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A Study of Marichev-Saigo-Maeda Fractional Integral Operators Associated with the S-Generalized Gauss Hypergeometric Function

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Abstract. In this work, we evaluate the Mellin transform of the Marichev-Saigo-Maeda fractional integral operator with Appell's function F_3 type kernel. We then discuss six special cases of the result involving the Saigo fractional integral operator, the Erdélyi-Kober fractional integral operator, the Riemann-Liouville fractional integral operator and the Weyl fractional integral operator. We obtain new and known results as special cases of our main results. Finally, we obtain the images of S-generalized Gauss hypergeometric function under the operators of our study.

1. Introduction and Definitions

Fractional calculus is an interesting and useful branch of mathematical analysis that studies differential and integral operators of arbitrary order. In recent years, several schemes of fractional calculus– namely Riemann-Liouville, Weyl,

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Gruunwald-Letnikow and Caputo– have been studied, and several researchers, such as Miller and Ross [9], Podlubny [10], Kilbas et al. [3], Yang et al.[22], Kumar [4, 5, 6], Srivastava et al. [18], Singh et al. [12, 13] and Choudhary et al. [1], have made valuable contributions in the area. Srivastava and Saigo [19] studied multiplication, and its applictions, on certain classes of operators of fractional calculus which include the Gaussian hypergeometric function. Then in [14, 21], Srivastava et al. made basic contributions to the development of Marichev-Saigo-Maeda fractional integral operators, which generalise both Riemann-Liouville and Erdélyi-Kober operators.

1.1. Marichev-Saigo-Maeda Fractional Integral Operators

The generalized integral operators of fractional order [7, 11] involving Appell's hypergeometric function F_3 [17] in two variables are expressed in the following forms:

(1.1)
$$\left(I_{0,z}^{\varrho,\varrho',\tau,\tau',\eta}f\right)(z)=\frac{z^{-\varrho}}{\Gamma(\eta)}\int\limits_0^z(z-w)^{\eta-1}w^{-\varrho'}F_3\left(\varrho,\varrho',\tau,\tau';\eta;1-\frac{w}{z},1-\frac{z}{w}\right)f(w)dw$$

and

(1.2)
$$\left(I_{z,\infty}^{\varrho,\varrho',\tau,\tau',\eta}f\right)(z)=\frac{z^{-\varrho'}}{\Gamma(\eta)}\int\limits_z^\infty(w-z)^{\eta-1}w^{-\varrho}F_3\left(\varrho,\varrho',\tau,\tau';\eta;1-\frac{z}{w},1-\frac{w}{z}\right)f(w)dw,$$

provided that $z>0$ and $\varrho,\varrho',\tau,\tau',\eta\in\mathbb{C},\Re(\eta)>0$ and the above integrals exist.

1.2. S-generalized Gauss Hypergeometric Function (S-GGHF)

The S-generalized Gauss hypergeometric function (S-GGHF) $F_q^{(\sigma,\varkappa;\vartheta,\mu)}(\lambda,\nu;\varphi;u)$ was proposed and studied by Srivastava et al. [15, p. 350, Eq. (1.12)]. It is expressed in the following form:

$$F_q^{(\sigma,\varkappa;\vartheta,\mu)}(\lambda,\nu;\varphi;u)=\sum_{\ell=0}^{\infty}(\lambda)_\ell\frac{B_q^{(\sigma,\varkappa;\vartheta,\mu)}(\nu+\ell,\varphi-\nu)}{B(\nu,\varphi-\nu)}\frac{u^\ell}{\ell!},\qquad(|u|<1)\tag{1.3}$$

provided that $\Re(q)\geq0;\quad\min\{\Re(\sigma),\Re(\varkappa),\Re(\vartheta),\Re(\mu)\}>0;\quad\Re(\varphi)>\Re(\nu)>0$ in terms of the S-generalized beta function $B_q^{(\sigma,\varkappa;\vartheta,\mu)}(u,v)$, which was also given by Srivastava et al. [15, p.350, Eq.(1.13)] in the following form:

$$B_q^{(\sigma,\varkappa;\vartheta,\mu)}(u,v)=\int_0^1t^{u-1}(1-t)^{v-1}{}_1F_1\left(\sigma;\varkappa;-\frac{q}{t^\vartheta(1-t)^\mu}\right)dt,\tag{1.4}$$

$\Re(q) \geq 0; \quad \min\{\Re(u), \Re(v), \Re(\sigma), \Re(\varkappa), \Re(\vartheta), \Re(\mu)\} > 0.$
Special cases of the S-GGHF and S-generalized beta function were given by Srivastava et al. [16].

