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ON STEFFENSEN INEQUALITY IN p-CALCULUS

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Abstract. In this paper, we provide a new version of Steffensen inequality for p-calculus analogue in [17, 18] which is a generalization of previous results. Also, the conditions for validity of reverse to p-Steffensen inequalities are given. Lastly, we will obtain a generalization of p-Steffensen inequality to the case of monotonic functions.

1. Introduction

Many applied sciences and engineering problems, for instance, can be pursued without their explicit mention. Nevertheless, a facility with inequalities seem to be necessary for an understanding of much of mathematics at intermediate and higher levels. Inequalities serve a natural purpose of comparison, and they sometimes afford us indirect routes of reasoning or problem solving when more direct routes might be inconvenient or unavailable.

The Steffensen inequality is of great interest in differential and difference equations [20]. Many authors have dealt with this renowned inequality [3, 4, 8, 12, 21]. Steffensen inequality is important not only in the theory of inequalities also in many applications such as statistics, functional equations, special functions, time scales etc. Some of these applications can be found in [5, 6, 9–11].

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The purpose of this paper is to find p-generalization of the classical Steffensen inequality. We will present some of the necessary conditions for validity of reverse to p-Steffensen inequality. As well as we will show that by using a quite different form, we can to access to p-Steffensen inequality. Before beginning the main subject of the paper, let us to present definitions and facts from the quantum calculus and p-calculus necessary for understanding of this paper.

Quantum calculus is usually known as "calculus without limit". In 1750 Euler introduced a type of quantum calculus called the q-calculus. The notions of the q-derivative and the definite q-integral were (re)introduced by Jackson in the early twentieth century [13]. The following expression,

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

is called the q-derivative of the function $f(x)$, where q is a fixed number different from 1. The q-calculus has developed into an interdisciplinary subject and has a lot of applications in different mathematical areas and physics and chemical physics [7,15,16]. For more details about quantum calculus, we refer the readers to [1,2,7,14,19].

Recently, the authors presented a new type of quantum calculus, called the p-calculus involving two concepts of p-derivative and p-integral in [17]. Moreover, some new properties of functions in p-calculus such as effects of a convex or monotone function on the p-derivative, the behavior of p-derivative in a neighborhood of a local extreme point and mean value theorems for p-derivatives and p-integrals were proposed in [18].

Throughout this paper, we assume that p is a fixed number different from 1 and domain of function $f(x)$ is $[0, +\infty)$. Here, we recall some definitions and fundamental results on p-calculus that is needed to prove our results (see [17,18]).

Definition 1.1. Let $f(x)$ be an arbitrary function. Then the p-differential is defined as

$$d_p f(x) = f(x^p) - f(x).$$

In particular, $d_p(x) = x^p - x$. By the p-differential, we can define p-derivative of a function.

Definition 1.2. For an arbitrary function $f(x)$, the p-derivative is defined by

$$D_p f(x) = \frac{d_p f(x)}{d_p(x)} = \frac{f(x^p) - f(x)}{x^p - x}, \qquad \text{if } \ x \neq 0, 1.$$

Corollary 1.3. *If $f(x)$ is differentiable, then* $\lim\limits_{p\to 1} D_p f(x) = f'(x)$, *and also if $f'(x)$ exists in a neighborhood of $x = 0$, $x = 1$ and is continuous at $x = 0$ and $x = 1$, then we have*

$$D_p f(0) = f'_+(0), \qquad\qquad D_p f(1) = f'(1).$$

Definition 1.4. The p-derivative of higher order of function f is defined by

$$(D_p^0 f)(x) = f(x), \qquad\qquad (D_p^n f)(x) = D_p(D_p^{n-1} f)(x), n \in N.$$

Notice that the p-derivative is a linear operator, i.e., for any constants a and b, and arbitrary functions $f(x)$ and $g(x)$, we have

$$D_p(af(x) + bg(x)) = aD_p f(x) + bD_p g(x).$$

Definition 1.5. A function $F(x)$ is a p-antiderivative of $f(x)$ if $D_p F(x) = f(x)$. It is denoted by

$$F(x) = \int f(x) d_p x.$$

To constructing the p-antiderivative, we define an operator $\hat{M}_p$, by $\hat{M}_p(F(x)) = F(x^p)$. Then we have:

$$\frac{1}{x^p - x}(\hat{M}_p - 1)F(x) = \frac{F(x^p) - F(x)}{x^p - x} = D_p F(x) = f(x).$$

