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The Convolution Sum $\sum_{\mathbf{al+bm=n}} \sigma(\mathbf{l})\sigma(\mathbf{m})$ for $(\mathbf{a},\mathbf{b}) = (\mathbf{1},\mathbf{28}),(\mathbf{4},\mathbf{7}),(\mathbf{1},\mathbf{14}),(\mathbf{2},\mathbf{7}),(\mathbf{1},\mathbf{7})$

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Abstract. We evaluate the convolution sum $W_{a,b}(n) := \sum_{al+bm=n} \sigma(l)\sigma(m)$ for $(a,b) = (1,28),(4,7),(2,7)$ for all positive integers n. We use a modular form approach. We also re-evaluate the known sums $W_{1,14}(n)$ and $W_{1,7}(n)$ with our method. We then use these evaluations to determine the number of representations of n by the octonary quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 7(x_5^2 + x_6^2 + x_7^2 + x_8^2)$. Finally we express the modular forms $\Delta_{4,7}(z)$, $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$ (given in [10, 14]) as linear combinations of eta quotients.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. For $k,n \in \mathbb{N}$ the sum of divisors function $\sigma_k(n)$ is defined by $\sigma_k(n) = \sum_{d|n} d^k$, where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. For $a,b \in \mathbb{N}$ with $a \leq b$ we define the convolution sum $W_{a,b}(n)$ by

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$$W_{a,b}(n) := \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ al+bm=n}} \sigma(l)\sigma(m). \tag{1.1}$$

Set $g = \gcd(a,b)$. Clearly

$$W_{a,b}(n) = \begin{cases} W_{a/g,b/g}(n/g) & \text{if } g \mid n, \\ 0 & \text{if } g \nmid n. \end{cases}$$

Hence we may suppose that $\gcd(a,b) = 1$. The convolution sum $W_{a,b}(n)$ has been evaluated for

$$\begin{aligned}(a,b) = &(1,b) \text{ for } 1 \le b \le 16, 18, 20, 23, 24, 25, 27, 32, 36, \\ &(2,3), (2,5), (2,9), (3,4), (3,5), (3,8), (4,5), (4,9).\end{aligned}$$

See, for example, [2, 3, 4, 5, 7, 10, 13, 14, 17, 18].

In this paper we evaluate the convolution sum $W_{a,b}(n)$ for

$$(a,b) = (1,28), (4,7), (2,7).$$

We use a modular form approach. The sum $W_{1,14}(n)$ has been evaluated by Royer [14], and the sum $W_{1,7}(n)$ has been evaluated by Lemire and Williams [10] and later by Cooper and Toh [4]. We re-evaluate the sums $W_{1,14}(n)$ and $W_{1,7}(n)$ with our method.

For $l, n \in \mathbb{N}$ let $R_l(n)$ denote the number of representations of n by the octonary quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 + l(x_5^2 + x_6^2 + x_7^2 + x_8^2)$, namely

$$\begin{aligned} R_l(n) := \operatorname{card}\{&(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid \\ &n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + l(x_5^2 + x_6^2 + x_7^2 + x_8^2)\}. \end{aligned} \tag{1.2}$$

Explicit formulas for $R_l(n)$ are known for $l = 1, 2, 3, 4, 5, 6, 8$, see, for example, [1, 2, 5, 11, 13]. We use the evaluations of the convolution sums $W_{1,28}(n)$, $W_{4,7}(n)$ and $W_{1,7}(n)$ to determine an explicit formula for $R_7(n)$.

Finally we express the modular forms $\Delta_{4,7}(z)$, $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$ (given in [10, 14]) as linear combinations of eta quotients.

2. Preliminary Results

Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

We write $M_k(\Gamma_0(N))$ to denote the space of modular forms of weight k and level N. It is known (see for example [15, p. 83]) that

$$M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)), \tag{2.1}$$

where $E_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ are the corresponding subspaces of Eisenstein forms and cusp forms of weight k with trivial multiplier system for the modular subgroup $\Gamma_0(N)$.

