w)$, for all $w \in N(v_j)$, which implies that v_i is not MMD v_j.
Thus u and v are the only *MMD* vertices in P_n. Hence, $(P_n)_{sr} \cong K_2$, implies that $dim_s(P_n) = \alpha(P_n)_{sr} = 1$.

On the other hand, let G be not a path, then either G is a tree (except path) or contains a cycle. Since in either case $dim(G) \geq 2$ and hence $dim_s(G) \geq 2$, as paths are the only graphs whose dimension is 1, a contradiction.

Further, any vertex v_i, $2 \leq i \leq n - 1$ does not strongly resolve the end vertices $u = v_1$ and $v = v_n$ of P_n. Therefore, only the end vertex forms a strong metric basis. □

The converse part also follows from the fact that $1 \leq dim(G) \leq dim_s(G)$, implying $dim(G) = 1$. Therefore, by [[19], Lemma 2.1], $G \cong P_n$.

THEOREM 2.4. *A connected graph G of order $n \geq 2$ has strong metric dimension $n-1$ if and only if $G \cong K_n$.*

Proof. First, assume that $G \cong K_n$. Since $dim(G) \leq dim_s(G)$ and $dim(G) = n-1$, it follows that $dim_s(G) \geq n-1$. Also, by definition, $dim_s(G) \leq n-1$. Combining, we have $dim_s(G) = n-1$.

For the converse, assume that $dim_s(G) = n-1$. Let $G' = K_n - e$, where $e = uv$ is an edge and let uu_iv be a path of length 2 in $G-e$. For the strong resolving set of $G-e$, we consider the following three cases. (i) $W_1 = V(G) \setminus \{u, v\}$ (ii) $W_2 = V(G) \setminus \{u_i, u_j\}$, $(|V(G)| \geq 4)$ (iii) $W_3 = V(G) \setminus \{u_i, u\}$ or $V(G) \setminus \{u_i, v\}$

Clearly, W_1 is not a strong resolving set. If so, then $u \in I[u_i, v]$ or $v \in I[u, u_i]$, for any $u_i \in W_1$ which is not true as u_i *adj* v. Also W_2 is not a strong resolving set, because neither $u_i \notin I[u, u_j]$ nor $u_j \notin I[u_i, u]$, because u *adj* u_i, u_j. So, W_3 is a strong resolving set, where u_i and u are strongly resolved by v; and u_i and v are strongly resolved by u. Thus, $dim_s(G-e) \leq n-2$. Therefore, $G \cong K_n$. □

By using the fact $(K_n)_{SR} = K_n$ and Theorem 2.2, we note that $dim_s(G) = n-1$ if $G \cong K_n$.

PROPOSITION 2.5. *For a graph G, $dim(G) = dim_s(G)$ if*

(i) $G \cong P_n$.
(ii) $G \cong K_n$.

THEOREM 2.6. *For any complete bipartite graph $K_{m,n}$, $dim_s(K_{m,n}) = m-n-2$.*

Proof. Consider a complete bipartite graph $G = K_{m,n}$ with partite sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Let W be a strong resolving set of G. Then $W = W_1 \cup W_2$, $W_i \subseteq V_i$, $(i = 1, 2)$ with $|W_1| = m-1$ and $|W_2| = n-1$. We claim that $W = V(G) \setminus \{u_m, v_n\}$. For if, $W^* = V(G) \setminus \{u_m, v_n, u_k\}$ or $W^* = V(G) \setminus \{u_m, v_n, v_k\}$, then both do not form a strong metric basis of G. If $W^* = V(G) \setminus \{u_m, v_n, u_k\}$ forms a strong metric basis, then a pair of vertices u_m and u_k are not strongly resolved by any vertex of W^*. The same argument applies to the other case. Hence, W is a strong metric basis of G. □

Definition 2.7. Two distinct vertices u and v of a connected graph G with $|V(G)| \geq 2$ are distance similar if $d(u, x) = d(v, x)$, for all $x \in V(G) \setminus \{u, v\}$. It can be easily seen that two distinct vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$ or $uv \in E(G)$ and $N[u] = N[v]$.

Theorem 2.8. *Let G be a connected graph whose vertex set is partitioned into k distinct distance similar classes $V_1, V_2, \ldots, V_k$ and m is the number of distance similar equivalence classes that consist of a single vertex. Then $|V(G)| \setminus k \leq dim_s(G) \leq |V(G)| - k + m$.*

Proof. Let $V(G)$ be partitioned into k distinct distance similar classes $V_1, V_2, \ldots, V_k$. Clearly, each V_i, $1 \leq i \leq k$, is either an independent set or induces a complete subgraph of G. If W is a strong resolving set of G, then W contains all except one vertex in each of the equivalence class V_i, otherwise there exists a pair of vertices u, v $(u \sim v)$ not resolved by any vertex $w \in W$. Thus, $dim_s(G) \geq |V(G)| - k$.

