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Super Theta Vectors and Super Quantum Theta Operators

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Abstract. Theta functions are the sections of line bundles on a complex torus. Noncommutative versions of theta functions have appeared as theta vectors and quantum theta operators. In this paper we describe a super version of theta vectors and quantum theta operators. This is the natural unification of Manin's result on bosonic operators, and the author's previous result on fermionic operators.

1. Introduction

In physics, one considers two related notions: observables and states. In classical theory, the observables are real valued functions on a phase space (position with momentum), and the states are probability measures on the phase space. In quantum theory, the observables are self-adjoint (Hermitian) operators on a Hilbert space, and the pure states are vectors in the Hilbert space with length one, mixed states are a mixture of pure states. Note that pure states correspond to Dirac measures on the phase space and mixed states are correspond to probability measures. Roughly speaking, in both classical and quantum theories, observables contain states.

Theta functions are functions on complex spaces, but more precisely, are sections of line bundles on the complex torus. The Noncommutative torus was introduced in [7]; however, the concept of the noncommutative torus had already been developed in terms of the Heisenberg group and Schrödinger representation, in [6]. Noncommutative tori are used in physics in the toroidal compactification by Connes, Douglas and Schwarz in [2]. Later the concept of theta vectors was introduced by Schwarz in [8]. Finally, Manin studied an operator version of theta functions,

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called quantum theta functions in [4]. Classically, theta functions play the role of observables (and states) or a Hilbert space in the geometric quantization. Theta vectors are the vacuum states which have the minimum energy and from which the other states are constructed, in the Hilbert space and quantum theta functions are observables. For more explanation and interpretation, see [10].

A classical theta function of $z \in \mathbb{C}^n$ is

$$\theta(z, T) = \sum_{l \in \mathbb{Z}^n} e^{\pi i l^t T l + 2\pi i l^t z},$$

where T is a symmetric complex matrix of size n with $\operatorname{Im} T > 0$. This function satisfies

$$\begin{aligned} \theta\,(z + k, T) &= \theta\,(z, T) \\ \theta\,(z + Tk, T) &= e^{-\pi i k^t T k - 2\pi i k^t z}\,\theta\,(z, T) \end{aligned}$$

for all $k \in \mathbb{Z}^n$, and

$$\theta(T^{-1}z, -T^{-1}) = (\det(T/i))^{\frac{1}{2}}\, e^{\pi i z^t T^{-1} z}\theta(z, t).$$

The corresponding theta vectors are defined as $f_T(x) = e^{\pi i x^t T x}$ with the same T which is considered as a vacuum vector in $L^2(\mathbb{R}^n)$. A quantum theta function is an element of a noncommutative algebra $C^\infty(D, \chi)$ which consists of $\sum_{h \in D} a_h e_{D,\chi}(h)$ over a lattice D in $\mathbb{R}^{2n}$ with $a_h \in \mathbb{C}$ satisfying the Schwarz condition, where $e_{D,\chi}(h)$'s are generators satisfying

$$e_{D,\chi}(h) e_{D,\chi}(g) = \chi(h, g) e_{D,\chi}(h + g)$$

and χ is a skew symmetric bilinear operator on D with value in $U(1) = S^1$. This $C^\infty(D, \chi)$ is called a quantum torus or a noncommutative torus.

Two questions were raised by Schwarz in [8]. The first one was of the connection between quantum theta functions and theta vectors, and the second one was of the existence of a quantum analogue of the classical functional equation for thetas. Manin answered both of these questions in [5].

Let $f, g \in L^2(\mathbb{R}^n)$ and π be the Heisenberg group representation of $Heis(\mathbb{R}^{2n})$ on $L^2(\mathbb{R}^n)$, where

$$(\pi_{(t,x,y)} f)(s) = e^{2\pi i (t + s^t y) + \pi i (x^t y)} \cdot f(s + x).$$

Rieffel [7] defined $C^\infty(D, \chi)$ valued inner product on $L^2(\mathbb{R}^n)$ as

$$\langle\!\langle f, g \rangle\!\rangle = \sum_{h \in D} \langle f, \pi_h g \rangle e_{D,\chi}(h).$$

Manin showed the follwing in [5].

Theorem 1.1. *For $f_T(x) = e^{\pi i x^t T x}$, with $T^t = T$, $\mathrm{Im}(T) > 0$,*

$$\langle\langle f_T, f_T \rangle\rangle = \frac{1}{\sqrt{2^n \det \mathrm{Im}\, T}} \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h})}\, e_{D,\chi}(h).$$

Moreover,

$$\Theta_D := \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h})}\, e_{D,\chi}(h)$$

is a quantum theta function in the ring $C^\infty(D, \chi)$ satisfying the following functional equations:

$$\forall\, g \in D,\;\; c_g\, e_{D,\chi}(g) s_g^*(\Theta_D) = \Theta_D,$$

where

$$c_g = e^{-\frac{\pi}{2} H(\underline{g}, \underline{g})},\;\; s_g^*(e_{D,\chi}(h)) = e^{-\pi H(\underline{g}, \underline{h})}\, e_{D,\chi}(h),$$

and

$$H(\underline{g}, \underline{h}) = (Tg_1 + g_2)^t\, T_2^{-1} (Th_1 + h_2)^*$$

with $g = (g_1, g_2)$, $h = (h_1, h_2)$ and $T = T_1 + iT_2$. Here $(g_1, g_2) = (0, g_1, g_2)$ and $(h_1, h_2) = (0, h_1, h_2)$ are in $Heis(\mathbb{R}^{2n})$, so that $g_1, g_2, h_1, h_2 \in \mathbb{R}^n$, and $T_1 = \mathrm{Re}(T)$, $T_2 = \mathrm{Im}(T)$ for $T \in M_{n\times n}(\mathbb{C})$.