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{2.2}$$

We set $q := q(z) = e^{2\pi i z}$. Then we can express the Dedekind eta function $\eta(z)$ in (2.2) as

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.3}$$

A product of the form

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z),$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an eta quotient. We define the following nine eta quotients

$$C_1(q) := \frac{\eta^5(z)\eta^5(7z)}{\eta(2z)\eta(14z)}, \tag{2.4}$$

$$C_2(q) := \eta^2(z)\eta^2(2z)\eta^2(7z)\eta^2(14z), \tag{2.5}$$

$$C_3(q) := \frac{\eta^6(z)\eta^6(14z)}{\eta^2(2z)\eta^2(7z)}, \tag{2.6}$$

$$C_4(q) := \frac{\eta^6(2z)\eta^6(7z)}{\eta^2(z)\eta^2(14z)}, \tag{2.7}$$

$$C_5(q) := \eta^2(4z)\eta^4(14z)\eta^2(28z), \tag{2.8}$$

$$C_6(q) := \frac{\eta^6(2z)\eta^6(28z)}{\eta^2(4z)\eta^2(14z)}, \tag{2.9}$$

$$C_7(q) := \frac{\eta^4(2z)\eta^6(28z)}{\eta^2(4z)}, \tag{2.10}$$

$$C_8(q) := \frac{\eta(z)\eta(2z)\eta(7z)\eta^8(28z)}{\eta^3(14z)}, \tag{2.11}$$

$$C_9(q) := \frac{\eta(2z)\eta(4z)\eta^9(28z)}{\eta^3(14z)}, \tag{2.12}$$

and integers $c_r(n)$ ($n \in \mathbb{N}$) for $r \in \{1,2,3,4,5,6,7,8,9\}$ by

$$C_r(q) = \sum_{n=1}^{\infty} c_r(n) q^n. \tag{2.13}$$

We use the following theorem to determine if a given eta quotient $f(z)$ is in $M_k(\Gamma_0(N))$. See [8, Theorem 5.7, p. 99] and [9, Corollary 2.3, p. 37].

Theorem 2.1.(Ligozat) *Let $N \in \mathbb{N}$ and $f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z)$ be an eta quotient. Let $k = \frac{1}{2} \sum_{1 \le \delta | N} r_\delta$ and $s = \prod_{1 \le \delta | N} \delta^{r_\delta}$. Suppose that the following conditions are satisfied:*

(i) $\sum_{1 \le \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{1 \le \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(iii) $\sum_{1 \le \delta | N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \ge 0$ *for each positive divisor d of N,*

(iv) *k is an even integer,*

(v) *s is the square of a rational integer.*

Then $f(z)$ is in $M_k(\Gamma_0(N))$.

(iii)′ *In addition to the above conditions, if the inequality in* (iii) *is strict for each positive divisor d of N, then $f(z)$ is in $S_k(\Gamma_0(N))$.*

We note that we have used MAPLE [12] to find the above eta quotients $C_j(q)$ for $1 \le j \le 9$ in a way that they satisfy Theorem 2.1 for $N = 28$ and $k = 4$.

The Eisenstein series $L(q)$ and $M(q)$ are defined as

$$L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{2.14}$$

$$M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{2.15}$$

respectively. We use Theorem 2.1 and the Eisenstein series $M(q)$ to give a basis for the modular space $M_4(\Gamma_0(28))$ in the following theorem.

Theorem 2.2.

(a) *$\{M(q^t) \mid t = 1,2,4,7,14,28\}$ is a basis for $E_4(\Gamma_0(28))$.*

(b) $\{C_j(q) \mid 1 \le j \le 9\}$ *is a basis for* $S_4(\Gamma_0(28))$.

(c) $\{M(q^t) \mid t = 1, 2, 4, 7, 14, 28\} \cup \{C_j(q) \mid 1 \le j \le 9\}$ *is a basis for* $M_4(\Gamma_0(28))$.