If W is a minimal strong resolving set for G, we prove that W contains at most $|V_i| - 1$ vertices of V_i, $|V_i| > 1$. Without loss of generality, suppose that $|V_1| > 1$ and W be a strong resolving set for G such that $V_1 \subset W$. Let $x \in V_1$. We show that either (a) $W' = W - \{x\}$ is a strong resolving set for G or (b) there exists an element $t \in V(G) - V_1$ such that $W^* = W' \cup \{t\} = W \cup \{t\} - \{x\}$ is a strong resolving set for G. That is, W^* is a strong resolving set of cardinality no larger than W, where $V_1 \subsetneq W$.

Define $W' = W - \{x\}$ and without loss of generality, choose $W = \{x, w_1, w_2, \ldots\}$ and $W' = \{w_1, w_2, \ldots\}$. Let $u, v \in V(G)$. If both $u, v \in W'$, then clearly u and v are strongly resolved by a vertex of W', that is, for any $w \in W'$ either $u \in I[w, v]$ or $v \in I[u, w]$. Again, if u or $v \in W'$, then by definition, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path containing u.

Suppose $u, v \notin W$. Then u and v are strongly resolved by a vertex of W. If u and v are not strongly resolved by a vertex of W', it must be the case that u and v are strongly resolved by x. However, there exist some $z \in W' \cap V_1$ such that u and v are not strongly resolved, would imply u and v are not strongly resolved by z. Since $x, z \in V_1$, imply u and v are strongly resolved by a vertex of W'.

If there does not exist any $u \in W'$ such that u and x are not strongly resolved by a vertex of W', then W' is a strong resolving set of G.

So, assume there does not exist any $u \in W'$ such that u and x are not strongly resolved by a vertex of W'. Let $r \in W' \cap V_1$ and let there be some other element $v \in W'$ such that v and x and u and x are not strongly resolved by a vertex of W'. Thus, $v \in I[u, x]$ or $u \in I[v, x]$. Since W is a strong resolving set, u, v are strongly resolved by a vertex of W. However, u and v not being strongly resolved by a vertex of W' and $x, r \in V_1$ imply x, r are not strongly resolved by u and v. Also,u, v are not strongly resolved by r, a contradiction. Hence, if there exists an element $v \in W'$ such that v and x are not strongly resolved by a vertex of W', then v is unique.

Case 1. Suppose $|V_i| > 1$ for each i. For any $y \in V(G) - \{x, u\}$, choose $q \in W'$ such that $y \sim q$. Then $d(y, u) = d(q, u) = d(q, x) = d(y, x)$. Thus, $u \sim x$, which is a contradiction.

Case 2. Suppose $|V_i| = 1$ for some i. Since v_i and x are not distance similar, there is some $s \in V(G) - \{x, u\}$ with $d(s, x) \neq d(u, x)$. Note that $V_j = \{s\}$ for some $j, s \in V_j$, because if not, there is some $t \in V_j \cap W'$ and $d(u, s) = d(u, t) = d(x, t) = d(x, s)$. Define, $W^* = W' \cup \{s\}$. Then $|W'^*| = |W|$ and u and x are strongly resolved by a vertex of W'^*. Since $W' \subset W'^*$ and using the same argument as above, a and b are strongly resolved by a vertex of W' for any two distinct vertices $a, b \in V(G)$. Hence, W'^* is a strong resolving set for G. Combining these facts we have $dim_s(G) \leq |V(G)| - k + m$. □

3. Strong metric dimension of zero divisor graphs of rings

We start this section with the following observation.

THEOREM 3.1. *Let R be a finite commutative ring. Then*

(i) $dim_s(\Gamma(R))$ *is finite if and only if R is finite.*
(ii) $dim_s(\Gamma(R))$ *is undefined if and only if R is an integral domain.*

Proof. (i) If R is finite, then $|Z^*(R)|$ is finite and therefore $dim_s(\Gamma(R))$ is finite. Now assume that $dim_s(\Gamma(R))$ is finite. Let W be a minimal strong metric basis for $\Gamma(R)$ with $|W| = k$, where k is some positive integer. Then $dim(\Gamma(R)) \leq dim_s(\Gamma(R)) = k$ implies that $dim(\Gamma(R)) \leq k$. Now, since the diameter of $\Gamma(R)$ is not more than 3, so by [[5], Theorem 2.3] $\Gamma(R)$ is finite. Therefore, $d(x, y) \in \{0, 1, 2, 3\}$, for every $x, y \in Z^*(R)$. Hence, $|Z^*(R)| \leq 4^k$. This implies that $Z^*(R)$ is finite and hence R is finite.