Also he showed that

$$\sum_{h \in D} e^{-\pi H(\underline{h}, \underline{h}) - \pi H(\underline{s}, \underline{h})} = \sum_{g \in D^!} e^{-\pi H(\underline{g}, \underline{g}) - \pi H(\underline{s}, \underline{g})}$$

as functions of variable s, where $D^! = \{x \in \mathbb{R}^{2n} \mid 2\psi(x, y) = x_1^t y_2 - y_1^t x_2 \in \mathbb{Z}, \forall\, y \in D\}$.

What the author obtained in [3] on $\mathbb{R}^{0|2m}$, as an analogue of the theorem by Manin is as follows. For a lattice D in odd space $\mathbb{R}^{0|2m}$, $g_R(\eta) = e^{-\pi i \xi^t R \xi}$ with $R^t = -R$, where R_2 is nondegenerate, we have

$$\begin{aligned}\langle\langle g_R(\eta), g_R(\eta) \rangle\rangle &= \sum_{\delta \in D} \langle g_R(\eta), \pi_\delta\, g_R(\eta) \rangle\, e_{D,\chi}(\delta) \\ &= \sum_{\delta \in D} 2^{\frac{m}{2}}\, \mathrm{Pf}(R_2) e^{-\frac{\pi}{2} K(\underline{\delta}, \underline{\delta})} e_{D,\chi}(\delta)\end{aligned}$$

with $R = R_1 + iR_2$, $\delta = (\delta_1, \delta_2)$ and $K(\underline{\delta}, \underline{\epsilon}) = (R\delta_1 + \delta_2)^t R_2^{-1} (R\epsilon_1 + \epsilon_2)^*$, without knowing the meaning of the super theta vector defined. Note that if the dimension is not even, $\mathrm{Pf}(R_2) = 0$ (see the second section.) In this paper we find a general construction from the super theta vector naturally defined.

Let $L^2(\mathbb{R}^{n|m}) = L^2(\mathbb{R}^n) \otimes \Lambda^\bullet(\mathbb{R}^m)$, which is the completion of the Schwarz space $S(\mathbb{R}^n) \otimes \Lambda^\bullet(\mathbb{R}^m)$. Here $\Lambda^\bullet(\mathbb{R}^m)$ is the Grassmann algebra spanned by $\{\eta_1 \wedge \cdots \wedge \eta_l \mid \eta_i \in \mathbb{R}^m, l \le m\}$. For $\mathcal{Z}$ with $\mathcal{Z}^{st} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \mathcal{Z}$ and $\mathcal{Z}_2 = \mathrm{Im}(\mathcal{Z}) > 0$, where $e^{\pi i (s^t,\eta^t)(\mathcal{Z})\binom{s}{\eta}}$ is in $L^2(R^{n|m})$. Here $\mathcal{Z}_2 > 0$ means $\mathcal{Z}_2$ can be expressed as $\mathcal{Z}_2 = W^{st}\begin{pmatrix} I & \\ & J \end{pmatrix} W$ for some nondegenerate W, where $J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$, so that m must be even for $\mathcal{Z}_2$ nondegenerate. We call $F_{\mathcal{Z}}(x,\eta) = e^{\pi i (x^t,\eta^t)\mathcal{Z}\binom{x}{\eta}}$ as a super theta vector generalizing the theta vector $f_T(x) = e^{\pi i x^t T x}$ with $T^t = T$, $\mathrm{Im}(T) > 0$.

As in the bosonic case we define $C^\infty(D,\chi)$ valued inner product on $L^2(\mathbb{R}^{n|m})$ as

$$\langle\langle f, g\rangle\rangle = \sum_{(h,\delta)\in D} \langle f, \pi_{(h,\delta)} g\rangle e_{D,\chi}(h,\delta).$$

Then we get our main theorem.

Theorem 1.2.

(1)

$$\begin{aligned}&\langle\langle F_{\mathcal{Z}}(s,\eta), F_{\mathcal{Z}}(s,\eta)\rangle\rangle\\ &= \frac{1}{\sqrt{\mathrm{sdet}(2\mathcal{Z}_2)}} \sum_{(h,\delta)\in D} e^{-\frac{\pi}{2} H\left(\underline{(h,\delta)},\underline{(h,\delta)}\right)} e_{D,\chi}(h,\delta),\end{aligned}$$

where $H\left(\underline{(s,\eta)},\underline{(t,\epsilon)}\right) = \underline{(s,\eta)}^t \mathcal{Z}_2^{-st}\underline{(t,\epsilon)}^*$, $\underline{(s,\eta)} = \mathcal{Z}\binom{s_1}{\eta_1} + \binom{s_2}{-\eta_2}$, *and* $*$ *is the complex conjugation.*