Since $\hat{M}_p^j(F(x)) = F(x^{p^j})$ for $j \in \{0, 1, 2, 3, ...\}$, and also by the geometric series expansion, we formally have

(1.1)
$$F(x) = \frac{1}{1 - \hat{M}_p}((x - x^p) f(x)) = \sum_{j=0}^{\infty} \hat{M}_p^j((x - x^p) f(x)) = \sum_{j=0}^{\infty} (x^{p^j} - x^{p^{j+1}}) f(x^{p^j}).$$

It is worth mentioning that we say that (1.1) is formal because the series does not always converge.

Definition 1.6. The p-integral of $f(x)$ is defined to be the series

$$\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}).$$

We now want to define the definite p-integral. We consider the following three cases. Then, the definite p-integral related to each case is given.

Case 1. Let $1 < a < b$ where $a, b \in R$, $p \in (0,1)$ and function f is defined on $(1, b]$. Notice that for any $j \in \{0, 1, 2, 3, ...\}$, $b^{p^j} \in (1, b]$. We now define the definite p-integral of $f(x)$ on interval $(1, b]$.

Definition 1.7. The p-integral of a function $f(x)$ on the interval $(1, b]$ is defined as

$$\int_1^b f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}), \tag{1.2}$$

and

$$\int_a^b f(x)d_px := \int_1^b f(x)d_px - \int_1^a f(x)d_px.$$

Note 1.8. Geometrically, the integral in (1.2) corresponds to the area of the union of an infinite number of rectangles. On $[1+\epsilon, b]$, where ϵ is a small positive number, the sum consists of finitely many terms, and is a Riemann sum. Therefore, as $p \to 1$, the norm of partition approaches zero, and the sum tends to the Riemann integral on $[1+\epsilon, b]$. Since ϵ is arbitrary, provided that $f(x)$ is continuous in the interval $[1, b]$, thus we have

$$\lim_{p\to 1}\int_1^b f(x)d_px = \int_1^b f(x)dx.$$

Case 2. Let $0 < b < 1$ and $p \in (0,1)$. It should be noted that for any $j \in \{0, 1, 2, 3, ...\}$, $b^{p^j} \in [b, 1)$ and $b^{p^j} < b^{p^{j+1}}$. We will define the definite p-integral of $f(x)$ on interval $[b, 1)$ as follows.

Definition 1.9. The p-integral of a function $f(x)$ on the interval $[b, 1)$ is defined as

$$\int_b^1 f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}).$$

NOTE 1.10. The p-integrals defined above are also denoted by

$$\int_1^b f(x)d_px = I_{p^+}f(b), \qquad \int_b^1 f(x)d_px = I_{p^-}f(b).$$

Case 3. Let $0 < a < b < 1$ and $p \in (0,1)$. Then for any $j \in \{0,1,2,3,...\}$, $b^{p^{-j}} \in (0,b]$ and $b^{p^{-j-1}} < b^{p^{-j}}$. Let us to state the definite p-integral of $f(x)$ on interval $(0,b]$.

DEFINITION 1.11. The p-integral of a function $f(x)$ on the interval $(0,b]$ $(b < 1)$ is defined as

$$I_pf(b) = \int_0^b f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}})$$
$$= \sum_{j=0}^{\infty}(b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}}),$$

and

$$\int_a^b f(x)d_px = \int_0^b f(x)d_px - \int_0^a f(x)d_px.$$

NOTE 1.12. We can also apply Note 1.8 for the p-integrals defined in the cases 2 and 3 on the intervals $[b, 1-\epsilon]$ and $[\epsilon, b]$ respectively, and by it define the Riemann integral.