(ii) This follows from the fact that the strong metric basis of $\Gamma(R)$ is undefined if and only if the vertex set of $\Gamma(R)$ is empty. □

THEOREM 3.2. *Let R be a commutative ring with unity. Then $dim_s(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings.*

(i) $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2$

(ii) $\frac{\mathbb{Z}_3[x]}{(x^2)}, \frac{\mathbb{Z}_2[x]}{(x^3)}$ *or* $\frac{\mathbb{Z}_4[x]}{(2x, x^2-2)}$

Proof. Observe that the zero-divisor relation is not transitive for these rings, implies their $\Gamma(R)$ is a path P_2 or P_3. Therefore, by Lemma 2.3, $dim_s(G) = 1$. On the other hand, since paths are the only graphs for which the strong metric dimension is 1, so $|Z^*(R)| \leq 3$. The only if direction follows. □

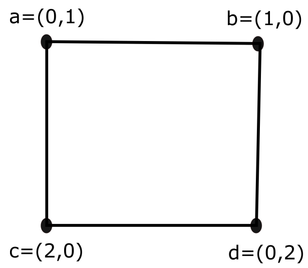

FIGURE 1. $dim_s(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$

PROPOSITION 3.3. *Let R be a commutative ring with $1 \neq 0$. Then $dim_s(\Gamma(R)) = 2$ if*

(i) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$

(ii) $R \cong \frac{\mathbb{F}_4[x]}{(x^2)}, \frac{\mathbb{Z}_4[x]}{(x^2+x+1)}, \frac{\mathbb{Z}_4[x]}{(2,x)^2}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$.

Proof. (i) If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $\Gamma(R)$ is a cycle on four vertices as shown in Figure 1. The set $W = \{a, b\}$ is in fact a strong resolving set of $\Gamma(R)$. Since all the possible sets $I[u, v]$, where $u \in \Gamma(R)$ and $v \in W$ have the form $I[a, b] = \{a, b\}$, $I[a, c] = \{a, c\}$, $I[a, d] = \{a, b, c, d\}$, $I[b, c] = \{a, b, c, d\}$, therefore each pair of vertices which contain vertex a or vertex b is strongly resolved by a or b. Vertices c and d are strongly resolved by both a and b, since $c \in [a, d]$ and $d \in [b, c]$. Hence, $2 = dim(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) \leq dim_s(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) \leq 2$. Thus, $dim_s(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$.
(ii) We know that $\Gamma(R) \cong K_3$ only if R is isomorphic to the rings mentioned above. Now, by Example 2.1, $(K_3)_{SR} = K_3$ and it is easy to see that $\alpha(K_3) = 2$. Hence, $dim_s(K_3) = 2$, by Theorem 2.2. □

For any zero-divisor graph $\Gamma(R)$ of a commutative ring R with vertex set $V(\Gamma(R))$ containing at least four vertices, $dim(\Gamma(R)) = |\Gamma(R)| - 2$ implies that $dim_s(\Gamma(R)) = |\Gamma(R)| - 2$. However, the converse is not true. For example, consider the ring $R \cong \frac{\mathbb{Z}_4[x]}{(x^2)}$. Then its $\Gamma(R)$ is shown in Figure 2(b) with $3 = dim(\Gamma(R)) \leq dim_s(\Gamma(R)) = 5$.

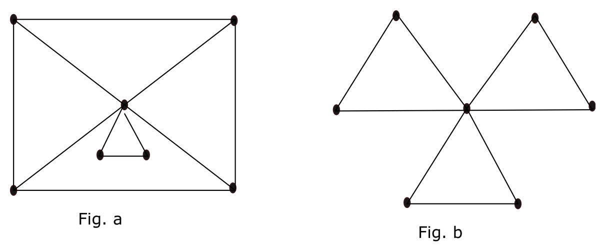

FIGURE 2. $3 = dim(\Gamma(\frac{\mathbb{Z}_4[x]}{(x^2)})) \leq dim_s(\Gamma(\frac{\mathbb{Z}_4[x]}{(x^2)})) = 5$

THEOREM 3.4. *Let R be a finite commutative ring with unity 1 such that $|Z^*(R)| \geq 3$. If $\Gamma(R)$ has a cut vertex but no degree 1 vertex, then $dim_s(\Gamma(R)) = 5$*

Proof. By [[23], Theorem 3], if $\Gamma(R)$ has a cut vertex but no degree one vertex, then R is isomorphic to one of the following rings $\mathbb{Z}_4[x,y]/(x^2,y^2,xy-2,2x,2y), \mathbb{Z}_2[x,y]/(x^2,y^2), \mathbb{Z}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2+2x),$ $\mathbb{Z}_8[x]/(2x,x^2+4), \mathbb{Z}_2[x,y]/(x^2,y^2-xy), \mathbb{Z}_4[x]/(x^2,y^2-xy,xy-2,2x,2y)$. The zero-divisor graphs associated to these rings with $dim_s(\Gamma(R)) = 5$ are shown in Figure 2. □

THEOREM 3.5. *Let R be a commutative ring with unity. Then $(\Gamma(R))_{sr} \cong K_{1,t}$ if and only if $R \cong \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2, \frac{\mathbb{Z}_3[x]}{(x^2)}, \frac{\mathbb{Z}_2[x]}{(x^3)}$ or $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$ and $t = 1$.*

Proof. Let R be isomorphic to one of these rings. Then $\Gamma(R)$ is a path with at most three vertices. Since the only end vertices of a path are MMD from each other, see Lemma 2.3, it follows that $|V(\Gamma(R))_{sr}| = |\partial(\Gamma(R))| = 2$. Thus, $(\Gamma(R))_{sr} \cong K_2$. On the other hand, let $(\Gamma(R))_{sr} \cong K_{1,t}$. Then, by Lemma 2.3, $dim_s(\Gamma(R)) = 1$ implies $\Gamma(R)$ is a path. Hence R is isomorphic to one of the rings mentioned above. □

The above discussions lead to the following problem.