(2) *Let*

$$\Theta_D = \sum_{(h,\delta)\in D} e^{-\frac{\pi}{2} H\left(\underline{(h,\delta)},\underline{(h,\delta)}\right)} e_{D,\chi}(h,\delta).$$

Then $\forall (g,\mu)\in D$,

$$c_{(g,\mu)} e_{D,\chi}(g,\mu)(s,\eta)^*_{(g,\mu)}\Theta_D = \Theta_D,$$

where

$$\begin{aligned}c_{(g,\mu)} &= e^{-\frac{\pi}{2} H\left(\underline{(g,\mu)},\underline{(g,\mu)}\right)}\\ (s,\eta)^*_{g,\mu} e_{D,\chi}(h,\delta) &= e^{-\pi H\left(\underline{(g,\mu)},\underline{(h,\delta)}\right)} e_{D,\chi}(h,\delta).\end{aligned}$$

(3)

$$\begin{aligned}&\sum_{(h,\delta)\in D} e^{-\pi H\left(\underline{(h,\delta)},\underline{(h,\delta)}\right)-\pi H\left(\underline{(s,\eta)},\underline{(h,\delta)}\right)}\\ &\qquad = \sum_{(g,\mu)\in D^!} e^{-\pi H\left(\underline{(g,\mu)},\underline{(g,\mu)}\right)-\pi H\left(\underline{(s,\eta)},\underline{(g,\mu)}\right)},\end{aligned}$$

as functions of variables (s,η).

There are several different approaches to define super theta functions [1, 9]. However their approaches and the concepts are different from ours in the sense that they deal with classical super theta functions while our quantum super theta functions are operators (observables) coming from super theta vectors which are vacuum states.

The contents of this paper are as follows. In section 2, we describe the necessary materials on superspaces for this paper. In section 3, we discuss the super Heisenberg groups and the super theta vector which is a vacuum state generalizing the classical theta vector. In section 4, we construct the super quantum theta functions on a superspace $\mathbb{R}^{n|2m}$ coming from the super theta vector constructed in the previous section, which generalizes Manin's result on even spaces and our previous result on odd spaces.

2. Superspaces

In this section, we explain necessary materials for later sections including superspace, superlinear algebra, integration on superspace.

On $\mathbb{R}^n$, also denoted by $\mathbb{R}^{n|0}$, we define polynomials

$$\mathbb{R}[x_1,\ldots,x_n] = \mathbb{R}\langle x_1,\ldots,x_n\rangle \big/ (x_ix_j - x_jx_i),\ 1 \le i,j \le n$$

as the quotient of the free algebra generated by $x_1,\ldots,x_n$ by the ideals generated by $(x_ix_j - x_jx_i)$ for $1 \le i,j \le n$. It is a commutative algebra. On odd $\mathbb{R}^m$, denoted by $\mathbb{R}^{0|m}$, we can similarly define the algebra as the quotient of $\mathbb{R}\langle \xi_1,\ldots,\xi_m\rangle$ by the ideal generated by $(\xi_\alpha\xi_\beta + \xi_\beta\xi_\alpha)$ for $1 \le \alpha,\beta \le m$. It is an anticommutative algebra, called a Grassmann algebra. Combining these two concepts, we can define $\mathbb{R}^{n|m}$, where the polynomial functions are $\mathbb{R}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_m] = \mathbb{R}\langle x_1,\ldots,x_n,\xi_1,\ldots,\xi_m\rangle \big/ I$, where I is the ideal generated by $(x_ix_j - x_jx_i)$, $(\xi_\alpha\xi_\beta + \xi_\beta\xi_\alpha)$, $(x_i\xi_\alpha - \xi_\alpha x_i)$ for $1 \le i,j \le n$, $1 \le \alpha,\beta \le m$, which is supercommutative, in the sense that $x_ix_j = x_jx_i$, $\xi_\alpha\xi_\beta = -\xi_\beta\xi_\alpha$, $x_i\xi_\alpha = \xi_\alpha x_i$.

Let $A = \{a_{i,j}\}$ be a $2m \times 2m$ skew-symmetric matrix. The Pfaffian of A is defined by the equation

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1),\sigma(2i)}$$

where S_{2m} is the symmetric group of the dimension $(2m)!$ and $\mathrm{sgn}(\sigma)$ is the signature of σ. Then $\mathrm{Pf}(A)^2 = \det(A)$. The Pfaffian of a $m \times m$ skew-symmetric matrix for m odd is defined to be zero, as the determinant of an odd skew-symmetric matrix is zero.

For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the supermatrix representing a linear map from $\mathbb{R}^{n|m}$ to $\mathbb{R}^{n|m}$, where A of size $n \times n$ and D of size $m \times m$ have even entries and B of size $n \times m$ and C of size $m \times n$ have odd entries, $\mathrm{sdet}(X)$ is defined by $\det(A - BD^{-1}C)\det(D)^{-1}$ or, equivalently, by $\det(A)\det(D - CA^{-1}B)^{-1}$, where A and D

are invertible. This is the generalization of the determinant of even matrices and the Pfaffian of odd matrices.