REMARK 1.13. If $p \in (0,1)$, then for any $j \in \{0,\pm1,\pm2,...\}$, we have $p^{p^j} \in (0,1)$, $p^{p^j} < p^{p^{j+1}}$ and

$$\int_0^1 f(x)d_px = \sum_{j=-\infty}^{\infty}\int_{p^{p^j}}^{p^{p^{j+1}}} f(x)d_px = \sum_{j=-\infty}^{\infty}(p^{p^{j+1}} - p^{p^j})f(p^{p^j}).$$

PROPERTY 1.14. Suppose $0 \le a < 1 < b$. Then by Note 1.8 and Note 1.12, we have

$$\int_a^b f(x)d_px = \int_a^1 f(x)d_px + \int_1^b f(x)d_px.$$

DEFINITION 1.15. The p-integral of higher order of a function f is given by

$$(I_p^0 f)(x) = f(x), \quad (I_p^n f)(x) = I_p(I_p^{n-1}f)(x), \ \ n \in N.$$

LEMMA 1.16. *[17] If $x > 1$ and $p \in (0,1)$, then $D_pI_{p^+}f(x) = f(x)$, and also if function f is continuous at $x = 1$, then we have $I_{p^+}D_pf(x) = f(x) - f(1)$.*

Lemma 1.17. If $x, p \in (0,1)$ *and* $I_p f(x) = \int_0^x f(s) d_p s$, *then* $D_p I_p f(x) = f(x)$, *and also if function* f *is continuous at* $x = 0$, *then we have* $I_p D_p f(x) = f(x) - f(0)$.

An important difference between the definite p-integral and its ordinary counterpart is that if we are integrating a function on an interval like $[2,3]$ or $[\frac{1}{3}, \frac{1}{2}]$, we have to care about its behavior at $x = 1$ or $x = 0$, respectively.

The definite p-integrals defined above are too general for our purpose of studying inequalities. For example, if $f(x) \geq g(x) \geq 0$, it is not necessarily true $\int_a^b f(x) d_p x \geq \int_a^b g(x) d_p x \geq 0$ $(a \neq 0, 1)$. From now on, we will use a special type of the definite p-integral, or in other words we will study the definite p-integrals on interval $[a,b]$, where $a = b^{p^n}$ and $b > 1$, or on interval $[b,a]$, where $a = b^{p^n}$, $b < 1$ and $n \in Z^+$. The following formulae are concluded as follows:

$$\int_a^b f(x) d_p x = \int_{b^{p^n}}^b f(x) d_p x = \sum_{j=0}^{n-1} (b^{p^j} - b^{p^{j+1}}) f(b^{p^j}), \quad \text{if } 1 \leq a < b,$$

and

$$\int_b^a f(x) d_p x = \int_b^{b^{p^n}} f(x) d_p x = \sum_{j=0}^{n-1} (b^{p^{n-j}} - b^{p^{n-j-1}}) f(b^{p^{n-j-1}}), \quad \text{if } 0 < b < a < 1.$$

Obviously, if $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x) d_p x \geq \int_a^b g(x) d_p x$.

Definition 1.18. $f(x)$ is called p-increasing (respectively, p-decreasing) on $[a,b]$ if $f(x^p) \leq f(x)$ (respectively, $f(x^p) \geq f(x)$) for all $x^p < x$, or $f(x^p) \geq f(x)$ (respectively, $f(x^p) \leq f(x)$) for all $x^p > x$, whenever $x \in [a,b]$ and $x^p \in [a,b]$.

Note 1.19. If $f(x)$ is increasing (decreasing), then it is also p-increasing (p-decreasing).

2. Steffensen Inequality

In 1918, Steffensen proved the following theorem [20]:

Theorem 2.1. *Suppose that* f *and* g *are integrable functions defined on* (a,b), f *is decreasing and for each* $x \in (a,b)$, $0 \leq g(x) \leq 1$. *Set*

$\lambda = \int_a^b g(x)dx$. *Then,*

$$\int_{b-\lambda}^{b} f(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^{a+\lambda} f(x)dx.$$

We now establish p-generalization of Theorem 2.1.

Theorem 2.2. (p-*Steffensen inequality*). *Suppose* $0 < p < 1$, $b > 1$ *and* $a = b^{p^n}$, *where* $n \in Z^+$. *Let* $f, g, h : [1, b] \to R$ *be three functions such that* f *is* p-*decreasing and* $0 \le g(x) \le h(x)$ *on* $[a, b]$. *Assume that* k,$l \in \{0, 1, 2, ..., n\}$, *such that*

$$\int_{b^{p^l}}^{b} h(x)d_px \le \int_a^b g(x)d_px \le \int_a^{b^{p^k}} h(x)d_px, \qquad \text{if } f \ge 0, \text{ on } [a,b],$$

and

$$\int_a^{b^{p^k}} h(x)d_px \le \int_a^b g(x)d_px \le \int_{b^{p^l}}^{b} h(x)d_px, \qquad \text{if } f \le 0, \text{ on } [a,b].$$