Proof. (a) Appealing to [8, Theorem 3.8, p. 50] or [15, Proposition 6.1], we see that $\dim(E_4(\Gamma_0(28))) = 6$. Then we see from [15, Theorem 5.9] that $\{M(q^t) \mid t = 1, 2, 4, 7, 14, 28\}$ is a basis for $E_4(\Gamma_0(28))$.

(b) It follows from Theorem 2.1 that each $C_j(q)$ is in $S_4(\Gamma_0(28))$ for $1 \le j \le 9$. By [8, Theorem 3.8, p. 50] or [15, Proposition 6.1], we have $\dim(S_4(\Gamma_0(28))) = 9$. One can see that there is no linear relationship among the eta quotients $C_j(q)$ ($1 \le j \le 9$). Thus, $\{C_j(q) \mid 1 \le j \le 9\}$ constitute a basis for $S_4(\Gamma_0(28))$.

(c) The assertion follows from (a), (b) and (2.1). □

We note that $\{C_1(q), C_2(q), C_3(q), C_4(q)\}$ is a basis for $S_4(\Gamma_0(14))$.

We use the Sturm bound $S(N)$ to show the equality of two modular forms in the same modular space. The following theorem gives $S(N)$ for $M_4(\Gamma_0(N))$, see [8, Theorem 3.13 and Proposition 2.11] for a general case.

Theorem 2.3. *Let* $f(z),\ g(z) \in M_4(\Gamma_0(N))$ *with the Fourier series expansions* $f(z) = \sum_{n=0}^{\infty} a_n q^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n q^n$. *The Sturm bound* $S(N)$ *for the modular space* $M_4(\Gamma_0(N))$ *is given by*

$$S(N) = \frac{N}{3} \prod_{p \mid N} \left(1 + 1/p\right),$$

and so if $a_n = b_n$ *for all* $n \le S(N)$ *then* $f(z) = g(z)$.

By Theorem 2.3, the Sturm bounds for the modular spaces $M_4(\Gamma_0(14))$ and $M_4(\Gamma_0(28))$ are

$$S(14) = 8,\ S(28) = 16, \tag{2.16}$$

respectively. Using (2.16) and Theorem 2.2 we prove Theorem 2.4. We then use Theorem 2.4 to determine explicit formulas for our convolution sums in the next section.

Theorem 2.4. *We have*

$$\begin{aligned}
\left(L(q) - 28L(q^{28})\right)^2 = &\frac{118}{125}M(q) - \frac{21}{125}M(q^2) - \frac{112}{125}M(q^4) - \frac{343}{125}M(q^7) \\
&- \frac{1029}{125}M(q^{14}) + \frac{92512}{125}M(q^{28}) - \frac{13452}{25}C_1(q) - \frac{86004}{25}C_2(q) \\
&+ 252C_3(q) + \frac{40188}{25}C_4(q) + \frac{407232}{25}C_5(q) + \frac{68544}{5}C_6(q) \\
&- \frac{52416}{25}C_7(q) + \frac{2327808}{25}C_8(q) + \frac{2731008}{25}C_9(q),
\end{aligned}$$

$$\begin{aligned}
\big(4L(q^4)-7L(q^7)\big)^2 =& -\frac{7}{125}M(q)-\frac{21}{125}M(q^2)+\frac{1888}{125}M(q^4)+\frac{5782}{125}M(q^7)\\
&-\frac{1029}{125}M(q^{14})-\frac{5488}{125}M(q^{28})-\frac{8364}{175}C_1(q)-\frac{5004}{25}C_2(q)\\
&+324C_3(q)+\frac{10716}{175}C_4(q)-\frac{24768}{25}C_5(q)+\frac{28224}{5}C_6(q)\\
&-\frac{138816}{25}C_7(q)+\frac{676608}{25}C_8(q)+\frac{273408}{25}C_9(q),\\
\big(L(q)-14L(q^{14})\big)^2 =& \frac{111}{125}M(q)-\frac{56}{125}M(q^2)-\frac{686}{125}M(q^7)+\frac{21756}{125}M(q^{14})\\
&-\frac{4608}{25}C_2(q)+\frac{672}{25}C_3(q)+\frac{10272}{25}C_4(q),\\
\big(2L(q^2)-7L(q^7)\big)^2 =& -\frac{14}{125}M(q)+\frac{444}{125}M(q^2)+\frac{5439}{125}M(q^7)-\frac{2744}{125}M(q^{14})\\
&-\frac{4608}{25}C_2(q)+\frac{10272}{25}C_3(q)+\frac{672}{25}C_4(q),\\
\big(L(q)-7L(q^7)\big)^2 =& \frac{18}{25}M(q)+\frac{882}{25}M(q^7)+\frac{576}{5}(C_1(q)+4C_2(q)).
\end{aligned}$$