For any supermatrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we define the supertrace

$$X^{st} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}.$$

Here we introduce how to integrate on odd space called Berezin integral. Let Λ^m be the exterior algebra of polynomials in anticommuting elements $\theta_1, \dots, \theta_m$ over the field of complex numbers. (The ordering of the generators $\theta_1, \dots, \theta_m$ is fixed and defines the orientation of the exterior algebra.) The Berezin integral on Λ^m is the linear functional $\int_{\Lambda^m} \cdot d\theta$ with the following properties:

$$\int_{\Lambda^m} \theta_m \cdots \theta_1 \, d\theta = 1,$$
$$\int_{\Lambda^m} \frac{\partial f}{\partial \theta_i} \, d\theta = 0, \;\; i = 1, \dots, m$$

for any $f \in \Lambda^m$, where $\partial / \partial \theta_i$ means the left or the right partial derivative. These properties define the integral uniquely. The formula

$$\int_{\Lambda^m} f(\theta) \, d\theta = \int_{\Lambda^1} \left(\cdots \int_{\Lambda^1} \left(\int_{\Lambda^1} f(\theta) \, d\theta_1 \right) d\theta_2 \cdots \right) d\theta_n$$

expresses the Fubini law.

3. Super Theta Vectors

In this section, we want to define a super theta vector which is a vacuum state. First we want to define the super Heisenberg group $sHeis(\mathbb{R}^{2n|2m}, \psi)$ as follows, generalizing the Heisenberg group $Heis(\mathbb{R}^{2n}, \psi)$. For $t, t' \in \mathbb{R}$, and $(x, \alpha), (y, \beta), (x', \alpha'), (y', \beta') \in \mathbb{R}^{n|m}$, we define the multiplication of (t, x, y, α, β), $(t', x', y', \alpha', \beta') \in \mathbb{R} \times \mathbb{R}^{2n|2m}$ by

$$\begin{aligned}&(t, x, y, \alpha, \beta) \cdot (t', x', y', \alpha', \beta') \\ &\quad = (t + t' + \psi(x, y, \alpha, \beta;\; x', y', \alpha', \beta'), x + x', y + y', \alpha + \alpha', \beta + \beta'),\end{aligned}$$

where $\psi : \mathbb{R}^{2n|2m} \times \mathbb{R}^{2n|2m} \to \mathbb{R}$, satisfies the cocycle condition

$$\begin{aligned}&\psi(x, y, \alpha, \beta;\; x', y', \alpha', \beta') + \psi(x + x', y + y', \alpha + \alpha', \beta + \beta';\; x'', y'', \alpha'', \beta'') \\ &= \psi(x, y, \alpha, \beta;\; x' + x'', y' + y'', \alpha' + \alpha'', \beta' + \beta'') + \psi(x', y', \alpha', \beta';\; x'', y'', \alpha'', \beta''),\end{aligned}$$

a necessary and sufficient for the associative multiplication. Then there is a central extension

$$0 \to \mathbb{R} \xrightarrow{i} sHeis(\mathbb{R}^{2n|2m}, \psi) \xrightarrow{j} \mathbb{R}^{2n|2m} \to 0,$$

which is an exact sequence, with the inclusion $i(t) = (t, 0)$, the projection $j(t, z) = z$, for $z \in \mathbb{R}^{2n|2m}$, where $i(\mathbb{R})$ is the center in $sHeis(\mathbb{R}^{2n|2m}, \psi)$. As in the bosonic case, we can introduce the unitary representation of the super Heisenberg group.

Let

$$\psi(x, y, \alpha, \beta;\ x', y', \alpha', \beta') = \frac{1}{2}(x^t y' - y^t x' - \alpha^t \beta' - \beta^t \alpha').$$

Then ψ satisfies the cocycle condition. We define

$$(\pi_{(t,x,y,\alpha,\beta)} f)(s, \eta) = e^{2\pi i(t + s^t y - \eta^t \beta) + \pi i(x^t y - \alpha^t \beta)} \cdot f(s + x, \eta + \alpha).$$

Then

$$\begin{aligned}&\pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)} \pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)} \\ &\qquad = e^{\pi i(x_1^t y_2 - y_1^t x_2 - \alpha_1^t \beta_2 - \beta_1^t \alpha_2)} \pi_{(t_1+t_2, x_1+x_2, y_1+y_2, \alpha_1+\alpha_2, \beta_1+\beta_2)},\end{aligned}$$

so that

$$\begin{aligned}&\pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)} \pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)} \\ &\qquad = e^{2\pi i(x_1^t y_2 - y_1^t x_2 - \alpha_1^t \beta_2 - \beta_1^t \alpha_2)} \pi_{(t_2,x_2,y_2,\alpha_2,\beta_2)} \pi_{(t_1,x_1,y_1,\alpha_1,\beta_1)}.\end{aligned}$$

Let D be a lattice in $\mathbb{R}^{2n|2m}$. Let ψ be a $\mathbb{R}$ valued bililnear form on D. We define $C^\infty(D, \chi)$ of infinite series

$$\sum_{(h,\delta) \in D} a_{(h,\delta)} e_{D,\chi}(h, \delta),$$

where

$$e_{D,\chi}(g, \mu) e_{D,\chi}(h, \delta) = \chi((g, \mu), (h, \delta)) e_{D,\chi}(g + h, \mu + \delta),$$

with

$$\chi((g, \mu), (h, \delta)) = e^{2\pi i \psi(g,\mu;\ h,\delta)}.$$

This $C^\infty(D, \chi)$ is a super quantum torus, generalizing the notion of a quantum torus by Rieffel.