Then,

$$\int_{b^{p^l}}^{b} f(x)h(x)d_px \le \int_a^b f(x)g(x)d_px \le \int_a^{b^{p^k}} f(x)h(x)d_px. \tag{2.1}$$

Proof. We prove only the left inequality in (2.1) in the case $f \ge 0$. The proofs of the other cases are similar. Since f is p-decreasing and g is nonnegative, we have

$$\begin{aligned}
&\int_a^b f(x)g(x)d_px - \int_{b^{p^l}}^{b} f(x)h(x)d_px \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px + \int_{b^{p^l}}^{b} f(x)g(x)d_px - \int_{b^{p^l}}^{b} f(x)h(x)d_px \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px - \int_{b^{p^l}}^{b} f(x)(h(x) - g(x))d_px
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^{b^{p^l}} f(x)g(x)d_px - \sum_{j=0}^{l-1}(b^{p^j} - b^{p^{j+1}})[f(b^{p^j})(h(b^{p^j}) - g(b^{p^j}))] \\
&\geq \int_a^{b^{p^l}} f(x)g(x)d_px - \sum_{j=0}^{l-1}(b^{p^j} - b^{p^{j+1}})[f(b^{p^l})(h(b^{p^j}) - g(b^{p^j}))] \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px - \sum_{j=0}^{l-1}(b^{p^j} - b^{p^{j+1}})f(b^{p^l})h(b^{p^j}) \\
&\qquad + \sum_{j=0}^{l-1}(b^{p^j} - b^{p^{j+1}})f(b^{p^l})g(b^{p^j}) \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px - f(b^{p^l})\int_{b^{p^l}}^{b} h(x)d_px + f(b^{p^l})\int_{b^{p^l}}^{b} g(x)d_px \\
&\geq \int_a^{b^{p^l}} f(x)g(x)d_px - f(b^{p^l})\int_a^{b} g(x)d_px + f(b^{p^l})\int_{b^{p^l}}^{b} g(x)d_px \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px - f(b^{p^l})[\int_a^{b} g(x)d_px - \int_{b^{p^l}}^{b} g(x)d_px] \\
&= \int_a^{b^{p^l}} f(x)g(x)d_px - f(b^{p^l})\int_a^{b^{p^l}} g(x)d_px \\
&= \int_a^{b^{p^l}} [f(x) - f(b^{p^l})]g(x)d_px \\
&= \sum_{j=0}^{n-l-1}(b^{p^{l+j}} - b^{p^{l+j+1}})[f(b^{p^{l+j}}) - f(b^{p^l})]g(b^{p^{l+j}}) \geq 0.
\end{aligned}
$$

□

REMARK 2.3. Setting $h(x) = 1$ in Theorem 2.2, we obtain the special case of (2.1), that is

$$\int_{b^{p^l}}^{b} f(x)d_px \leq \int_a^{b} f(x)g(x)d_px \leq \int_a^{b^{p^k}} f(x)d_px.$$

The Theorem 2.2 is also true for the case $0 < b < 1$ that we state it as follows.

THEOREM 2.4. *Suppose* $p, b \in (0,1)$ *and* $a = b^{p^n}$, *where* $n \in Z^+$. *Let* $f, g, h : [0,a] \to R$ *be three functions such that* f *is* p*-decreasing and* $0 \le g(x) \le h(x)$ *on* $[b,a]$. *Assume that* $k, l \in \{0,1,2,...n\}$ *such that*

$$\int_{b^{p^l}}^{a} h(x)d_px \le \int_{b}^{a} g(x)d_px \le \int_{b}^{b^{p^k}} h(x)d_px, \qquad \text{if } f \ge 0, \text{ on } [b,a],$$

and

$$\int_{b}^{b^{p^k}} h(x)d_px \le \int_{b}^{a} g(x)d_px \le \int_{b^{p^l}}^{a} h(x)d_px, \qquad \text{if } f \le 0, \text{ on } [b,a].$$

