

Approximation by Generalized Kantorovich Sampling Type Series

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ABSTRACT. In the present article, we analyse the behaviour of a new family of Kantorovich type sampling operators $(K_w^\varphi f)_{w>0}$. First, we give a Voronovskaya type theorem for these Kantorovich generalized sampling series and a corresponding quantitative version in terms of the first order of modulus of continuity. Further, we study the order of approximation in $C(\mathbb{R})$, the set of all uniformly continuous and bounded functions on \mathbb{R} for the family $(K_w^\varphi f)_{w>0}$. Finally, we give some examples of kernels such as B-spline kernels and the Blackman-Harris kernel to which the theory can be applied.

1. Introduction

The theory of generalized sampling series was first initiated by P. L. Butzer and his school [13] and [14]. In recent years, it has become an attractive topic in approximation theory due to its wide range of applications, especially in signal and image processing. For $w > 0$, a generalized sampling series of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(T_w^\varphi f)(x) = \sum_{k=-\infty}^{\infty} \varphi(wx - k) f\left(\frac{k}{w}\right), \quad x \in \mathbb{R},$$

where φ is a kernel function on \mathbb{R} . These type of operators have been studied by

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many authors (e.g. [7, 9, 11, 23, 24, 25, 26, 27]).

The study of Kantorovich type generalizations of approximation operators is an important subject in approximation theory, as they can be used to approximate Lebesgue integrable functions. In the last few decades, Kantorovich modifications of several operators have been constructed and their approximation behavior studied, we mention some of the work in this direction e.g., [1, 2, 3, 22] etc.

In [12], the authors introduced the sampling Kantorovich operators and studied their rate of convergence in the general settings of Orlicz spaces. After that, Danilo and Vinti [16] extended their study in the multivariate setting and obtained the rate of convergence for functions in Orlicz spaces. Danilo and Vinti [18], obtained the rate of approximation for the family of sampling Kantorovich operators in the uniform norm, for bounded uniformly continuous functions belonging to Lipschitz classes and for functions in Orlicz spaces. Also, the nonlinear version of sampling Kantorovich operators have been studied in [17] and [29].

Altomare and Leonessa [5] considered a new sequence of positive linear operators acting on the space of Lebesgue-integrable functions on the unit interval. Such operators include the Kantorovich operators as a particular case. Later, in order to obtain an approximation process for spaces of locally integrable functions on unbounded intervals, Altomare et. al. introduced and studied the generalized Szász-Mirakjan-Kantorovich operators in [4]. Also in [15], the authors obtained some qualitative properties and an asymptotic formula for such a sequence of operators.

We consider the generalized Kantorovich type sampling series. Let $\{a_k\}_{k \in \mathbb{Z}}$ and $\{b_k\}_{k \in \mathbb{Z}}$ be two sequences of real numbers such that $b_k - a_k = \Delta_k > 0$ for $k \in \mathbb{Z}$. In this paper, we analyse the approximation properties of the following type of generalized Kantorovich sampling series

$$(1.1) \quad (K_w^\varphi f)(x) = \sum_{k=-\infty}^{\infty} \frac{w}{\Delta_k} \varphi(wx - k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} f(u) du,$$

where f is in $C(\mathbb{R})$, the class of all uniformly continuous and bounded functions on \mathbb{R} .

In the present paper, first we obtain asymptotic formula and their quantitative estimate for the operators $(K_w^\varphi f)_{w>0}$. Further, the order of approximation of these operators is analysed in $C(\mathbb{R})$. Finally, we give some examples of kernels such as B-spline kernels and the Blackman-Harris kernel, to which the theory can be applied.