PROBLEM 3.6. Do there exist rings R whose strong resolving graph $\Gamma(R))_{sr}$ is isomorphic to P_3.

THEOREM 3.7. *If R is a finite commutative ring with unity and $R \cong \mathbb{Z}_2 \times \mathbb{F}$ for some finite field $\mathbb{F}$, then $dim_s(\Gamma(R)) = |\Gamma(R)| - 2$. Moreover, if R is a local ring such that $\Gamma(R)$ has no cycles, then $dim_s(\Gamma(R)) = 1$.*

Proof. Firstly, if R is a local ring, the only zero divisor graphs with no cycles have three or fewer vertices [[24], Theorem 2.1]. Hence, $dim_s(\Gamma(R)) = 1$ in this case. Now, if R is a non local ring and $R \cong \mathbb{Z}_2 \times \mathbb{F}$, then its zero-divisor graph has a vertex adjacent to all other vertices, that is, $\Gamma(R)$ is a star graph $K_{1,|Z^*(R)|-1}$ of order $|Z^*(R)|$. Let u be the center vertex adjacent to the set of all other $|Z^*(R)|-1$ vertices v_i, $(1 \leq i \leq n)$, $n = |Z^*(R)| - 1$ which is an independent set. Clearly, the path between the two vertices v_i and v_j is not contained in any other shortest path and therefore every strong resolving set must contain at least one of them. In other words, each v_i is mutually maximally distant with v_j, $i \neq j$, $(1 \leq i, j \leq n)$, as $d(v_i, v_j) \geq d(v_j, u)$ for every $u \in N(v_i)$ and $d(v_i, v_j) \geq d(u, v_i)$ for every $u \in N(v_j)$. Therefore, any strong resolving set of $K_{1,|Z^*(R)|-1}$ must contain either v_i or v_j, $i \neq j$.

We claim that $W = \{v_1, v_2, \ldots, v_{n-1}\}$ is a strong resolving set. For, if $W' = \{v_1, v_2, \ldots, v_{n-2}\}$ is a strong resolving set, then by definition each pair of vertices is resolved by any vertex of W'. Choose v_{n-1} and v_n. Then $v_{n-1} \in I[v_i, v_n]$ or $v_n \in I[v_i, v_{n-1}]$ for any $v_i \in W'$ which is not true. Thus, W is a strong resolving set. Hence, $dim_s(\Gamma(R)) = dim_s(K_{1,|Z^*(R)|-1}) = |Z^*(R)| - 2 = |\Gamma(R)| - 2$. □

COROLLARY 3.8. *If R is a reduced ring and $\Gamma(R)$ has a vertex adjacent to every other vertex, then either $\Gamma(R) \cong K_2$ or $dim_s(\Gamma(R)) = |\Gamma(R)| - 2$.*

THEOREM 3.9. *Let R be a ring and let $\Gamma(R)$ be a regular graph. Then $dim_s(\Gamma(R)) = |Z^*(R)| - 1$ if and only if either $R \cong \mathbb{F} \times A$, where A is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime).*

Proof. Suppose that $R \cong \mathbb{F} \times A$, where A is an integral domain. Then, for $0 \neq a$, vertex $(a, 0)$ is adjacent to every other vertex. But $\Gamma(R)$ is regular graph, therefore $\Gamma(R)$ is complete regular and hence $dim_s(\Gamma(R)) = |Z^*(R)| - 1$. Conversely, assume that $dim_s(\Gamma(R)) = |Z^*(R)| - 1$. Since $\Gamma(R)$ is regular, so $\Gamma(R)$ is a complete graph. Thus there exists a vertex adjacent to every other vertex. Now, let $Z(R)$ be not an annihilator ideal (and hence is prime). Then, by [[5], Theorem 2.5], the result follows. □

PROPOSITION 3.10. *If R is a finite commutative ring with unity 1 such that $R = \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are finite fields with $|\mathbb{F}_1| = m \geq 3$ and $|\mathbb{F}_2| = n \geq 3$, then $dim_s(\Gamma(R)) = |\mathbb{F}_1| + |\mathbb{F}_2| - 2\omega(\Gamma(R))$.*

Proof. If $R = \mathbb{F}_1 \times \mathbb{F}_2$, then the vertex set of $\Gamma(R)$ can be partitioned into two distinct vertex sets $V_1 = \{(u, 0) : u \in \mathbb{F}_1^*\}$ and $V_2 = \{(0, v) : v \in \mathbb{F}_2^*\}$, where each $(u, 0)$ is adjacent to every vertex $(0, v)$. Thus, $\Gamma(R)$ is a complete bipartite graph $K_{m-1,n-1}$. Since $\omega(\Gamma(R)) = 2$, by Theorem 2.6, $dim_s(\Gamma(R)) = |\mathbb{F}_1| + |\mathbb{F}_2| - 2\omega(\Gamma(R))$. □