Then,

$$\int_{b^{p^l}}^{a} f(x)h(x)d_px \le \int_{b}^{a} f(x)g(x)d_px \le \int_{b}^{b^{p^k}} f(x)h(x)d_px. \tag{2.2}$$

Proof. The proof is very similar to the proof of Theorem 2.2. We prove only the right inequality in (2.2) in the case $f \ge 0$. In the following calculation, we use of the fact that f is p-decreasing and $g(x)$ is nonnegative.

$$\begin{aligned}
&\int_{b}^{a} f(x)g(x)d_px - \int_{b}^{b^{p^k}} f(x)h(x)d_px \\
&= \int_{b}^{b^{p^k}} f(x)g(x)d_px + \int_{b^{p^k}}^{a} f(x)g(x)d_px - \int_{b}^{b^{p^k}} f(x)h(x)d_px \\
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - \int_{b}^{b^{p^k}} f(x)(h(x)-g(x))d_px \\
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - \sum_{j=0}^{k-1}(b^{p^{k-j}} - b^{p^{k-j-1}})[f(b^{p^{k-j-1}})(h(b^{p^{k-j-1}}) - g(b^{p^{k-j-1}}))] \\
&\le \int_{b^{p^k}}^{a} f(x)g(x)d_px - \sum_{j=0}^{k-1}(b^{p^{k-j}} - b^{p^{k-j-1}})[f(b^{p^k})(h(b^{p^{k-j-1}}) - g(b^{p^{k-j-1}}))] \\
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - \sum_{j=0}^{k-1}(b^{p^{k-j}} - b^{p^{k-j-1}})f(b^{p^k})h(b^{p^{k-j-1}}) \\
&\qquad + \sum_{j=0}^{k-1}(b^{p^{k-j}} - b^{p^{k-j-1}})f(b^{p^k})g(b^{p^{k-j-1}})
\end{aligned}$$

$$\begin{aligned}
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - f(b^{p^k})\int_{b}^{b^{p^k}} h(x)d_px + f(b^{p^k})\int_{b}^{b^{p^k}} g(x)d_px\\
&\leq \int_{b^{p^k}}^{a} f(x)g(x)d_px - f(b^{p^k})\int_{b}^{a} g(x)d_px + f(b^{p^k})\int_{b}^{b^{p^k}} g(x)d_px\\
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - f(b^{p^k})[\int_{b}^{a} g(x)d_px - \int_{b}^{b^{p^k}} g(x)d_px]\\
&= \int_{b^{p^k}}^{a} f(x)g(x)d_px - f(b^{p^k})\int_{b^{p^k}}^{a} g(x)d_px\\
&= \int_{b^{p^k}}^{a} [f(x) - f(b^{p^k})]g(x)d_px \leq 0.
\end{aligned}$$

□

The above result is also true for the case that $b = 0$ and $a \in (0,1)$ is arbitrary as follows.

THEOREM 2.5. *Suppose $p, a \in (0,1)$ and $f, g, h : [0,a] \to R$ be three functions such that f is p-decreasing and $0 \leq g(x) \leq h(x)$ on $[0,a]$. Assume that k,$l \in \{0,1,2...\}$ such that*

$$\int_{a^{p^{-l}}}^{a} h(x)d_px \leq \int_{0}^{a} g(x)d_px \leq \int_{0}^{a^{p^{-k}}} h(x)d_px, \qquad \text{if } f \geq 0, \text{ on } [0,a],$$

and

$$\int_{0}^{a^{p^{-k}}} h(x)d_px \leq \int_{0}^{a} g(x)d_px \leq \int_{a^{p^{-l}}}^{a} h(x)d_px, \qquad \text{if } f \leq 0, \text{ on } [b,a].$$

Then,

$$\int_{a^{p^{-l}}}^{a} f(x)h(x)d_px \leq \int_{0}^{a} f(x)g(x)d_px \leq \int_{0}^{a^{p^{-k}}} f(x)h(x)d_px. \tag{2.3}$$

Proof. The proof is analogous to the one of Theorem 2.4. □

In Theorem 2.2, we used a special type of definite p-integral, namely $a = b^{p^n}$. The lower limit of integral, namely $a \geq 1$, can be arbitrary, but in this case the result holds for some $p \in (0,1)$. We explain this as follows.