We choose $\mathcal{Z}$ such that $\mathcal{Z}^{st} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \mathcal{Z}$ and $\mathcal{Z}_2 > 0$. We define $\mathcal{H}^2_\phi(\mathbb{C}^{n|m}, \mathcal{Z})$ as the space of super holomorphic functions $F\left(\underline{(s, \eta)}\right)$ on $\mathbb{C}^{n|m}$ such that

$$\|F\|^2 = \int_{\mathbb{C}^{n|m}} \left| F\left(\underline{(s, \eta)}\right) \right|^2 e^{-\pi H\left(\underline{(s,\eta)}, \underline{(s,\eta)}\right)}\, d\underline{(s, \eta)} < \infty,$$

where $H\left(\underline{(s, \eta)}, \underline{(t, \epsilon)}\right) = \underline{(s, \eta)}^t \mathcal{Z}_2^{-st} \underline{(t, \epsilon)}^*$, $\underline{(s, \eta)} = \mathcal{Z} \left(\begin{smallmatrix} s_1 \\ \eta_1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} s_2 \\ -\eta_2 \end{smallmatrix} \right)$, and $*$ is the complex conjugation. Here super holomorphic on $\mathbb{C}^{n|m}$ means holomorphic on $\mathbb{C}^n$ and integration on $\mathbb{C}^{n|m}$ is the Berezin integral exaplained in the second section. We define the Heisenberg group representation as

$$(U_{\lambda,(h,\delta)} F)\left(\underline{(s, \eta)}\right) = \lambda^{-1}\, e^{-\pi H\left(\underline{(s,\eta)}, \underline{(h,\delta)}\right) - \frac{\pi}{2} H\left(\underline{(h,\delta)}, \underline{(h,\delta)}\right)} F\left(\underline{(s, \eta)} + \underline{(h, \delta)}\right).$$

Now we determine the representation of the Heisenberg algebra associated to the above Heisenberg group representation. Let A_i, A_μ, B_i, B_μ, C denote the basis of the Heisenberg Lie algebra $1 \le i \le n$, $1 \le \mu \le m$, such that

$$\begin{gathered}
\exp\left(\sum s_i^{(1)} A_i\right) = \left(1, \begin{pmatrix} s_i^{(1)} \\ 0 \end{pmatrix}\right) \\
\exp\left(\sum \eta_\mu^{(1)} A_\mu\right) = \left(1, \begin{pmatrix} \eta_\mu^{(1)} \\ 0 \end{pmatrix}\right) \\
\exp\left(\sum s_i^{(2)} B_i\right) = \left(1, \begin{pmatrix} 0 \\ s_i^{(2)} \end{pmatrix}\right) \\
\exp\left(\sum \eta_\mu^{(2)} B_\mu\right) = \left(1, \begin{pmatrix} 0 \\ \eta_\mu^{(2)} \end{pmatrix}\right) \\
\exp(tC) = (e^{2\pi i t}, 0).
\end{gathered}$$

In realization, we have

$$\begin{aligned}
\delta U_{A_i}(F)(s,\eta) &= \frac{\partial F}{\partial s_i} \\
\delta U_{A_\mu}(F)(s,\eta) &= \frac{\partial F}{\partial \eta^\mu} \\
\delta U_{B_i}(F)(s,\eta) &= 2\pi i s_i F(s,\eta) \\
\delta U_{B_\mu}(F)(s,\eta) &= 2\pi i \eta_\mu F(s,\eta),
\end{aligned}$$

so that

$$\begin{aligned}
\delta(U_A)F\left(\underline{(s,\eta)}\right) &= -\pi\overline{\mathcal{Z}}\mathcal{Z}_2^{-st}\left(\underline{(s,\eta)}\right)F\left(\underline{(s,\eta)}\right) + \mathcal{Z}\frac{\partial F}{\partial \underline{(s,\eta)}} \\
\delta(U_B)F\left(\underline{(s,\eta)}\right) &= -\pi\mathcal{Z}_2^{-st}\left(\underline{(s,\eta)}\right)F\left(\underline{(s,\eta)}\right) + \frac{\partial F}{\partial \underline{(s,\eta)}}.
\end{aligned}$$

This implies that $W_{\mathcal{Z}}$ defined as the span of $\{(\delta U_{A_i}, \delta U_{A_\mu}) - \mathcal{Z}(\delta U_{B_i}, \delta U_{B_\mu})\}$ is equal to the span of $\{F \to \underline{(s,\eta)}F\}$, $W_{\overline{\mathcal{Z}}}$ defined as the span of $\{(\delta U_{A_i}, \delta U_{A_\mu}) - \overline{\mathcal{Z}}(\delta U_{B_i}, U_{B_\mu})\}$ is equal to the span of $\{F \to \frac{\partial F}{\partial \underline{(s,\eta)}}\}$.