2. Main Results

Let $\varphi \in C(\mathbb{R})$ be fixed. For every $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $u \in \mathbb{R}$ we define the algebraic moments as

$$m_\nu(\varphi, u) := \sum_{k=-\infty}^{\infty} \varphi(u - k)(k - u)^\nu$$

and the absolute moments by

$$M_\nu(\varphi) := \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |\varphi(u-k)|(k-u)^\nu.$$

Note that for $\mu, \nu \in \mathbf{N}_0$ with $\mu < \nu$, $M_\nu(\varphi) < +\infty$ implies $M_\mu(\varphi) < +\infty$. Indeed for $\mu < \nu$, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |\varphi(u-k)|(k-u)^\nu &= \sum_{|u-k|<1} |\varphi(u-k)|(k-u)^\nu + \sum_{|u-k|\geq 1} |\varphi(u-k)|(k-u)^\nu \\ &\leq 2\|\varphi\|_\infty + \sum_{|u-k|\geq 1} |\varphi(u-k)| \frac{|(k-u)^\nu|}{|(k-u)^{\nu-\mu}} \\ &\leq 2\|\varphi\|_\infty + M_\nu(\varphi). \end{aligned}$$

When φ is compactly supported, we immediately have that $M_\nu(\varphi) < +\infty$ for every $\nu \in \mathbb{N}_0$.

We suppose that the following assumptions hold:

(i) for every $u \in \mathbb{R}$, we have

$$\sum_{k=-\infty}^{\infty} \varphi(u-k) = 1,$$

(ii) $M_2(\varphi) < +\infty$ and there holds

$$\lim_{r \rightarrow \infty} \sum_{|u-k|>r} |\varphi(u-k)|(k-u)^2 = 0$$

uniformly with respect to $u \in \mathbb{R}$,

(iii) for every $u \in \mathbb{R}$, we have

$$m_1(\varphi, u) = m_1(\varphi) = \sum_{k=-\infty}^{\infty} \varphi(u-k)(k-u) = 0,$$

(iv) $\sup_k \{|a_k|, |b_k|\} = M^* < \infty$.

Theorem 2.1. *Let $f \in C(\mathbb{R})$ and $\{a_k\}$ and $\{b_k\}$ be two bounded sequences of real numbers such that $a_k + b_k = \alpha$ and $b_k - a_k > \Delta^* > 0$. If $f'(x)$ exists at a point $x \in \mathbb{R}$ then,*

$$\lim_{w \rightarrow \infty} w[(K_w^\varphi f)(x) - f(x)] = \frac{\alpha}{2} f'(x).$$

Proof. Let $M^* = \sup_k \{|a_k|, |b_k|\}$. From the Taylor's theorem, we have

$$f(u) = f(x) + f'(x)(u-x) + h(u-x)(u-x),$$

for some bounded function h such that $h(t) \rightarrow 0$ as $t \rightarrow 0$.

Thus, we have

$$\begin{aligned} (K_w^\varphi f)(x) - f(x) &= f'(x) \sum_{k=-\infty}^{\infty} \frac{w}{\Delta_k} \varphi(wx-k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (u-x) du \\ &\quad + \sum_{k=-\infty}^{\infty} \frac{w}{\Delta_k} \varphi(wx-k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} h(u-x)(u-x) du \\ &= I_1 + I_2, \text{ (say).} \end{aligned}$$

First, we obtain I_1 .

$$\begin{aligned} I_1 &= f'(x) \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} \varphi(wx-k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (u-x) du \\ &= \frac{f'(x)}{2} \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} \varphi(wx-k) \left[\left(\frac{k+b_k}{w} - x \right)^2 - \left(\frac{k+a_k}{w} - x \right)^2 \right] \\ &= \frac{f'(x)}{2w} \sum_{k=-\infty}^{\infty} \varphi(wx-k) [(b_k + a_k) + 2(k-wx)] \\ &= \frac{\alpha f'(x)}{2w} \sum_{k=-\infty}^{\infty} \varphi(wx-k) + \frac{f'(x)}{w} \sum_{k=-\infty}^{\infty} \varphi(wx-k)(k-wx) = \frac{\alpha f'(x)}{2w}. \end{aligned}$$

Next, we obtain I_2 .

$$I_2 = \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} \varphi(wx-k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} h(u-x)(u-x) du.$$

In order to obtain an estimate of I_2 , let $\epsilon > 0$ be fixed. Then, there exists $\delta > 0$ such that $|h(t)| \leq \epsilon$ for $|t| \leq \delta$. Then, we have

$$\begin{aligned} |I_2| &\leq \sum_{|k-wx| < \delta w/2} \frac{w}{b_k - a_k} |\varphi(wx-k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |h(u-x)|(u-x) du \\ &\quad + \sum_{|k-wx| \geq \delta w/2} \frac{w}{b_k - a_k} |\varphi(wx-k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |h(u-x)|(u-x) du = J_1 + J_2, \text{ (say).} \end{aligned}$$