THEOREM 2.6. *Let $1 \leq a < b$ and $f, g, h : [1,b] \to R$ be three functions such that f is p-decreasing and $0 \leq g(x) \leq h(x)$ on $[a,b]$. Then,* *there exists $p \in (0,1)$ such that (2.1) hold (k,l hold in the assumptions of Theorem 2.2).*

Proof. Since $1 \leq a < b$, there exists $p \in (0,1)$ and $n \in Z^+$ such that $a = b^{p^n}$. Using Theorem 2.2, we obtain (2.1). □

3. Reverse Inequality

Here we want to state reverse to p-Steffensen inequalities. By the use of identities similar to those in (2.2), the conditions for validity of reverse to p-Steffensen inequalities are given. Let us to present it as follows.

THEOREM 3.1. *Suppose $0 < p < 1$, $b > 1$ and $a = b^{p^n}$, where $n \in Z^+$. Let $f, g, h : [1, b] \to R$ be three functions such that f is p-decreasing on $[a, b]$ and $l \in \{0, 1, 2, ...n\}$ such that $g(x) \leq 0$ for $a \leq x \leq b^{p^l}$ and $g(x) \geq h(x)$ for $b^{p^l} \leq x \leq b$ and also*

$$\int_a^b g(x) d_p x \leq \int_{b^{p^l}}^b h(x) d_p x, \qquad \text{if } f \geq 0, \text{ on } [a, b],$$

and

$$\int_{b^{p^l}}^b h(x) d_p x \leq \int_a^b g(x) d_p x, \qquad \text{if } f \leq 0, \text{ on } [a, b].$$

Then,

$$\int_a^b f(x) g(x) d_p x \leq \int_{b^{p^l}}^b f(x) h(x) d_p x.$$

THEOREM 3.2. *Suppose $0 < p < 1$, $b > 1$ and $a = b^{p^n}$, where $n \in Z^+$. Let $f, g, h : [1, b] \to R$ be three functions such that f is p-decreasing on $[a, b]$ and $k \in \{0, 1, 2, ...n\}$ such that $g(x) \geq h(x)$ for $a \leq x \leq b^{p^k}$ and $g(x) \leq 0$ for $b^{p^k} \leq x \leq b$ and also*

$$\int_a^{b^{p^k}} h(x) d_p x \leq \int_a^b g(x) d_p x, \qquad \text{if } f \geq 0, \text{ on } [a, b],$$

and

$$\int_a^b g(x) d_p x \leq \int_a^{b^{p^k}} h(x) d_p x, \qquad \text{if } f \leq 0, \text{ on } [a, b].$$

Then,

$$\int_a^{b^{p^k}} f(x) h(x) d_p x \leq \int_a^b f(x) g(x) d_p x.$$

Theorem 3.3. *Suppose* $0 < p < 1$, $b > 1$ *and* $a = b^{p^n}$, *where* $n \in Z^+$. *Let* $f, g, h : [1, b] \to R$ *be three functions such that* f *is* p*-increasing and* $0 \le g(x) \le h(x)$ *on* $[a, b]$. *If* $k,l \in \{0, 1, 2, ..., n\}$, *such that*

$$\int_a^{b^{p^k}} h(x)d_px \le \int_a^b g(x)d_px \le \int_{b^{p^l}}^b h(x)d_px, \qquad \text{if } f \ge 0, \text{ on } [a, b],$$

and

$$\int_{b^{p^l}}^b h(x)d_px \le \int_a^b g(x)d_px \le \int_a^{b^{p^k}} h(x)d_px, \qquad \text{if } f \le 0, \text{ on } [a, b].$$

Then,

$$\int_a^{b^{p^k}} f(x)h(x)d_px \le \int_a^b f(x)g(x)d_px \le \int_{b^{p^l}}^b f(x)h(x)d_px.$$

4. Access to p-Steffensen Inequality

The following theorem helps us to achieve to p-Steffensen inequality.

Theorem 4.1. *Suppose* $0 < p < 1$, $b > 1$ *and* $a = b^{p^n}$. *Let* G *be increasing and* $f : [1, A] \to R$ *decreasing* ($A \in R$ *such that* $A > 1$ *and also* a, b, $G(a)$, $G(b) \in [1, A]$). *Assume that there exists* $j \in \{0, 1, 2, ..., n\}$ *such that* $G(a) = G(b)^{p^j}$.