In the Heisenberg representation $\mathcal{H}$ of $sHeis(\mathbb{R}^{2n|2m}, \psi)$, there is an element $F_{\mathcal{Z}}$, unique up to scalar, such that $\delta U_X(F_{\mathcal{Z}})$ is defined and equal to 0 for all $X \in W_{\overline{\mathcal{Z}}}$. In $\mathcal{H}_\phi^2(\mathbb{C}^{n|m})$, there is a unique $F_{\mathcal{Z}}$ killed by $W_{\overline{\mathcal{Z}}}$ and it is $F_{\mathcal{Z}} = 1$, the vacuum state. Hence $\mathcal{H}_\phi^2$ is irreducible and in the conjugate linear isomorphism with $L^2(\mathbb{R}^{n|m})$, 1 corresponds to

$$F_{\mathcal{Z}}(s,\eta) = e^{\pi i (s^t,\eta^t)\mathcal{Z}\binom{s}{\eta}},$$

where $\mathcal{Z} = \begin{pmatrix} T & \Delta \\ \nabla & S \end{pmatrix}$ with $T^t = T$, $\Delta^t = -\nabla$, $S^t = -S$ and $\mathcal{Z}_2 > 0$, with the above condition.

Definition 3.1.(Super Theta Vector) We define this $F_{\mathcal{Z}}(s,\eta)$ as the super theta vector.

4. Super theta operator

In this section, we generalize Manin's result on quantum theta function for $\mathbb{R}^{n|0}$ and our result on quantum theta function for $\mathbb{R}^{0|2m}$. We construct the super quantum theta operators for a superspace $\mathbb{R}^{n|2m}$ coming from the super theta vector constructed in the previous section.

Definition 4.1.(Super Quantum Theta Operator) Let $F_{\mathcal{Z}}(x,\eta) = e^{\pi i(x^t,\eta^t)\mathcal{Z}\binom{x}{\eta}}$, the super theta vector. As in the bosonic case we define $C^\infty(D,\chi)$ valued inner product on $L^2(\mathbb{R}^{n|2m})$ as

$$\langle\langle f,g\rangle\rangle = \sum_{(h,\delta)\in D} \langle f, \pi_{(h,\delta)}g\rangle e_{D,\chi}(h,\delta).$$

We define $\langle\langle F_{\mathcal{Z}}(x,\eta), F_{\mathcal{Z}}(x,\eta)\rangle\rangle$ as the super quantum theta operator.

The following is the proof of our main theorem (Theorem 1.2).

Proof of Theorem 1.2.

The proof of (1) is obtained by the computation of

$$\begin{aligned}
&\langle F_{\mathcal{Z}}(x,\eta), \pi_{h,\delta}F_{\mathcal{Z}}(x,\eta)\rangle \\
&\quad = \int F_{\mathcal{Z}}(x,\eta)\overline{\pi_{h,\delta}F_{\mathcal{Z}}(x,\eta)}\, dxd\eta \\
&\quad = \int e^{\pi i(x^t,\eta^t)\mathcal{Z}\binom{x}{\eta}} e^{-\pi i((x+h_1)^t,(\eta+\delta_1)^t)\overline{\mathcal{Z}}\binom{x+h_1}{\eta+\delta_1}} \\
&\qquad\quad \cdot e^{-2\pi i(x^t h_2 - \eta^t\delta_2)} e^{-\pi i(h_1^t h_2 - \delta_1^t\delta_2)}\, dxd\eta.
\end{aligned}$$

If we compute the exponent,

$$\begin{aligned}
&\pi i(x^t,\eta^t)\mathcal{Z}\begin{pmatrix}x\\ \eta\end{pmatrix} - \pi i(x^t,\eta^t)\overline{\mathcal{Z}}\begin{pmatrix}x\\ \eta\end{pmatrix} \\
&\qquad - 2\pi i(x^t,\eta^t)\left(\overline{\mathcal{Z}}\begin{pmatrix}h_1\\ \delta_1\end{pmatrix} + \begin{pmatrix}h_2\\ -\delta_2\end{pmatrix}\right) - \pi i(h_1^t,\delta_1^t)\left(\overline{\mathcal{Z}}\begin{pmatrix}h_1\\ \delta_1\end{pmatrix} + \begin{pmatrix}h_2\\ -\delta_2\end{pmatrix}\right) \\
&= -2\pi(x^t,\eta^t)\mathcal{Z}_2\begin{pmatrix}x\\ \eta\end{pmatrix} \\
&\qquad - 2\pi i(x^t,\eta^t)\left(\overline{\mathcal{Z}}\begin{pmatrix}h_1\\ \delta_1\end{pmatrix} + \begin{pmatrix}h_2\\ -\delta_2\end{pmatrix}\right) - \pi i(h_1^t,\delta_1^t)\left(\overline{\mathcal{Z}}\begin{pmatrix}h_1\\ \delta_1\end{pmatrix} + \begin{pmatrix}h_2\\ -\delta_2\end{pmatrix}\right) \\
&= -2\pi\left((x^t,\eta^t) + \frac{i}{2}\underline{(h,\delta)}^{*t}\mathcal{Z}_2^{-st}\right)\mathcal{Z}_2\left(\begin{pmatrix}x\\ \eta\end{pmatrix} + \frac{i}{2}\mathcal{Z}_2^{-1}\underline{(h,\delta)}^*\right) \\
&\qquad - \frac{\pi}{2}\underline{(h,\delta)}^{*t}\mathcal{Z}_2^{-st}\mathcal{Z}_2\mathcal{Z}_2^{-1}\underline{(h,\delta)}^* - \pi i(h_1^t,\delta_1^t)\underline{(h,\delta)}^*
\end{aligned}$$

$$= -2\pi \left((x^t, \eta^t) + \frac{i}{2} \underline{(h,\delta)}^{*t} \mathcal{Z}_2^{-st} \right) \mathcal{Z}_2 \left(\begin{pmatrix} x \\ \eta \end{pmatrix} + \frac{i}{2} \mathcal{Z}_2^{-1} \underline{(h,\delta)}^* \right)$$
$$- \frac{\pi}{2} \underline{(h,\delta)}^t \mathcal{Z}_2^{-st} \left(\underline{(h,\delta)}^* \right).$$