Proof. We prove only the first and fourth equalities as the remaining three can be proven similarly. Let us prove the first equality. By [15, Theorem 5.8] we have $L(q)-28L(q^{28})\in M_2(\Gamma_0(28))$, and so

$$\big(L(q)-28L(q^{28})\big)^2\in M_4(\Gamma_0(28)).$$

By Theorem 2.2(c) there exist coefficients $x_1,x_2,x_4,x_7,x_{14},x_{28},y_1,y_2,\ldots,y_9\in\mathbb{C}$ such that

$$\begin{aligned}
\big(L(q)-28L(q^{28})\big)^2 =& x_1M(q)+x_2M(q^2)+x_4M(q^4)+x_7M(q^7)+x_{14}M(q^{14})\\
&+x_{28}M(q^{28})+\sum_{i=1}^{9}y_iC_i(q). \qquad (2.17)
\end{aligned}$$

Appealing to (2.16), we equate the coefficients of q^n for $0\le n\le 16$ on both sides of (2.17), and have a system of linear equations with 17 equations and 15 unknowns. By using MAPLE we solve this system and find the asserted coefficients.

Let us now prove the fourth equality. We have

$$2L(q^2)-7L(q^7)=L(q)-7L(q^7)-(L(q)-2L(q^2)). \qquad (2.18)$$

By [15, Theorem 5.8], we have

$$L(q)-7L(q^7)\in M_2(\Gamma_0(7)) \text{ and } L(q)-2L(q^2)\in M_2(\Gamma_0(2)). \qquad (2.19)$$

Thus it follows from (2.18) and (2.19) that $2L(q^2)-7L(q^7)\in M_2(\Gamma_0(14))$, and so

$$\big(2L(q^2)-7L(q^7)\big)^2\in M_4(\Gamma_0(14)).$$

As $M_4(\Gamma_0(14)) \subset M_4(\Gamma_0(28))$, we have $\left(2L(q^2) - 7L(q^7)\right)^2 \in M_4(\Gamma_0(28))$. Thus by Theorem 2.2(c) there exist coefficients $x_1, x_2, x_4, x_7, x_{14}, x_{28}, y_1, y_2, \ldots, y_9 \in \mathbb{C}$ such that

$$\begin{aligned}\left(2L(q^2) - 7L(q^7)\right)^2 =& x_1 M(q) + x_2 M(q^2) + x_4 M(q^4) + x_7 M(q^7) + x_{14} M(q^{14}) \\ &+ x_{28} M(q^{28}) + \sum_{i=1}^{9} y_i C_i(q).\end{aligned} \tag{2.20}$$

We equate the coefficients of q^n for $0 \le n \le 16$ on both sides of (2.20) to obtain the asserted coefficients. Alternatively, one can use the fact that $\{C_j(q) \mid 1 \le j \le 4\}$ is a basis for $S_4(\Gamma_0(14))$, and find the asserted cofficients for the formulas of $W_{2,7}(n)$, $W_{1,14}(n)$ and $W_{1,7}(n)$ accordingly. □

3. Evaluating the Convolution Sum $W_{a,b}(n)$

We now present explicit formulas for the convolution sum $W_{a,b}(n)$ for $(a,b) = (1,28),(4,7),(1,14),(2,7),(1,7)$. We make use of Theorem 2.4 and the classical identity

$$L^2(q) = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n, \tag{3.1}$$

see for example [6] and [10, Lemma 3.1].