2. Mellin Transform

In the present section, we obtain the Mellin transform of Marichev-Saigo-Maeda fractional integral operator its particular cases.

2.1. Mellin Transform of Fractional Integral Operator

The Mellin transform of a function $f(t)$ is defined as [2]

$$\mathfrak{M}[f(z)](\mathfrak{s}) = \int_0^\infty z^{\mathfrak{s}-1} f(z)dz, \qquad \Re(\mathfrak{s}) > 0 \tag{2.1}$$

provided that the above integral exists.

Theorem 2.1. *If $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \varrho', \tau, \tau', \eta, \rho \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(\mathfrak{s}+\rho+\eta-\varrho-\varrho') > 1$ and $\Re(\rho) < \max\{\Re(2-\mathfrak{s}-\eta+\varrho), \Re(2-\mathfrak{s}-\tau+\varrho), \Re(2-\mathfrak{s})\}$. Then, the following Mellin transform formula holds:*

$$\mathfrak{M}\left[z^{\rho-1}\left(I_{0,z}^{\varrho,\varrho',\tau,\tau',\eta} f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(2-\mathfrak{s}-\rho-\eta+\varrho)\Gamma(2-\mathfrak{s}-\rho-\tau)}{\Gamma(2-\mathfrak{s}-\rho)\Gamma(2-\mathfrak{s}-\rho+\varrho-\tau)} {}_2F_1\left[\begin{matrix}\varrho', \tau'; \\ \\ \mathfrak{s}+\rho+\eta-\varrho-1;\end{matrix}\ 1\right] \mathfrak{M}[f(z)](\mathfrak{s}+\rho+\eta-\varrho-\varrho'-1). \tag{2.2}$$

Proof. To prove the result (2.2), we take the Mellin transform of (1.1), it gives (say $\Delta(\mathfrak{s})$)

$$\Delta(\mathfrak{s}) := \int_0^\infty z^{\mathfrak{s}+\rho-2}\left[\frac{z^{-\varrho}}{\Gamma(\eta)}\int_0^z (z-w)^{\eta-1} w^{-\varrho'} F_3\left(\varrho, \varrho', \tau, \tau'; \eta; 1-\frac{w}{z}, 1-\frac{z}{w}\right) f(w)dw\right] dz.$$

Next, we express the Appell's function F_3 in the series form and then interchange the order of w,z-integrals and series (which is permissible under the conditions stated), it gives

$$\Delta(\mathfrak{s}) := \sum_{\iota,\kappa=0}^\infty \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{\Gamma(\eta)(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa \int_0^\infty w^{-\varrho'-\kappa} f(w)\left[\int_w^\infty z^{\mathfrak{s}+\rho-\varrho-\iota-2}(z-w)^{\iota+\kappa+\eta-1} dz\right] dw.$$

On evaluating the resulting z-integral, we get

$$\Delta(\mathfrak{s}) := \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{\Gamma(\eta)(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa$$
$$\frac{\Gamma(\iota+\kappa+\eta)\Gamma(2-\mathfrak{s}-\rho-\kappa-\eta+\varrho)}{\Gamma(2-\mathfrak{s}-\rho+\varrho+\iota)} \int_0^\infty w^{\mathfrak{s}+\rho-\varrho'-\varrho+\eta-2} f(w)dw.$$

Finally, by using the result obtained by Srivastava and Karlsson [17, p. 17, Eq. (8)] and the result (2.1), we arrive at the required result (2.2). □