Here the first term gives $\dfrac{1}{\sqrt{\operatorname{sdet} 2\mathcal{Z}_2}}$ after integration.

The proof of (2) comes from the following observation.

$$\begin{aligned}
&\operatorname{Im} H \left(\underline{(g,\mu)}, \underline{(h,\delta)} \right) \\
&= (g_1^t, \mu_1^t) \left(\mathcal{Z}_1 \begin{pmatrix} h_1 \\ \delta_1 \end{pmatrix} + \begin{pmatrix} h_2 \\ -\delta_2 \end{pmatrix} \right) + \left((g_2^t, -\mu_2^t) + (g_1^t, \mu_1^t) \mathcal{Z}_1^{st} \right) \mathcal{Z}_2^{-st} (-\mathcal{Z}_2) \begin{pmatrix} h_1 \\ \delta_1 \end{pmatrix} \\
&= g_1^t h_2 - g_2^t h_1 - \mu_1^t \delta_2 - \mu_2^t \delta_1 \\
&= 2\Psi((g,\mu),(h,\delta)),
\end{aligned}$$

by using $\mathcal{Z}_i^{st} = \mathcal{Z}_i \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

$$\operatorname{Re} H \left(\underline{(g,\mu)}, \underline{(h,\delta)} \right) = \frac{1}{2} \left(H \left(\underline{(g,\mu)}, \underline{(h,\delta)} \right) + H \left(\underline{(h,\delta)}, \underline{(g,\mu)} \right) \right),$$

with the property $H \left(\underline{(g,\mu)}, \underline{(h,\delta)} \right) = \left(H \left(\underline{(h,\delta)}, \underline{(g,\mu)} \right) \right)^*$.

The proof of (3) is a generalization of Manin, by using the Fourier transform of

$$F_{s,\eta}(h,\delta) = e^{-\pi H \left(\underline{(h,\delta)}, \underline{(h,\delta)} \right) - \pi H \left(\underline{(s,\eta)}, \underline{(h,\delta)} \right)}$$

and checking that

$$\widehat{F}_{s,\eta}(g,\mu) = e^{-\pi H \left(\underline{(g,\mu)}, \underline{(g,\mu)} \right) - \pi H \left(\underline{(s,\eta)}, \underline{(g,\mu)} \right)}$$

and using the Poisson summation formula.

$$\widehat{F}_{s,\eta}(g,\mu) = \int e^{-\pi H \left(\underline{(h,\delta)}, \underline{(h,\delta)} \right) - \pi H \left(\underline{(s,\eta)}, \underline{(h,\delta)} \right) - 2\pi i \widetilde{A}((g,\mu),(h,\delta))} \, d\underline{(h,\delta)},$$

where

$$\widetilde{A}((g,\mu),(h,\delta)) = (g_1, \mu_1, g_2, -\mu_2) \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & 1 & & \end{array} \right) \begin{pmatrix} h_1 \\ \delta_1 \\ h_2 \\ -\delta_2 \end{pmatrix}.$$

In fact, $\widetilde{A} = 2\Psi$.

$$
\begin{aligned}
&H\left(\underline{(s,\eta)}, \underline{(h,\delta)}\right) + H\left(\underline{(h,\delta)}, \underline{(h,\delta)}\right) + 2i\widetilde{A}((g,\mu),(h,\delta)) \\
&= (s_1^t, \eta_1^t, s_2^t, -\eta_2^t) B \begin{pmatrix} h_1 \\ \delta_1 \\ h_2 \\ -\delta_2 \end{pmatrix} + i(s_1^t, \eta_1^t, s_2^t, -\eta_2^t) A \begin{pmatrix} h_1 \\ \delta_1 \\ h_2 \\ -\delta_2 \end{pmatrix} \\
&\quad + 2i(g_1^t, \mu_1^t, g_2^t, -\mu_2^t) A \begin{pmatrix} h_1 \\ \delta_1 \\ h_2 \\ -\delta_2 \end{pmatrix} + (h_1^t, \delta_1^t, h_2^t, -\delta_2^t) B \begin{pmatrix} h_1 \\ \delta_1 \\ h_2 \\ -\delta_2 \end{pmatrix},
\end{aligned}
$$

where

$$
A = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right) \text{ and } B = \left(\begin{array}{c|c} \mathcal{Z}_1^{st}\mathcal{Z}_2^{-st}\mathcal{Z}_1 + \mathcal{Z}_2 & \mathcal{Z}_1^{st}\mathcal{Z}_2^{-st} \\ \hline \mathcal{Z}_2^{-st}\mathcal{Z}_1 & \mathcal{Z}_2^{-st} \end{array}\right).
$$