First, we estimate J_1 . We have

$$\begin{aligned} J_1 &\leq \epsilon \sum_{|k-wx| < \delta w/2} \frac{w}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u-x)| du \\ &\leq \frac{\epsilon}{2} \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} |\varphi(wx - k)| \left[\left(\frac{k+b_k}{w} - x \right)^2 + \left(\frac{k+a_k}{w} - x \right)^2 \right] \\ &\leq \frac{\epsilon}{w\Delta^*} \left((M^*)^2 M_0(\varphi) + 2M^* M_1(\varphi) + M_2(\varphi) \right). \end{aligned}$$

Next, we obtain J_2 . Let $R > 0$ be such that

$$\sum_{|u-k| > R} |\varphi(u-k)|(u-k)^2 < \epsilon$$

uniformly with respect to $u \in \mathbb{R}$. Also, let w be such that $\delta w/2 > R$. Then

$$\sum_{|k-wx| > \delta w/2} |\varphi(wx - k)|(wx - k)^2 < \epsilon$$

for every $x \in \mathbb{R}$. The same inequality holds also for the series

$$\sum_{|k-wx| > \delta w/2} |\varphi(wx - k)||wx - k|^j < \epsilon$$

for $j = 0, 1$. Hence, we get

$$\begin{aligned} J_2 &\leq \|h\|_{\infty} \sum_{|k-wx| \geq \delta w/2} \frac{w}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u-x)| du \\ &\leq \frac{\|h\|_{\infty}}{2} \sum_{|k-wx| \geq \delta w/2} \frac{w}{b_k - a_k} |\varphi(wx - k)| \left[\left(\frac{k+b_k}{w} - x \right)^2 + \left(\frac{k+a_k}{w} - x \right)^2 \right] \\ &\leq \frac{\epsilon \|h\|_{\infty}}{w\Delta^*} (1 + M^*)^2. \end{aligned}$$

Hence, the proof is completed. □

Remark 2.1 The boundedness assumption on f can be relaxed by assuming that there are two positive constant a, b such that $|f(x)| \leq a + b|x|$, for every $x \in \mathbb{R}$.

We have

$$\begin{aligned} |K_w^{\varphi} f(x)| &\leq \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |f(u)| du \\ &\leq \sum_{k=-\infty}^{\infty} \frac{w}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (a + b|u|) du \\ &\leq M_0(\varphi)(a + b|x|) + \frac{b}{\Delta^* w} \left(M_2(\varphi) + 2(M^*) M_1(\varphi) + (M^*)^2 M_0(\varphi) \right) \end{aligned}$$

and hence the series $K_w^\varphi f$ is absolutely convergent for every $x \in \mathbb{R}$. Moreover, for a fixed $x_0 \in \mathbb{R}$,

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

the Taylor's polynomial of first order centered at the point x_0 , by the Taylor's formula we can write

$$\frac{f(x) - P_1(x)}{(x - x_0)} = h(x - x_0),$$

where h is a function such that $\lim_{t \rightarrow 0} h(t) = 0$. Then h is bounded on $[x_0 - \delta, x_0 + \delta]$, for some $\delta > 0$. For $|x - x_0| > \delta$, we have

$$|h(x - x_0)| \leq \frac{a + b|x|}{|x - x_0|} + \frac{|P_1(x)|}{|x - x_0|} \leq \frac{a + b|x|}{|x - x_0|} + \frac{|f(x_0)|}{|x - x_0|} + |f'(x_0)|,$$

and the terms on the right-hand side of the above inequality are all bounded for $|x - x_0| > \delta$. Hence, $h(\cdot - x_0)$ is bounded on \mathbb{R} . Along the lines of the proof of Theorem 2.1., the same Voronovskaya formula can be obtained.

Let C^m denote the set of all $f \in C(\mathbb{R})$ such that f is m times continuously differentiable and $\|f^{(m)}\|_\infty < \infty$.