Theorem 3.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}W_{1,28}(n) =& \frac{1}{2400}\sigma_3(n) + \frac{1}{800}\sigma_3(\frac{n}{2}) + \frac{1}{150}\sigma_3(\frac{n}{4}) + \frac{49}{2400}\sigma_3(\frac{n}{7}) \\ &+ \frac{49}{800}\sigma_3(\frac{n}{14}) + \frac{49}{150}\sigma_3(\frac{n}{28}) + (\frac{1}{24} - \frac{n}{112})\sigma(n) + (\frac{1}{24} - \frac{n}{4})\sigma(\frac{n}{28}) \\ &+ \frac{1121}{67200}c_1(n) + \frac{2389}{22400}c_2(n) - \frac{1}{128}c_3(n) - \frac{3349}{67200}c_4(n) - \frac{101}{200}c_5(n) \\ &- \frac{17}{40}c_6(n) + \frac{13}{200}c_7(n) - \frac{433}{150}c_8(n) - \frac{254}{75}c_9(n), \\ W_{4,7}(n) =& \frac{1}{2400}\sigma_3(n) + \frac{1}{800}\sigma_3(\frac{n}{2}) + \frac{1}{150}\sigma_3(\frac{n}{4}) + \frac{49}{2400}\sigma_3(\frac{n}{7}) \\ &+ \frac{49}{800}\sigma_3(\frac{n}{14}) + \frac{49}{150}\sigma_3(\frac{n}{28}) + (\frac{1}{24} - \frac{n}{28})\sigma(\frac{n}{4}) + (\frac{1}{24} - \frac{n}{16})\sigma(\frac{n}{7}) \\ &+ \frac{697}{470400}c_1(n) + \frac{139}{22400}c_2(n) - \frac{9}{896}c_3(n) - \frac{893}{470400}c_4(n) + \frac{43}{1400}c_5(n) \\ &- \frac{7}{40}c_6(n) + \frac{241}{1400}c_7(n) - \frac{881}{1050}c_8(n) - \frac{178}{525}c_9(n), \\ W_{1,14}(n) =& \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right) + \left(\frac{1}{24} - \frac{n}{56}\right)\sigma(n) \\ &+ \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{14}\right) + \frac{2}{175}c_2(n) - \frac{1}{600}c_3(n) - \frac{107}{4200}c_4(n),\end{aligned}$$

$$\begin{aligned}W_{2,7}(n) =&\frac{1}{600}\sigma_3(n)+\frac{1}{150}\sigma_3\left(\frac{n}{2}\right)+\frac{49}{600}\sigma_3\left(\frac{n}{7}\right)+\frac{49}{150}\sigma_3\left(\frac{n}{14}\right)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma\left(\frac{n}{2}\right)\\&+\left(\frac{1}{24}-\frac{n}{8}\right)\sigma\left(\frac{n}{7}\right)+\frac{2}{175}c_2(n)-\frac{107}{4200}c_3(n)-\frac{1}{600}c_4(n),\\W_{1,7}(n) =&\frac{1}{120}\sigma_3(n)+\frac{49}{120}\sigma_3\left(\frac{n}{7}\right)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{7}\right)\\&-\frac{1}{70}c_1(n)-\frac{2}{35}c_2(n).\end{aligned}$$

Proof. We prove the theorem only for the convolution sum $W_{1,28}(n)$ as the other four sums can be proven similarly. Replacing q by q^{28} in (3.1), we obtain

$$L^2(q^{28}) = 1+\sum_{n=1}^{\infty}\Big(240\sigma_3\left(\frac{n}{28}\right)-\frac{72}{7}n\sigma\left(\frac{n}{28}\right)\Big)q^n. \tag{3.2}$$

We have

$$\begin{aligned}L(q)L(q^{28}) &= \Big(1-24\sum_{n=1}^{\infty}\sigma(n)q^n\Big)\Big(1-24\sum_{n=1}^{\infty}\sigma(n)q^{28n}\Big)\\&= 1-24\sum_{n=1}^{\infty}\sigma(n)q^n-24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{28}\right)q^n+576\sum_{n=1}^{\infty}W_{1,28}(n)q^n. \qquad (3.3)\end{aligned}$$