PROPOSITION 3.11. *If R is a finite local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}^2 = \{0\}$, then $dim_s(\Gamma(R)) = |\Gamma(R)| - 1$.*

Proof. Recall that the Jacobian radical $\mathfrak{J}(\mathfrak{R})$ of R is the intersection of maximal ideals of R. Since R is finite local ring, so $\mathfrak{J}(\mathfrak{R}) = Z(R)$ and $Z(R) = \mathfrak{m}$. Thus $Z(R)$ is a nilpotent ideal and R is not a field, implies $ann(Z(R)) \neq \{0\}$. As $\mathfrak{m}^2 = \{0\}$, so $ann(Z(R)) = Z^*(R)$ and therefore $\Gamma(R)$ is complete and thus the result follows by Theorem 2.4. □

THEOREM 3.12. *Let R be a reduced ring and $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two distinct prime ideals such that $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \{0\}$. Then $dim_s(\Gamma(R)) = |\mathfrak{I}_1| + |\mathfrak{I}_2| - 4$.*

Proof. Let $x \in Z(R) \setminus \mathfrak{I}_1 \cup \mathfrak{I}_2$. Then there exists $0 \neq b \in R$ such that $ab = 0 \in \mathfrak{I}_1 \cap \mathfrak{I}_2$. So, $y \in \mathfrak{I}_1 \cap \mathfrak{I}_2$, a contradiction, because $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \{0\}$. Also, $\mathfrak{I}_1 \cap \mathfrak{I}_2 \subseteq Z(R)$. So, $Z(R) = \mathfrak{I}_1 \cup \mathfrak{I}_2$. Now, take $V_1 = |\mathfrak{I}_1| - \{0\}$ and $V_2 = |\mathfrak{I}_2| - \{0\}$. We claim that $\Gamma(R)$ is a complete bipartite graph with partite sets V_1 and V_2. Indeed, if $a, b \in V_1$ with $ab = 0$, then $ab \in \mathfrak{I}_2$ and therefore a or $b \in V_2$, a contradiction. Thus $\Gamma(R)$ is a bipartite graph. Now, we take $a \in V_1$ and $b \in V_2$. So $ab \in \mathfrak{I}_1$ and $ab \in \mathfrak{I}_2$, since $\mathfrak{I}_1$ is an ideal and $\mathfrak{I}_2$ is an ideal. Then $ab \in \mathfrak{I}_1 \cap \mathfrak{I}_2 = \{0\}$ implies that $ab = 0$. Thus, $\Gamma(R)$ is a complete bipartite graph. Hence, by Theorem 2.6, $dim_s(\Gamma(R)) = |\mathfrak{I}_1| + |\mathfrak{I}_2| - 4$. □

THEOREM 3.13. *Let R_1 and R_2 be commutative rings with $R_1 \cong \mathbb{Z}_{p^2}$ and $R_2 \cong \frac{\mathbb{Z}_p[x]}{(x^2)}$, where p is a prime. Then $dim_s(\Gamma(R_1)) = dim_s(\Gamma(R_2)) = p - 2$.*

Proof. Considering the ring $R_1 = \mathbb{Z}_{p^2}$, its set of non-zero zero-divisors is $Z^*(R_1) = \{kp : 1 \leq k \leq p - 1, k \in \mathbb{N}\}$ such that $k_1pk_2p = 0$ for all $1 \leq k_1k_2 \leq p - 1$. Thus, $\Gamma(R) \cong K_{p-1}$.
Now, consider $R_2 = \frac{\mathbb{Z}_p[x]}{(x^2)} = \{a + bx : a, b \in \mathbb{Z}_p\}$. So, $Z^*(R_2) = \{bx : 1 \leq b \leq p - 1\}$. We see that $\Gamma(R_2) \cong K_{p-1}$. Therefore, by Theorem 2.4, $dim_s(\Gamma(R_1)) = dim_s(\Gamma(R_2)) = p - 2$. □

From the above theorem we have the following consequence.

Corollary 3.14. *The graph* $\Gamma(\mathbb{Z}_n)$ *is Hamiltonian if and only if* $dim_s(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 1$.

Proof. By Corollary 1 of [2], we know that the graph $\Gamma(\mathbb{Z}_n)$ is Hamiltonian graph if and only if $n = p^2$, where p is a prime larger than 3 and $\Gamma(\mathbb{Z}_n)$ is isomorphic to K_{p-1}. Thus the result follows. □

Proposition 3.15. $dim_s(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 2$, *if* $n = 2p$, *where* p *is prime larger than* 2.