Let $\delta > 0$. For $f \in C(\mathbb{R})$, the Peetre's K -functional is defined as

$$K(\delta, f, C, C^1) := \inf\{\|f - g\|_\infty + \delta\|g'\|_\infty : g \in C^1\}.$$

For a given $\delta > 0$, the usual modulus of continuity of a given uniformly continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\omega(f, \delta) := \sup_{|x-y| \leq \delta} |f(x) - f(y)|.$$

It is well known that, for any positive constant $\lambda > 0$, the modulus of continuity satisfies the following property

$$(2.1) \quad \omega(f, \lambda\delta) \leq (\lambda + 1)\omega(f, \delta).$$

For a function $f \in C^m$, $x_0, x \in \mathbb{R}$ and $m \geq 1$, the Taylor's formula is given by

$$f(x) = \sum_{k=0}^m \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_m(f; x_0, x)$$

and the remainder term $R_m(f; x_0, x)$ is estimated by

$$|R_m(f; x_0, x)| \leq \frac{|x - x_0|^m}{m!} \omega(f^{(m)}; |x - x_0|).$$

For every $f \in C(\mathbb{R})$ there holds

$$(2.2) \quad K(\delta/2, f, C, C^1) = \frac{1}{2}\bar{\omega}(f, \delta),$$

where $\bar{\omega}(f, \cdot)$ denotes the least concave majorant of $\omega(f, \cdot)$ (see e.g. [6]).

The following estimate for the remainder $R_m(f; x_0, x)$ in terms of $\bar{\omega}$ was proved in [21].

Lemma 2.1. *Let $f \in C^m$, $m \in \mathbb{N}^0$ and $x_0, x \in \mathbb{R}$. Then, we have*

$$|R_m(f; x_0, x)| \leq \frac{|x - x_0|^m}{m!} \bar{\omega}\left(f^{(m)}; \frac{|x - x_0|}{m + 1}\right).$$

We have the following quantitative version of Theorem 2.1 in terms of the modulus $\bar{\omega}$, in case of $m = 1$.

Theorem 2.2. *Let $f \in C^1$ and $\{a_k\}$ and $\{b_k\}$ be two sequences of real numbers such that $a_k + b_k = \alpha$, $b_k - a_k \geq \Delta^* > 0$ and $\sup_k \{|a_k|, |b_k|\} \leq M^*$. Then, for every $x \in \mathbb{R}$, the following hold:*

$$\left|w[(K_w^\varphi f)(x) - f(x)] - \frac{\alpha f'(x)}{2}\right| \leq \frac{A}{\Delta^*} \bar{\omega}\left(f', \frac{\Delta^*}{2w}\right)$$

where $A = (M^*)^2 M_0(\varphi) + 2(M^*) M_1(\varphi) + M_2(\varphi)$.

Proof. Let $f \in C^1$ be fixed. Then, we can write

$$\begin{aligned} & \left|w[(K_w^\varphi f)(x) - f(x)] - \frac{\alpha f'(x)}{2}\right| \\ &= \left|f'(x) \sum_{k=-\infty}^{\infty} \frac{w^2}{b_k - a_k} \varphi(wx - k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (u - x) du \right. \\ & \quad \left. + \sum_{k=-\infty}^{\infty} \frac{w^2}{b_k - a_k} \varphi(wx - k) \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} h(u - x)(u - x) du - \frac{\alpha f'(x)}{2}\right| \\ &\leq \sum_{k=-\infty}^{\infty} \frac{w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |h(u - x)|(u - x) du. \end{aligned}$$

Using the relation (2.2) and Lemma 2.1, we obtain

$$\begin{aligned} & \left|w[(K_w^\varphi f)(x) - f(x)] - \frac{\alpha f'(x)}{2}\right| \\ &\leq \sum_{k=-\infty}^{\infty} \frac{w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u - x)| \bar{\omega}\left(f', \frac{|x - u|}{2}\right) du \\ &= 2 \sum_{k=-\infty}^{\infty} \frac{w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u - x)| K\left(\frac{|u - x|}{4}, f', C, C^1\right) du := I_1. \end{aligned}$$