We obtain from (3.1)-(3.3) that

$$\begin{aligned}\big(L(q)-28L(q^{28})\big)^2 =&L^2(q)+784L^2(q^{28})-56L(q)L(q^{28})\\=&729+\sum_{n=1}^{\infty}\Big(240\sigma_3(n)+188160\sigma_3\left(\frac{n}{28}\right)\\&+32256\Big(\frac{1}{24}-\frac{n}{112}\Big)\sigma(n)+32256\Big(\frac{1}{24}-\frac{n}{4}\Big)\sigma\left(\frac{n}{28}\right) \qquad (3.4)\\&-32256W_{1,28}(n)\Big)q^n.\end{aligned}$$

We equate the coefficients of q^n on the right hand sides of $\big(L(q)-28L(q^{28})\big)^2$ in (3.4) and the first part of Theorem 2.4, and solve for $W_{1,28}(n)$ to obtain the asserted formula. □

Theorem 3.2. *Let $n\in\mathbb{N}$. Then*

$$\begin{aligned}R_7(n) =&8\sigma(n)-32\sigma\Big(\frac{n}{4}\Big)+8\sigma\Big(\frac{n}{7}\Big)-32\sigma\Big(\frac{n}{28}\Big)\\&+64W_{(1,7)}(n)+1024W_{(1,7)}\Big(\frac{n}{4}\Big)-256\Big(W_{(4,7)}(n)+W_{(1,28)}(n)\Big).\end{aligned}$$

Proof. For $n \in \mathbb{N}_0$ let $r_4(n)$ denote the number of representations of n as sum of four squares, namely

$$r_4(n) = \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2\},$$

so that $r_4(0) = 1$. It is a classical result of Jacobi, see for example [16], that

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4 \nmid d}} d = 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big) \text{ for } n \in \mathbb{N}. \tag{3.5}$$

By (1.2) and (3.5) we have

$$\begin{aligned} R_7(n) = & \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+7m=n}} r_4(l)r_4(m) \\ = & r_4(n)r_4(0) + r_4(0)r_4\left(\frac{n}{7}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} r_4(l)r_4(m) \\ = & 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big) + 8\sigma\Big(\frac{n}{7}\Big) - 32\sigma\Big(\frac{n}{28}\Big) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} r_4(l)r_4(m). \end{aligned} \tag{3.6}$$

We need to determine the last sum in (3.6). Using (3.5) we obtain

$$\begin{aligned} \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} r_4(l)r_4(m) = & \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} \Big(8\sigma(l) - 32\sigma\Big(\frac{l}{4}\Big)\Big)\Big(8\sigma(m) - 32\sigma\Big(\frac{m}{4}\Big)\Big) \\ = & 64 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} \sigma(l)\sigma(m) + 1024 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} \sigma\Big(\frac{l}{4}\Big)\sigma\Big(\frac{m}{4}\Big) \\ & - 256 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} \sigma\Big(\frac{l}{4}\Big)\sigma(m) - 256 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+7m=n}} \sigma(l)\sigma\Big(\frac{m}{4}\Big) \\ = & 64W_{1,7}(n) + 1024W_{1,7}(n/4) - 256\Big(W_{4,7}(n) + W_{1,28}(n)\Big). \end{aligned} \tag{3.7}$$

The assertion now follows from (3.6) and (3.7). □

We deduce the following corollary from Theorems 3.1 and 3.2.