(*i*) *If* $G(x) \ge x$, *then*

$$\int_a^b f(x)D_pG(x)d_px \ge \int_{G(a)}^{G(b)} f(z)d_pz. \tag{4.1}$$

(*ii*) *If* $G(x) \le x$, *then the opposite inequality in (4.1) holds.*

Proof. Set $G(x) = z$. Then, we have

$$\int_a^b f(x)D_pG(x)d_px = \int_a^b f(x)d_pG(x) = \int_{G(a)}^{G(b)} f(G^{-1}(z))d_pz.$$

If $G(z) \ge z$, then $G^{-1}(z) \le z$ and $f(G^{-1}(z)) \ge f(z)$. Hence, we have

$$\int_{G(a)}^{G(b)} f(G^{-1}(z))d_pz \ge \int_{G(a)}^{G(b)} f(z)d_pz.$$

If $G(z) \le z$, the opposite inequality holds. □

NOTE 4.2. If $G(x) \leq x$ and $G(a) = 1$ in Theorem 4.1, then the condition $G(a) = G(b)^{p^j}$ is omitted.

REMARK 4.3. The above result is a generalization of Remark 2.3 by adding a condition of monotony. Consider $G(x) = 1 + \int_1^x g(t)d_pt$ on $[1, b]$, where g is the function from Theorem 2.2 with $h(x) = 1$ (Remark 2.3), including g is increasing. Then, $G(x) \leq x$ and G is increasing. Since $G(1) = 1$, by Note 4.2 the conditions of Theorem 4.1 are satisfied. Now if $k \in \{0, 1, 2, ..., n\}$ such that $1 + \int_1^b g(t)d_pt = b^{p^k}$, from the opposite inequality to inequality (4.1), the second inequality in (2.1) for the case $a = 1$ follows.

Let us notice that the results of Theorem 4.1 also hold for the case $b < 1$ and $a = b^{p^n}$. On the basis of the proof of Theorem 4.1, we can formulate the following results.

THEOREM 4.4. *Suppose $p, b \in (0, 1)$, $a = b^{p^n}(n \in Z^+)$. Consider functions f, G such that G is increasing and $f : [0, A] \to R$ decreasing, where $A \leq 1$. Assume that there exists $j \in \{0, 1, 2, ..., n\}$ such that $G(a) = G(b)^{p^j}(a, b, G(a), G(b) \in [0, A])$.*

(i) If $G(x) \geq x$, then

$$\int_b^a f(x)D_pG(x)d_px \geq \int_{G(b)}^{G(a)} f(z)d_pz. \tag{4.2}$$

(ii) If $G(x) \leq x$, then the opposite inequality in (4.2) holds.

Proof. The proof is analogous to the one of Theorem 4.1. □

The above result is also true for the case that $b = 0$ and $a \in (0, 1)$ is arbitrary as follows.

THEOREM 4.5. *Let $p, a \in (0, 1)$, G is increasing and $f : [0, A] \to R$ decreasing, where $A \leq 1$. Assume that there exists $j \in \{0, 1, 2, ..., n\}$ such that $G(a) = G(0)^{p^j}(0, a, G(0), G(a) \in [0, A])$.*

(i) If $G(x) \geq x$, then

$$\int_0^a f(x)D_pG(x)d_px \geq \int_{G(0)}^{G(a)} f(z)d_pz. \tag{4.3}$$

(ii) If $G(x) \leq x$, then the opposite inequality in (4.3) holds.

Proof. The proof is analogous to the one of Theorem 4.1. □

NOTE 4.6. If $G(0) = 0$ in Theorem 4.5, then the condition $G(a) = G(0)^{p^j}$ is omitted.

REMARK 4.7. The above result is a generalization of Theorem 2.5 by adding a condition of monotony. Consider function $G(x) = \int_0^x g(t)d_pt$ on $[0, 1]$, where g is the function from Theorem 2.5 with $h(x) = 1$, including g is increasing. Then, $G(x) \leq x$ and G is increasing. Since $G(0) = 0$, then by Note 4.6 the conditions of Theorem 4.5 are satisfied. Now if $k \in \{0, 1, 2...\}$ such that $\int_0^a g(t)d_pt = a^{p^{-k}}$, from the opposite inequality to inequality (4.3), the second inequality in (2.3) follows.

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