Theorem 2.2. *Let* $z > 0$*,* $\Re(\mathfrak{s}) > 0$ *and the parameters* $\varrho, \varrho', \tau, \tau', \eta, \rho \in \mathbb{C}$*, satisfying* $\Re(\eta) > 0$*,* $\Re(\mathfrak{s}+\rho+\eta-\varrho-\varrho') > 1$ *and* $\Re(\rho) > \max\{\Re(1+\varrho+\tau+\varrho'-\mathfrak{s}-\eta), \Re(1+\varrho+\varrho'-\mathfrak{s}-\eta), \Re(1+\tau+\varrho'-\mathfrak{s}-\eta)\}$*. Then the following Mellin transform result holds:*

(2.3)

$$\mathfrak{M}\left[z^{\rho-1}\left(I_{z,\infty}^{\varrho,\varrho',\tau,\tau',\eta} f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(\mathfrak{s}+\rho+\eta-\varrho'-\varrho-\tau-1)\Gamma(\mathfrak{s}+\rho-\varrho'-1)}{\Gamma(\mathfrak{s}+\rho+\eta-\varrho'-\varrho-1)\Gamma(\mathfrak{s}+\rho+\eta-\varrho'-\tau-1)}$$
$${}_2F_1\left[\begin{matrix}\varrho', \tau'; \\ \\ 2-\mathfrak{s}-\rho-\varrho';\end{matrix} \; 1\right] \mathfrak{M}[f(z)](\mathfrak{s}+\rho+\eta-\varrho-\varrho'-1).$$

Proof. To prove the result (2.3), we take the Mellin transform of (1.2), it gives (say $\Lambda(\mathfrak{s})$)

$$\Lambda(\mathfrak{s}) := \int_0^\infty z^{\mathfrak{s}+\rho-2}\left[\frac{z^{-\varrho'}}{\Gamma(\eta)}\int_z^\infty (w-z)^{\eta-1} w^{-\varrho} F_3\left(\varrho, \varrho', \tau, \tau'; \eta; 1-\frac{z}{w}, 1-\frac{w}{z}\right) f(w)dw\right] dz.$$

Next, we express the Appell's function F_3 in the series form and then interchange the order of w,z-integrals and series (which is permissible under the conditions stated), it gives

$$\Lambda(\mathfrak{s}) := \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{\Gamma(\eta)(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa$$
$$\int_0^\infty w^{-\varrho-\iota} f(w)\left[\int_0^w z^{\mathfrak{s}+\rho-\varrho'-\kappa-2}(z-w)^{\eta+\iota+\kappa-1} dz\right] dw.$$

On evaluating the resulting z-integral, we get

$$\Lambda(\mathfrak{s}) := \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{\Gamma(\eta)(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa$$
$$\frac{\Gamma(\iota+\kappa+\eta)\Gamma(\mathfrak{s}+\rho-\varrho'-\kappa-1)}{\Gamma(\mathfrak{s}+\rho-\varrho'+\iota+\eta-1)} \int_0^\infty w^{\mathfrak{s}+\rho-\varrho'-\varrho+\eta-2} f(w)dw.$$

Finally, with the help of the result obtained by Srivastava and Karlsson [17, p. 17, Eq. (8)] and the result (2.1), we arrive at the required result (2.3). □

Now, we present the following special cases of Theorems 2.1 and 2.2. On setting $\varrho' = 0$ in Theorem 2.1, we get the following result.

Corollary 2.1. *If $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \tau, \eta, \rho \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(\mathfrak{s}+\rho-\tau) > 1$ and $\Re(\rho) < \max\{\Re(2-\mathfrak{s}+\eta), \Re(2-\mathfrak{s}-\tau), \Re(2-\mathfrak{s}), \Re(2-\mathfrak{s}+\varrho+\tau+\eta)\}$. Then, the following Mellin transform of Saigo fractional integral operators formula holds:*

(2.4)
$$\mathfrak{M}\left[z^{\rho-1}\left(I_{0,z}^{\varrho,\tau,\eta}f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(2-\rho-\mathfrak{s}+\eta)\Gamma(2-\rho-\mathfrak{s}+\tau)}{\Gamma(2-\mathfrak{s}-\rho)\Gamma(2-\mathfrak{s}-\rho+\varrho+\tau+\eta)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho-\tau-1).$$

provided that each member of the result (2.4) exists.

On setting $\varrho' = 0$ in Theorem 2.2, we get the following result.