Then

$$
B^{-1} = \left(\begin{array}{c|c} \mathcal{Z}_2^{-1} & -\mathcal{Z}_2^{-1}\mathcal{Z}_1^{st} \\ \hline -\mathcal{Z}_1\mathcal{Z}_2^{-1} & \mathcal{Z}_2^{st} + \mathcal{Z}_1\mathcal{Z}_2^{-1}\mathcal{Z}_1^{st} \end{array}\right)
$$

and

$$
B^{-st} = \left(\begin{array}{c|c} \mathcal{Z}_2^{-st} & -\mathcal{Z}_2^{-st}\mathcal{Z}_1^{st} \\ \hline -\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\mathcal{Z}_1\mathcal{Z}_2^{-1} & \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\mathcal{Z}_2\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\mathcal{Z}_1\mathcal{Z}_2^{-1}\mathcal{Z}_1\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) \end{array}\right).
$$

Let

$$
\begin{pmatrix} a_1 \\ \alpha_1 \\ a_2 \\ \alpha_2 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} s_1 \\ \eta_1 \\ s_2 \\ -\eta_2 \end{pmatrix} + iB^{-1}A\left[\begin{pmatrix} s_1 \\ \eta_1 \\ s_2 \\ -\eta_2 \end{pmatrix} + 2\begin{pmatrix} g_1 \\ \mu_1 \\ g_2 \\ -\mu_2 \end{pmatrix}\right]\right].
$$

Then we can show that

$$
\begin{aligned}
&H\left(\underline{(s,\eta)}, \underline{(h,\delta)}\right) + H\left(\underline{(h,\delta)}, \underline{(h,\delta)}\right) + 2i\widetilde{A}((g,\mu),(h,\delta)) \\
&= (h_1^t + a_1^t, \delta_1^t + \alpha_1^t, h_2^t + a_2^t, -(\delta_2^t + \alpha_2^t)) B \begin{pmatrix} h_1 + a_1 \\ \delta_1 + \alpha_1 \\ h_2 + a_2 \\ -(\delta_2 + \alpha_2) \end{pmatrix} \\
&\quad - (a_1^t, \alpha_1^t, a_2^t, -\alpha_2^t) B \begin{pmatrix} a_1 \\ \alpha_1 \\ a_2 \\ -\alpha_2 \end{pmatrix}.
\end{aligned}
$$

By using $-A^{st}B^{-st}B = A$ and $A^{st}B^{-st}A = B$, we have

$$-(a_1^t, \alpha_1^t, a_2^t, -\alpha_2^t)B \begin{pmatrix} a_1 \\ \alpha_1 \\ a_2 \\ -\alpha_2 \end{pmatrix} = H\left(\underline{(g,\mu)}, \underline{(g,\mu)}\right) + H\left(\underline{(s,\eta)}, \underline{(g,\mu)}\right).$$

For

$$\begin{aligned} B &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} I & B_{12}B_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} B_{11} - B_{12}B_{22}^{-1}B_{21} & 0 \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix}, \end{aligned}$$

$$\operatorname{sdet} B = \operatorname{sdet} \begin{pmatrix} B_{11} - B_{12}B_{22}^{-1}B_{21} & 0 \\ 0 & B_{22} \end{pmatrix} = \operatorname{sdet} \begin{pmatrix} \mathcal{Z}_2 & \\ & \mathcal{Z}_2^{-st} \end{pmatrix} = 1.$$

Then after integration on (h, δ) space we get

$$\widehat{F}_{s,\eta}(g,\mu) = e^{-\pi H\left(\underline{(g,\mu)}, \underline{(g,\mu)}\right) - \pi H\left(\underline{(s,\eta)}, \underline{(g,\mu)}\right)}. \qquad \square$$

References

[1] M. Bergvelt and J. Rabin, *Supercurves, their Jacobians, and super KP equations*, Duke Math. J., **98(1)**(1999), 1–57.

[2] A. Connes, M. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory*, J. High Energy Phys., **JHEP9802**(1998), 003, 35 pp.

[3] H. Kim, *Quantum super theta vectors and theta functions*, Kyungpook Math. J., **56(1)**(2016), 249–256.

[4] Y. Manin, *Theta functions, quantum tori and Heisenberg Groups*, Lett. Math. Phys., **56(3)**(2001), 295–320.

[5] Y. Manin, *Functional equations for quantum theta functions*, Publ. Res. Inst. Math. Sci., **40(3)**(2004), 605–624.

[6] D. Mumford (with M. Nori and P. Norman), *Tata lectures on theta III*, Progress in Math. **97**, Birkhäuser, Boston, 1991.

[7] M. Rieffel, *Projective modules over higher dimensional noncommutative tori*, Canad. J. Math., **40(2)**(1988), 257–338.

[8] A. Schwarz, *Theta functions on noncommutative tori*, Lett. Math. Phys., **58**(2001), 81–90.

[9] Y. Tsuchimoto, *On super theta functions*, J. Math. Kyoto Univ., **34(3)**(1994), 641–694.

[10] A. Tyurin, *Quantization and "theta functions"*, arXiv:math/9904046.