Corollary 3.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}R_7(n) =& \frac{8}{25}\sigma_3(n) - \frac{16}{25}\sigma_3\Big(\frac{n}{2}\Big) + \frac{128}{25}\sigma_3\Big(\frac{n}{4}\Big) + \frac{392}{25}\sigma_3\Big(\frac{n}{7}\Big) - \frac{784}{25}\sigma_3\Big(\frac{n}{14}\Big)\\ &+ \frac{6272}{25}\sigma_3\Big(\frac{n}{28}\Big) - \frac{928}{175}c_1(n) - \frac{768}{25}c_2(n) + \frac{32}{5}c_3(n) + \frac{2272}{175}c_4(n)\\ &+ \frac{2304}{25}c_5(n) + \frac{768}{5}c_6(n) - \frac{1152}{25}c_7(n) + \frac{24576}{25}\Big(c_8(n) + c_9(n)\Big).\end{aligned}$$

Proof. We substitute the formulas of $W_{1,28}(n)$, $W_{4,7}(n)$, $W_{1,7}(n)$ and $W_{1,7}(n/4)$ from Theorem 3.1 into the right hand side of $R_7(n)$ in Theorem 3.2 to obtain the formula

$$\begin{aligned}R_7(n) =& \frac{8}{25}\sigma_3(n) - \frac{16}{25}\sigma_3\Big(\frac{n}{2}\Big) + \frac{128}{25}\sigma_3\Big(\frac{n}{4}\Big) + \frac{392}{25}\sigma_3\Big(\frac{n}{7}\Big) - \frac{784}{25}\sigma_3\Big(\frac{n}{14}\Big)\\ &+ \frac{6272}{25}\sigma_3\Big(\frac{n}{28}\Big) - \frac{6816}{1225}c_1(n) - \frac{5696}{175}c_2(n) + \frac{32}{7}c_3(n) + \frac{16224}{1225}c_4(n)\\ &+ \frac{21248}{175}c_5(n) + \frac{768}{5}c_6(n) - \frac{10624}{175}c_7(n) + \frac{166912}{175}\Big(c_8(n) + c_9(n)\Big)\\ &- \frac{512}{35}\Big(c_1(n/4) + 4c_2(n/4)\Big).\end{aligned} \tag{3.8}$$

By Theorem 2.1, one can see that $C_1(q^4)$ and $C_2(q^4)$ are in $S_4(\Gamma_0(56))$, and so we have $C_1(q^4) + 4C_2(q^4) \in S_4(\Gamma_0(56))$. By Theorem 2.2(b), we know that $C_j(q)$ $(1 \le j \le 9)$ are in $S_4(\Gamma_0(28)) \subset S_4(\Gamma_0(56))$. We want to see if there exist coefficients $x_1, x_2, \dots, x_9 \in \mathbb{C}$ such that

$$C_1(q^4) + 4C_2(q^4) = x_1C_1(q) + x_2C_2(q) + \cdots + x_9C_9(q). \tag{3.9}$$

By Theorem 2.3, the Sturm bound for the modular space $M_4(\Gamma_0(56))$ is $S(56) = 32$. By using MAPLE we equate the coefficients of q^n for $1 \le n \le 32$ on both sides of (3.9) and have a system of linear equations with 32 equations and nine unknowns. We then solve this system to obtain the identity

$$\begin{aligned}C_1(q^4) + 4C_2(q^4) =& -\frac{1}{56}C_1(q) - \frac{1}{8}C_2(q) - \frac{1}{8}C_3(q) + \frac{1}{56}C_4(q)\\ &+ 2C_5(q) - C_7(q) - 2C_8(q) - 2C_9(q).\end{aligned} \tag{3.10}$$

We deduce from (3.10) that, for $n \in \mathbb{N}$,

$$\begin{aligned}c_1(n/4) + 4c_2(n/4) =& -\frac{1}{56}c_1(n) - \frac{1}{8}c_2(n) - \frac{1}{8}c_3(n) + \frac{1}{56}c_4(n)\\ &+ 2c_5(n) - c_7(n) - 2c_8(n) - 2c_9(n).\end{aligned} \tag{3.11}$$

The asserted expression for $R_7(n)$ now follows by substituting (3.11) into (3.8). □

4. Expressing $\Delta_{4,7}(z)$, $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$ as Linear Combinations of Eta Quotients

In this section we express the modular forms $\Delta_{4,7}(z)$, $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$ as linear combinations of the eta quotients $C_i(q)$ $(1 \leq i \leq 4)$. We refer the reader to [14, Remark 1.2 and Tables 6 and 7] for the definitions of $\Delta_{4,7}(z)$, $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$.