Corollary 2.2. *Let $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \tau, \eta, \rho \in \mathbb{C}$, satisfying $\Re(\eta) > 0$, $\Re(\mathfrak{s}+\rho-\tau) > 1$ and $\Re(\rho) > \max\{\Re(1+\tau-\mathfrak{s}-\eta), \Re(1+\tau-\mathfrak{s}), \Re(1-\varrho-\mathfrak{s}-\eta)\}$. Then, the following Mellin transform of Saigo fractional integral operators formula holds:*

(2.5)
$$\mathfrak{M}\left[z^{\rho-1}\left(I_{z,\infty}^{\varrho,\tau,\eta}f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(\mathfrak{s}+\rho+\eta-\tau-1)\Gamma(\mathfrak{s}+\rho-1)}{\Gamma(\mathfrak{s}+\rho-\tau-1)\Gamma(\mathfrak{s}+\rho+\eta+\varrho-1)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho-\tau-1),$$

provided that each member of the result (2.5) exists.

Further, on setting $\tau = 0$ and $\tau = -\varrho$ in Corollary 2.1 and Corollary 2.2, we get the Mellin transforms of Erdélyi [20], Kober [20], Riemann-Liouville [20] and Weyl [20] fractional integral operators.

Corollary 2.3. *If $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \eta, \rho \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(\mathfrak{s}+\rho) > 1$ and $\Re(\rho) < \max\{\Re(2-\mathfrak{s}+\eta), \Re(2-\mathfrak{s}+\varrho+\eta)\}$. Then, the following Mellin transform of Erdélyi fractional integral operator formula holds:*

$$\mathfrak{M}\left[z^{\rho-1}\left(\varepsilon_{0,z}^{\varrho,\eta}f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(2-\rho-\mathfrak{s}+\eta)}{\Gamma(2-\rho-\mathfrak{s}+\varrho+\eta)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho-1), \tag{2.6}$$

provided that each member of the result (2.6) exists.

Corollary 2.4. *Let $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \eta, \rho \in \mathbb{C}$, satisfying $\Re(\eta) > 0$, $\Re(\mathfrak{s}+\rho) > 1$ and $\Re(\rho) > \max\{\Re(1-\mathfrak{s}-\eta), \Re(1-\varrho-\mathfrak{s}-\eta)\}$. Then, the following Mellin transform of Kober fractional integral operator formula holds:*

$$\mathfrak{M}\left[z^{\rho-1}\left(K_{z,\infty}^{\varrho,\eta} f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(\mathfrak{s}+\rho+\eta-1)}{\Gamma(\mathfrak{s}+\rho+\eta+\varrho-1)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho-1), \tag{2.7}$$

provided that each member of the result (2.7) exists.

Corollary 2.5. *If $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \rho \in \mathbb{C}$, such that $\Re(\mathfrak{s}+\varrho+\rho) > 1$ and $\Re(\rho) < \max\{\Re(2-\mathfrak{s}-\varrho), \Re(2-\mathfrak{s})\}$. Then, the following Mellin transform of Riemann-Liouville fractional integral operator formula holds:*

$$\mathfrak{M}\left[z^{\rho-1}\left(R_{0,z}^{\varrho} f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(2-\mathfrak{s}-\rho-\varrho)}{\Gamma(2-\mathfrak{s}-\rho)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho+\varrho-1), \tag{2.8}$$

provided that each member of the result (2.8) exists.

Corollary 2.6. *Let $z > 0$, $\Re(\mathfrak{s}) > 0$ and the parameters $\varrho, \rho \in \mathbb{C}$, satisfying $\Re(\mathfrak{s}+\rho+\varrho) > 1$ and $\Re(\rho) > \max\{\Re(1-\mathfrak{s}), \Re(1-\varrho-\mathfrak{s})\}$. Then, the following Mellin transform of Weyl fractional integral operator formula holds:*

$$\mathfrak{M}\left[z^{\rho-1}\left(W_{z,\infty}^{\varrho} f\right)(z)\right](\mathfrak{s}) = \frac{\Gamma(\mathfrak{s}+\rho-1)}{\Gamma(\mathfrak{s}+\rho+\varrho-1)}\mathfrak{M}[f(z)](\mathfrak{s}+\rho+\varrho-1), \tag{2.9}$$

provided that each member of the result (2.9) exists.