For $g \in C^2$, we have

$$\begin{aligned}
I_1 &\leq \sum_{k=-\infty}^{\infty} \frac{2w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u-x)| \left(\|(f-g)'\|_{\infty} + \frac{|u-x|}{4} \|g''\|_{\infty} \right) du \\
&\leq \|(f-g)\|_{\infty} \sum_{k=-\infty}^{\infty} \frac{2w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |(u-x)| du \\
&\quad + \|g''\|_{\infty} \sum_{k=-\infty}^{\infty} \frac{2w^2}{b_k - a_k} |\varphi(wx - k)| \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (u-x)^2 du \\
&\leq \|(f-g)'\|_{\infty} \sum_{k=-\infty}^{\infty} |\varphi(wx - k)| \frac{w^2}{b_k - a_k} \left[\left(\frac{k+b_k}{w} - x \right)^2 + \left(\frac{k+a_k}{w} - x \right)^2 \right] \\
&\quad + \|g''\|_{\infty} \sum_{k=-\infty}^{\infty} \frac{w^2}{6(b_k - a_k)} |\varphi(wx - k)| \left[\left(\frac{k+b_k}{w} - x \right)^3 - \left(\frac{k+a_k}{w} - x \right)^3 \right] \\
&\leq \frac{\|(f-g)'\|_{\infty}}{\Delta^*} \left(2(M^*)^2 M_0(\varphi) + 4M_1(\varphi)(M^*) + 2M_2(\varphi) \right) \\
&\quad + \|g''\|_{\infty} \frac{1}{6w} \left(3(M^*)^2 M_0(\varphi) + 6(M^*)M_1(\varphi) + 3M_2(\varphi) \right) \\
&\leq \|(f-g)'\|_{\infty} \frac{2}{\Delta^*} \left((M^*)^2 M_0(\varphi) + 2M_1(\varphi)(M^*) + M_2(\varphi) \right) \\
&\quad + \|g''\|_{\infty} \frac{1}{2w} \left((M^*)^2 M_0(\varphi) + 2M_1(\varphi)(M^*) + M_2(\varphi) \right) \\
&\leq \frac{2A}{\Delta^*} \left(\|(f-g)'\|_{\infty} + \|g''\|_{\infty} \frac{\Delta^*}{4w} \right).
\end{aligned}$$

Taking the infimum over all $g \in C^2$, we get

$$(2.3) \quad I_1 \leq \frac{A}{\Delta^*} \bar{\omega} \left(f', \frac{\Delta^*}{2w} \right).$$

Hence, the proof is completed. \square

Remark 2.2. As a consequence of Theorem 2.2, under the above assumptions we get the uniform convergence for $w[(K_w^\varphi f)(x) - f(x)]$ to $\frac{\alpha}{2} f'(x)$.

Remark 2.3. Note that when φ is supported in $I = [-R, R]$, $R > 0$ we can obtain a different estimate for I_1 .

$$\left| w[(K_w^\varphi f)(x) - f(x)] - \frac{\alpha}{2} f'(x) \right| \leq \frac{M_0(\varphi)(R^2 + 2RM^* + (M^*)^2)}{\Delta^*} \bar{\omega} \left(f', \frac{\Delta^*}{2w} \right).$$

Also, we obtain

$$I_1 \leq \frac{2M_0(\varphi)(R^2 + 2RM^* + (M^*)^2)}{\Delta^*} \left[\|(f-g)'\|_{\infty} + \|g''\|_{\infty} \frac{\Delta^*}{4w} \right].$$

Next, we study the order of approximation for the generalized sampling Kantorovich series. The order of approximation for the generalized sampling Kantorovich series has been extensively studied by many authors (see [10, 18, 19, 20]).

Theorem 2.3. *Let φ be a kernel satisfying an additional condition that $M_\beta(\varphi) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\varphi(u - k)|(k - u)^\beta < +\infty$ for some $0 < \beta < 1$ and $\{a_k\}$ and $\{b_k\}$ be two bounded sequences of real numbers. Then, for any $f \in C(\mathbb{R})$, we have*

$$\begin{aligned} |(K_w^\varphi f)(x) - f(x)| &\leq \omega(f, w^{-\beta}) \left(M_\beta(\varphi) + (M^*)^\beta M_0(\varphi) + M_0(\varphi) \right) \\ &\quad + 2^{\beta+1} \|f\|_\infty w^{-\beta} M_\beta(\varphi), \end{aligned}$$

for every $x \in \mathbb{R}$ and $w > 2M^*$.