Proof. If $p > 2$, the zero-divisor set of $\mathbb{Z}_n$ is $Z^*(\mathbb{Z}_n) = \{2k, 1 \leq k \leq p; k \in \mathbb{N}\}$ such that $2k_1 2k_2 = 0$. It follows that p is adjacent to all other vertices. Thus, $\Gamma(\mathbb{Z}_n) \cong K_{1,|Z^*(\mathbb{Z}_n)|-1}$. Therefore, by Theorem 2.6, $dim_s(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 2$. □

Theorem 3.16. *Let* p *be a prime number and* $n \in \mathbb{N}$. *Then* $dim_s(\Gamma(\mathbb{Z}_n)) = |Z^*(\mathbb{Z}_n)| - 2$ *if*

(i) $n = pq$, *where* p *and* q *are distinct primes.*
(ii) $n = 2^2 p$, *where* p *is any odd prime.*

Proof. (i) If $n = pq$, we partition the zero-divisor set of $\mathbb{Z}_n$ into sets $V_1 = \{kp : (k, q) = 1\}$ and $V_2 = \{kq : (k, p) = 1\}$. Clearly, $\Gamma(\mathbb{Z}_n)$ is a bipartite graph. Also, $uv = 0$ for every $u \in V_1$ and $v \in V_2$. Hence, $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph. Therefore, by Theorem 2.6, the result follows.
(ii) If $n = 2^2 p$, where p is any odd prime, we partition the vertex set into sets $V_1 = \{2k, 1 \leq k \leq n, k \neq p\}$ and $V_2 = \{kp : kp < n\} = \{p, 2p, 3p\}$. Since, $p \nmid 2k$ for any $1 \leq k \leq n$, none of the elements of V_1 are adjacent. Also, since $2 \nmid p$ and $2 \nmid 3p$, no elements of V_2 are adjacent. Furthermore, we see that $uv = 0$ for every $u \in V_1$ and $v \in V_2$. Hence, $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph. Therefore, by Theorem 2.6, the result follows. □

Remark 3.17. To construct the zero-divisor graph of $\mathbb{Z}_n$ and hence to find strong metric dimension of $\Gamma(\mathbb{Z}_n)$, it is best to break down n into prime factorization. Here we discuss some cases, when $1 < n < 100$.

Case 1. If n is a single prime, the graph $\Gamma(\mathbb{Z}_n)$ is trivial with no vertices or edges.
Case 2. If $n = pq$. The numbers in this case are 6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87,

91, 93, 94, 95. The zero-divisor graph is the complete bipartite graph by taking all the multiples of p in one partite set and the remaining zero-divisors as the multiples of q in another partite set. Thus, by Theorem 2.6, $dim_s(\Gamma(\mathbb{Z}_n))$ = number of multiples of p + number of multiples of q - 2. Clearly this case is discussed in Theorem 3.16.
Case 3. If $n = p^2$, the numbers in this case are $9, 25$ and 49. This case is also discussed in Theorem 3.13.
Case 4. If $n = p^3$, the zero-divisor graph is a complete bipartite graph by taking the vertices which are multiples of p^2 in one class and the remaining vertices being all multiples of p in the other. Thus this case also follows from Theorem 2.6. □

Definition 3.18. The set of Gaussian integers is denoted by $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$. Clearly $\mathbb{Z}[i]$ is a ring under the usual complex operations. The factor ring $\mathbb{Z}[i]/\langle n\rangle$ is isomorphic to $\mathbb{Z}_n[i] = \{a+ib \mid a, b \in \mathbb{Z}_n\}$, where $\langle n\rangle$ is a principal ideal generated by n for some positive integer larger than 1 in $\mathbb{Z}[i]$. Obviously, $\mathbb{Z}_n[i]$ is a ring with addition and multiplication modulo n. This ring is called the ring of Gaussian integers modulo n.

We now determine the strong metric dimension of $\Gamma(\mathbb{Z}_n[i])$.

Theorem 3.19. (i) *$dim_s(\Gamma(\mathbb{Z}_n[i])) = 0$, if $n = 2$.*
(ii) *$dim_s(\Gamma(\mathbb{Z}_n[i]))$ is undefined, if $n = q \equiv 3$ modulo 4.*
(iii) *$dim_s(\Gamma(\mathbb{Z}_n[i])) = q^2 - 2$, if $n = q^2$.*
(iv) *$dim_s(\Gamma(\mathbb{Z}_n[i])) = 2p - 4$, if $n = p \equiv 1$ modulo 4.*
(v) *$dim_s(\Gamma(\mathbb{Z}_{q_1q_2}[i])) = {q_1}^2 + {q_2}^2 - 4$, if $q_j \equiv 3$ modulo 4, $j = 1, 2$.*