Remark 2.1. It can be noticed that the above six corollaries are also quite general in nature and reduce to several interesting results. Thus, the six interesting results recorded in the text by Mathai et al. [8] follow as simple special cases of these corollaries.

3. Images

In this section, we establish some image formulas for the S-GGHF under the Marichev-Saigo-Maeda fractional integral operator (1.1) and (1.2).

3.1. Images of S-generalized Gauss Hypergeometric Function under the Marichev-Saigo-Maeda Fractional Integral Operator

Theorem 3.1. *If $z > 0$, $\Re(\varphi) > \Re(\nu) > 0$, $\Re(q) \geq 0$ and the parameters $\varrho, \varrho', \tau, \tau', \eta, \delta \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(\delta-\varrho') > 0$ and $\Re(\delta+\eta-\varrho') > 0$. Then* *the following fractional integral result holds:*

(3.1)

$$\left[I_{0,z}^{\varrho,\varrho',\tau,\tau',\eta} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{w}{z}\right)\right](z) = \frac{z^{\delta+\eta-\varrho'-\varrho-1}\Gamma(\delta-\varrho')}{\Gamma(\delta+\eta-\varrho')} \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{(1+\varrho'-\delta)_\kappa(\delta+\eta-\varrho')_\iota \iota!\kappa!} {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix}\lambda,\nu,\eta+\iota+\kappa; \\ \varphi,\delta+\eta+\iota-\varrho';\end{matrix} 1\right].$$

Proof. In order to prove the result (3.1), we take the Marichev-Saigo-Maeda fractional integral (1.1) of (1.3), it gives (say, Ξ)

$$\Xi = \frac{z^{-\varrho}}{\Gamma(\eta)} \int_0^z (z-w)^{\eta-1} w^{\delta-\varrho'-1} F_3\left(\varrho,\varrho',\tau,\tau';\eta;1-\frac{w}{z},1-\frac{z}{w}\right) F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{w}{z}\right) dw.$$

Next, we express the S-generalized Gauss hypergeometric function $F_q^{(\sigma,\varkappa;\vartheta,\mu)}(.)$ in the series form with the help of (1.3) and then change the order of integration and summation (which is permissible under the conditions presented), it gives

$$\Xi = \sum_{\iota,\kappa,\ell=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa \frac{(\lambda)_\ell B_q^{\sigma,\varkappa;\vartheta,\mu}(\nu+\ell,\varphi-\nu)}{B(\nu,\varphi-\nu)\ell!} \frac{z^{-\varrho-\ell-\iota}}{\Gamma(\eta)} \int_0^z w^{\delta-\varrho'-\kappa-1}(z-w)^{\eta+\iota+\kappa+\ell-1} dw.$$

Finally, using the beta function and result (1.3), we get the required result (3.1). □

Theorem 3.2. *If $z>0$, $\Re(\varphi)>\Re(\nu)>0$, $\Re(q)\geq 0$ and the parameters $\varrho,\varrho',\tau,\tau',\eta,\delta\in\mathbb{C}$, such that $\Re(\eta)>0$, $\Re(1+\varrho-\eta-\delta)>0$ and $\Re(1+\varrho-\delta)>0$. Then, the following fractional integral result holds:*

(3.2)

$$\left[I_{z,\infty}^{\varrho,\varrho',\tau,\tau',\eta} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{z}{w}\right)\right](z) = \frac{z^{\delta+\eta-\varrho'-\varrho-1}\Gamma(1+\varrho-\eta-\delta)}{\Gamma(1+\varrho-\delta)} \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{(\delta+\eta-\varrho)_\kappa(1+\varrho-\delta)_\iota \iota!\kappa!} {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix}\lambda,\nu,\eta+\iota+\kappa; \\ \varphi,1+\varrho+\iota-\delta;\end{matrix} 1\right].$$

Proof. To prove the result (3.2), we consider the Marichev-Saigo-Maeda fractional integral (1.2) of (1.3), it gives

$$\Xi = \frac{z^{-\varrho'}}{\Gamma(\eta)} \int_z^\infty (w-z)^{\eta-1} w^{\delta-\varrho-1} F_3\left(\varrho,\varrho',\tau,\tau';\eta;1-\frac{z}{w},1-\frac{w}{z}\right) F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{z}{w}\right) dw$$