We first express the cusp form $\Delta_{4,7}(z)$ as a linear combination of two eta quotients. The sum $W_{1,7}(n)$ has been given by Lemire and Williams [10, Theorem 2] as

$$\begin{aligned}W_{1,7}(n) =&\frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) + \left(\frac{1}{24} - \frac{n}{28}\right)\sigma(n)\\ &+ \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{7}\right) - \frac{1}{70}u(n),\end{aligned} \tag{4.1}$$

where $u(n)$ is given by (with our notation)

$$\begin{aligned}\sum_{n=1}^{\infty} u(n)q^n &= \left(\eta^{16}(z)\eta^{8}(7z) + 13\eta^{12}(z)\eta^{12}(7z) + 49\eta^{8}(z)\eta^{16}(7z)\right)^{1/3}\\ &=\Delta_{4,7}(z) \text{ (with the notation in [14, Remark 1.2])}\\ &=\sum_{n=1}^{\infty} \tau_{4,7}(n)q^n.\end{aligned} \tag{4.2}$$

Equating the right hand sides of (4.1) and $W_{1,7}(n)$ in Theorem 3.1, we obtain

$$u(n) = c_1(n) + 4c_2(n) \text{ for } n \in \mathbb{N}, \tag{4.3}$$

where $c_1(n)$ and $c_2(n)$ are given by (2.13). Then, by (4.3), (2.4) and (2.5), we obtain

$$\Delta_{4,7}(z) = \sum_{n=1}^{\infty} \tau_{4,7}(n)q^n = \sum_{n=1}^{\infty} u(n)q^n = C_1(q) + 4C_2(q). \tag{4.4}$$

As for $\Delta_{4,14,1}(z)$ and $\Delta_{4,14,2}(z)$, which are in $S_4(\Gamma_0(14))$ (see [14, p. 236]), we use the values of $\tau_{4,14,1}(n)$ and $\tau_{4,14,2}(n)$ in [14, Tables 6 and 7] and the Sturm bound given in (2.16) to obtain

$$\Delta_{4,14,1}(z) := \sum_{n=1}^{\infty} \tau_{4,14,1}(n)q^n = -C_3(q) + C_4(q), \tag{4.5}$$

$$\Delta_{4,14,2}(z) := \sum_{n=1}^{\infty} \tau_{4,14,2}(n)q^n = -4C_2(q) + C_3(q) + C_4(q). \tag{4.6}$$

The sum $W_{1,14}(n)$ has been given by Royer [14, Theorem 1.7] as

$$W_{1,14}(n) = \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right) + \left(\frac{1}{24} - \frac{n}{56}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{14}\right) - \frac{3}{350}\tau_{4,7}(n) - \frac{6}{175}\tau_{4,7}(n/2) - \frac{1}{84}\tau_{4,14,1}(n) - \frac{1}{200}\tau_{4,14,2}(n), \tag{4.7}$$

where $\tau_{4,7}(n)$ and $\tau_{4,7}(n/2)$ are given by (4.4), and the values of $\tau_{4,14,1}(n)$ and $\tau_{4,14,2}(n)$ are given in (4.5) and (4.6), respectively. Finally, equating the right hand sides of (4.7) and $W_{1,14}(n)$ in Theorem 3.1, we obtain

$$\frac{2}{175}c_2(n) - \frac{1}{600}c_3(n) - \frac{107}{4200}c_4(n) = -\frac{3}{350}\tau_{4,7}(n) - \frac{6}{175}\tau_{4,7}(n/2) - \frac{1}{84}\tau_{4,14,1}(n) - \frac{1}{200}\tau_{4,14,2}(n) \text{ for } n \in \mathbb{N},$$

from which, by (4.4)–(4.6) and (2.13), we obtain the identity

$$4C_1(q^2) + 16C_2(q^2) = -C_1(q) - 3C_2(q) + C_3(q) + C_4(q).$$

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