Proof. (i). $\mathbb{Z}_2[i]$ is isomorphic to the local ring $\mathbb{Z}[i]/\langle(1 + i)^2\rangle$, with unique maximal ideal $\{0, 1+i\}$. So we have $V(\Gamma(\mathbb{Z}_2[i])) = \{1+i\}$, which implies that $\Gamma(\mathbb{Z}_2[i])$ is a graph on a single vertex and no edge and the result holds.
(ii). In this case $\mathbb{Z}_q[i]$ is a field, therefore $\Gamma(\mathbb{Z}_q[i])$ is an empty graph. So $dim_s(\Gamma(\mathbb{Z}_q[i]))$ is undefined.
(iii). $\Gamma(\mathbb{Z}_{q^2}[i])$ is a complete graph isomorphic to K_{q^2-1} implies that $dim_s(\Gamma(\mathbb{Z}_{q^2}[i])) = q^2 - 2$.
(iv). $\Gamma(\mathbb{Z}_p[i])$ is a complete bipartite graph $K_{p-1,p-1}$ with partite sets $V_1 =< a + ib > -\{0\}$ and $V_2 =< a - ib > -\{0\}$, since $\mathbb{Z}_p[i] \cong \mathbb{Z}[i] \cong \mathbb{Z}[i]/\langle a + ib\rangle \times \mathbb{Z}[i]/\langle a - ib\rangle$.
(v). Since $\mathbb{Z}_{q_j}[i]$ is a field and $\Gamma(\mathbb{Z}_{q_1q_2}[i]) \cong \Gamma(\mathbb{Z}_{q_1}[i]) \times \Gamma(\mathbb{Z}_{q_2}[i])$, therefore $\Gamma(\mathbb{Z}_{q_1q_2}[i]) \cong K_{{q_1}^2-1,{q_2}^2-1}$ is a complete bipartite graph. □

The following result gives a relation between the maximum degree, diameter and metric dimension of $\Gamma(R)$.

THEOREM 3.20. *[19] Let R be a finite commutative ring with unity 1 such that $|Z^*(R)| \geq 2$ with diameter d. Then*

$$\lceil log_3(\Delta + 1)\rceil \leq dim(\Gamma(R)) \leq |Z^*(R)| - d,$$

where Δ is the maximum degree of $\Gamma(R)$.

We observe that the lower and upper bounds of Theorem 3.20 also hold when $dim(\Gamma(R))$ is replaced by $dim_s(\Gamma(R))$.

THEOREM 3.21. *Let R be a finite commutative ring with unity 1 such that $|Z^*(R)| \geq 2$ with diameter d. Then*

$$\lceil log_3(\Delta + 1)\rceil \leq dim_s(\Gamma(R)) \leq |Z^*(R)| - d,$$

where Δ is the maximum degree.

Proof. We first establish the upper bound. Let u and v be the vertices for which $d(u, v) = sup\{d(x, y) \mid x, y \in Z^*(R)\}$, that is, $d(u, v)$ is the diameter of $\Gamma(R)$ and let $u = v_1, v_2, \ldots, v_d = v$ be $u - v$ path of length d. Since $W = V(\Gamma(R)) - \{u_i \mid 1 \leq i \leq d\}$ forms a strong resolving set for $\Gamma(R)$ with $|W| = n - d$, so $dim_s(\Gamma(R)) \leq n - d$.

Now, for the lower bound, since $\lceil log_3(\Delta+1)\rceil \leq dim(\Gamma(R)) \leq dim_s(\Gamma(R))$, it follows that $dim_s(\Gamma(R)) \geq \lceil log_3(\Delta + 1)\rceil$. □

THEOREM 3.22. *If R is a finite commutative ring, then $dim_s(\Gamma(R)) \leq |\partial(\Gamma(R))| - 1$.*

Proof. If R is a finite commutative ring and $\Gamma(R)$ be its corresponding zero-divisor graph with vertex set $|Z^*(R)|$, then $dim(\Gamma(R)) \leq |Z^*(R)| - 1$ implies $dim_s(\Gamma(R)) = \alpha((\Gamma(R))_{SR}) \leq |\partial(\Gamma(R))| - 1$. □

DEFINITION 3.23. For a commutative ring R with $1 \neq 0$, a compressed zero-divisor graph of a ring R is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$.

The authors in [18] have discussed the metric dimension of compressed zero-divisor graphs $\Gamma_E(R)$. We have the following observations.

THEOREM 3.24. *If R is a finite commutative ring, then* $\dim_E(\Gamma(R)) \leq dim(\Gamma(R)) \leq dim_s(\Gamma(R))$.

Proof. Since $\Gamma_E(R)$ has a vertex set constructed by taking equivalence of zero-divisors of a ring R, therefore $[x] \subset Z(R) \setminus \{0\}$ implies that each vertex of $\Gamma_E(R)$ is a representative of a distinct class of zero-divisors actually in R. Hence, $dim(\Gamma_E(R)) \leq dim(\Gamma(R))$. Also, we know that $dim(\Gamma(R)) \leq dim_s(\Gamma(R))$. □

PROPOSITION 3.25. (i) $dim_s(\Gamma_E(R)) = 0$ *if* $\Gamma_E(R) \cong K_n$ *unless* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(ii) $dim_s(\Gamma_E(R)) = 1$ *if* $\Gamma_E(R) \cong K_{m,n}$, m *or* $n \geq 2$.
(iii) $dim_s(\Gamma_E(R)) = n - 1$ *if* $\Gamma_E(R) = K_{1,n}$, $n \geq 2$.