Next, we express the S-generalized Gauss hypergeometric function $F_q^{(\sigma,\varkappa;\vartheta,\mu)}(.)$ in the series form with the help of (1.3) and then changing the order of integration and summation (which is permissible under the conditions presented), it gives

$$\Xi = \sum_{\iota,\kappa,\ell=0}^{\infty} \frac{(\varrho)_\iota(\varrho')_\kappa(\tau)_\iota(\tau')_\kappa}{(\eta)_{\iota+\kappa}\iota!\kappa!}(-1)^\kappa$$
$$\frac{(\lambda)_\ell B_q^{\sigma,\varkappa;\vartheta,\mu}(\nu+k,\varphi-\nu)}{B(\nu,\varphi-\nu)\ell!}\frac{z^{-\varrho'-\kappa}}{\Gamma(\eta)}\int_z^\infty w^{\delta-\varrho-\iota-\ell-1}(w-z)^{\eta+\iota+\kappa+\ell-1}dw.$$

Finally, using the beta function and the result (1.3), we arrive at the desired result (3.2). □

Now, we give six interesting special cases of the Theorems 3.1 and 3.2. On setting $\varrho'=0$ in Theorem 3.1, we get the following result.

Corollary 3.1. *If $z>0$, $\Re(\varphi)>\Re(\nu)>0$, $\Re(q)\geq 0$ and the parameters $\varrho,\tau,\eta,\delta\in\mathbb{C}$, such that $\Re(\eta)>0$, $\Re(\delta)>0$ and $\Re(\delta+\varrho)>0$. Then, the image of Saigo fractional integral operator involving S-generalized Gauss hypergeometric function is expressed as:*

$$(3.3)\quad \left(I_{0,z}^{\varrho,\tau,\eta} z^{\delta-1}F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{w}{z}\right)\right)(z)$$
$$=\frac{z^{\delta-\tau-1}\Gamma(\delta)}{\Gamma(\delta+\varrho)}\sum_{\iota,\kappa=0}^{\infty}\frac{(\varrho+\tau)_\iota(-\eta)_\iota}{(\delta+\varrho)_\iota\iota!}\,{}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix}\lambda,\nu,\varrho+\iota; \\ \varphi,\delta+\varrho+\iota;\end{matrix}\ 1\right],$$

provided that each member of the result (3.3) exists.

On setting $\varrho'=0$ in Theorem 3.2, we get the following result.

Corollary 3.2. *Let $z>0$, $\Re(\varphi)>\Re(\nu)>0$, $\Re(q)\geq 0$ and the parameters $\varrho,\tau,\eta,\delta\in\mathbb{C}$, such that $\Re(\eta)>0$, $\Re(1+\tau-\delta)>0$ and $\Re(1+\varrho+\tau-\delta)>0$. Then, the image of Saigo fractional integral operator involving S-generalized Gauss hypergeometric function is expressed as:*

(3.4)

$$\left(I_{z,\infty}^{\varrho,\tau,\eta} z^{\delta-1}F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{z}{w}\right)\right)(z)$$
$$=\frac{z^{\delta-\tau-1}\Gamma(1+\tau-\delta)}{\Gamma(1+\varrho+\tau-\delta)}\sum_{\iota=0}^{\infty}\frac{(\varrho+\tau)_\iota(-\eta)_\iota}{(1+\varrho+\tau-\delta)_\iota\iota!}\,{}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix}\lambda,\nu,\varrho+\iota; \\ \varphi,1+\iota+\varrho+\tau-\delta;\end{matrix}\ 1\right],$$

provided that each member of the assertions (3.4) exists.

Further, on setting $\tau=0$ and $\tau=-\varrho$ in Corollary 3.1 and Corollary 3.2, we get the following images of S-generalized Gauss hypergeometric function under the

Erdélyi [20], Kober [20], Riemann-Liouville [20] and Weyl [20] fractional integral operators.

Corollary 3.3. *If $z > 0$, $\Re(\varphi) > \Re(\nu) > 0$, $\Re(q) \geq 0$ and the parameters $\varrho, \eta, \delta \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(\delta) > 0$ and $\Re(\delta + \varrho) > 0$. Then, the image of Erdélyi fractional integral operator involving S-generalized Gauss hypergeometric function is expressed as:*

$$\left(\varepsilon_{0,z}^{\varrho,\eta} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{w}{z}\right)\right)(z) = \frac{z^{\delta-1}\Gamma(\delta)}{\Gamma(\delta+\varrho)} \sum_{\iota,\kappa=0}^{\infty} \frac{(\varrho)_\iota(-\eta)_\iota}{(\delta+\varrho)_\iota \iota!} \, {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix} \lambda,\nu,\varrho+\iota; \\ \varphi,\delta+\varrho+\iota; \end{matrix} \; 1\right], \tag{3.5}$$

provided that each member of the assertions (3.5) exists.

Corollary 3.4. *Let $z > 0$, $\Re(\varphi) > \Re(\nu) > 0$, $\Re(q) \geq 0$ and the parameters $\varrho, \eta, \delta \in \mathbb{C}$, such that $\Re(\eta) > 0$, $\Re(1-\delta) > 0$ and $\Re(1+\varrho-\delta) > 0$. Then, the image of Kober fractional integral operator involving S-generalized Gauss hypergeometric function is expressed as:*

$$\left(K_{z,\infty}^{\varrho,\eta} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{z}{w}\right)\right)(z) = \frac{z^{\delta-1}\Gamma(1-\delta)}{\Gamma(1+\varrho-\delta)} \sum_{\iota=0}^{\infty} \frac{(\varrho)_\iota(-\eta)_\iota}{(1+\varrho-\delta)_\iota \iota!} \, {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix} \lambda,\nu,\varrho+\iota; \\ \varphi,1+\iota+\varrho-\delta; \end{matrix} \; 1\right], \tag{3.6}$$

provided that each member of the assertions (3.6) exists.

Corollary 3.5. *If $z > 0$, $\Re(\varphi) > \Re(\nu) > 0$, $\Re(q) \geq 0$ and the parameters $\varrho, \delta \in \mathbb{C}$, such that $\Re(\delta) > 0$ and $\Re(\delta + \varrho) > 0$. Then, the image of Riemann-Liouville fractional integral operator involving S-generalized Gauss hypergeometric function is expressed as:*

$$\left(R_{0,z}^{\varrho} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{w}{z}\right)\right)(z) = \frac{z^{\delta+\varrho-1}\Gamma(\delta)}{\Gamma(\delta+\varrho)} {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{matrix} \lambda,\nu,\varrho; \\ \varphi,\delta+\varrho; \end{matrix} \; 1\right], \tag{3.7}$$

provided that each member of the result (3.7) exists.

Corollary 3.6. *Let $z > 0$, $\Re(\varphi) > \Re(\nu) > 0$, $\Re(q) \geq 0$ and the parameters $\varrho, \delta \in \mathbb{C}$, such that $\Re(1-\varrho-\delta) > 0$ and $\Re(1-\delta) > 0$. Then, the image of Weyl* *fractional integral operator containing S-generalized Gauss hypergeometric function is expressed as:*

(3.8)

$$\left(W^{\varrho}_{z,\infty} z^{\delta-1} F_q^{(\sigma,\varkappa;\vartheta,\mu)}\left(\lambda,\nu;\varphi;1-\frac{z}{w}\right)\right)(z) = \frac{z^{\delta+\varrho-1}\Gamma(1-\varrho-\delta)}{\Gamma(1-\delta)}\, {}_1F_{q,1}^{(\sigma,\varkappa;\vartheta,\mu)}\left[\begin{array}{c} \lambda,\nu,\varrho; \\ \\ \varphi, 1-\delta; \end{array} 1\right],$$

provided that each member of the result (3.8) exists.

4. Conclusions

In this article, we evaluated the Mellin transform of the Marichev-Saigo-Maeda fractional integral operator, the kernel of which is the Appell's function F_3. We then obtained the image of S-generalized hypergeometric function. On account of the more general nature of the S-generalized hypergeometric function, our findings yield a large number of new and known results involving simpler fractional derivative formulas associated with several special functions. Thus, we assert that the results obtained in the present study are basic in nature.

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