4. Strong metric dimension of Cartesian products

The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ whose vertex set is $V = V(G_1) \times V(G_2)$ and the two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V ; $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ are adjacent in $G_1 \times G_2$ if and only if (a) either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.
Let S be a set of vertices in the Cartesian product $G_1 \times G_2$. The *projection* of S onto G_1 is the set of vertices $a \in V(G_1)$ for which there exists a vertex $(a, v) \in S$. The same is defined similarly for G_2.

We have the following observation about cartesian product of two graphs.

LEMMA 4.1. *For any graphs G_1 and G_2,* $\partial(G_1 \times G_2) = \partial(G_1) \times \partial(G_2)$.

Proof. Suppose $(u, v) \in \partial(G_1 \times G_2)$ and $u \notin \partial(G_1)$. Then, for every $u_1 \in V(G_1)$, there exists $u_2 \in N_{G_1}(u)$ such that $d_{G_1}(u, u_1) < d_{G_1}(u_1, u_2)$. Now, consider a vertex $(u_2, v) \in N_{G_1 \times G_2}(u, v)$. Then, for arbitrary $v_1 \in V(G_2)$, we have $d_{G_1 \times G_2}((u_1, v_1), (u_2, v)) = d_{G_1}(u_1, u_2) + d_{G_2}(v_1, v) > d_{G_1}(u_1, u) + d_{G_2}(v_1, v) = d_{G_1 \times G_2}((u_1, v_1), (u, v))$, a contradiction to the assumption $(u, v) \in \partial(G_1 \times G_2)$. Thus, $u \in \partial(G_1)$. Similarly, we can prove that $v \in \partial(G_2)$.

Now, let $u \in \partial(G_1)$ and $v \in \partial(G_2)$. Thus there exists a vertex $u_1 \in V(G_1)$ such that for every $u_2 \in N_{G_1}(u)$, we have $d_{G_1}(u, u_1) \geq d_{G_1}(u_1, u_2)$ and there is a vertex $v_1 \in V(G_2)$ such that for every $v_2 \in N_{G_2}(v)$, we have, $d_{G_2}(v, v_1) \geq d_{G_2}(v_1, v_2)$. Let (u_2, v_2) be an arbitrary vertex from $N_{G_1 \times G_2}(u, v)$. Without loss of generality, assume that $u_2 u \in E(G_1)$ and $v_2 = v$. Then $d_{G_1 \times G_2}((u_1, v_1), (u_2, v_2)) = d_{G_1}(u_1, u_2) + d_{G_2}(v_1, v_2) \leq d_{G_1}(u_1, u) + d_{G_2}(v_1, v) = d_{G_1 \times G_2}((u_1, v_1), (u, v))$ and $(u, v) \in \partial(G_1 \times G_2)$. □

Here, we observe that $V((G_1 \times G_2)_{SR}) = \partial(G_1 \times G_2) = \partial(G_1) \times \partial(G_2) = V((G_1)SR \times V(G_2)_{SR})$.

THEOREM 4.2. *Let R be a finite commutative ring with unity $1 \neq 0$. Then $dim_s(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$ if and only if $\Gamma(R)$ is a path.*

Proof. If $\Gamma(R)$ is a path, then

$$\begin{aligned} dim_s(\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) &= \alpha((\Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))_{SR}) \\ &= \alpha((\Gamma(R))_{SR} \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)_{SR}) \\ &= \alpha(K_2 \times K_2) = 2. \end{aligned}$$

Now, Let $G = \Gamma(R) \times \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and let $W = \{(u, v), (u_1, v_1)\}$ be a strong metric basis of G. We consider the following two cases.
Case 1. If $u \neq u_1$, let w_1 be a neighbor of u_1 on a $u - u_1$ path. Since W is a strong metric basis, each pair of vertices of G by definition is resolved by a vertex of W. We choose (u_1, v) and (w_1, v_1). Then we have $d_G((u_1, v), (u, v)) = d_{\Gamma(R)}(u, u_1) + d_{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}(v, v) = d_{\Gamma(R)}(u, w_1) + 1 + d_{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}(v, v_1) - 1 = d_{\Gamma(R)}(u, w_1) + d_{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}(v, v_1) = d_G((w_1, v_1), (u, v))$. Thus,$(u_1, v) \notin I_G[(w_1, v_1), (u, v)]$ and $(w_1, v_1) \notin I_G[(u, v), (u_1, v)]$. Moreover,

$$\begin{aligned} d_G((u_1, v), (u_1, v_1)) &= d_{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}(v, v_1) = 1 \\ &= d_{\Gamma(R)}(u_1, w_1) = d_G((w_1, v_1), (u_1, v_1)). \end{aligned}$$

Thus, $(u_1, v) \notin I_G[(w_1, v_1), (u_1, v_1)]$ and $(w_1, v_1) \notin I_G[(u_1, v_1), (u_1, v)]$. Therefore, $S = \{(u, v), (u_1, v_1)\}$ does not strongly resolve (u_1, v) and (w_1, v_1) and so $u = u_1$.
Case 2. If $u = u_1$, then the projection of W onto $\Gamma(R)$ is a single vertex and therefore the projection of W onto $\Gamma(R)$ strongly resolves $\Gamma(R)$. Hence $\Gamma(R)$ is a path. □

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