Proof. Let $x \in \mathbb{R}$ be fixed. Then, for $w > 0$, we can write

$$\begin{aligned} |(K_w^\varphi f)(x) - f(x)| &= \left| (K_w^\varphi f)(x) - f(x) \sum_{k \in \mathbb{Z}} \varphi(wx - k) \right| \\ &\leq \sum_{k \in \mathbb{Z}} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |f(u) - f(x)| du \right) |\varphi(wx - k)| =: J. \end{aligned}$$

Now, we estimate J .

$$\begin{aligned} J &\leq \sum_{|wx-k| \leq w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |f(u) - f(x)| du \right) |\varphi(wx - k)| \\ &\quad + \sum_{|wx-k| > w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} |f(u) - f(x)| du \right) |\varphi(wx - k)| =: I_1 + I_2. \end{aligned}$$

We observe that, for every $u \in \left[\frac{k+a_k}{w}, \frac{k+b_k}{w} \right]$ and suitable large w with $|wx - k| \leq w/2$, we get

$$|u - x| \leq \left| u - \left(\frac{k+a_k}{w} \right) + \left(\frac{k+a_k}{w} \right) - x \right| \leq \left| u - \frac{k}{w} \right| + \left| \frac{k}{w} - x \right| \leq \frac{M^*}{w} + \frac{1}{2} \leq 1,$$

for every $w \geq 2M^*$. Since $0 < \beta < 1$, we have

$$\omega(f, |u - x|) \leq \omega(f, |u - x|^\beta).$$

Using the property of modulus of continuity (2.1), we obtain

$$\begin{aligned}
 I_1 &\leq \sum_{|wx-k|\leq w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} \omega(f, |u-x|^\beta) du \right) |\varphi(wx-k)| \\
 &\leq \sum_{|wx-k|\leq w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} (1+w^\beta |u-x|^\beta) \omega(f, w^{-\beta}) du \right) |\varphi(wx-k)| \\
 &\leq \omega(f, w^{-\beta}) \left[\sum_{|wx-k|\leq w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} w^\beta |u-x|^\beta du \right) |\varphi(wx-k)| \right. \\
 &\quad \left. + \sum_{|wx-k|\leq w/2} |\varphi(wx-k)| \right] \\
 &=: \omega(f, w^{-\beta})(J_1 + J_2).
 \end{aligned}$$

First, we obtain J_1 . Using the property of sub-additivity of $|\cdot|^\beta$ with $0 < \beta < 1$, we have

$$\begin{aligned}
 J_1 &\leq \sum_{|wx-k|\leq w/2} \left(w^\beta \max_{u \in \left[\frac{k+a_k}{w}, \frac{k+b_k}{w} \right]} |u-x|^\beta \right) |\varphi(wx-k)| \\
 &\leq \sum_{|wx-k|\leq w/2} \left(w^\beta \max \left(\left| \frac{k+a_k}{w} - x \right|^\beta, \left| \frac{k+b_k}{w} - x \right|^\beta \right) \right) |\varphi(wx-k)| \\
 &\leq \sum_{|wx-k|\leq w/2} \left(w^\beta \max \left(\left| \frac{k}{w} - x \right|^\beta + \left| \frac{a_k}{w} \right|^\beta, \left| \frac{k}{w} - x \right|^\beta + \left| \frac{b_k}{w} \right|^\beta \right) \right) |\varphi(wx-k)| \\
 &\leq \sum_{|wx-k|\leq w/2} w^\beta \left(\left| \frac{k}{w} - x \right|^\beta + \left(\sup_k \{ |a_k|, |b_k| \} \right)^\beta w^{-\beta} \right) |\varphi(wx-k)| \\
 &\leq \sum_{|wx-k|\leq w/2} |k-wx|^\beta |\varphi(wx-k)| + \sum_{|wx-k|\leq w/2} (M^*)^\beta |\varphi(wx-k)| \\
 &\leq M_\beta(\varphi) + (M^*)^\beta M_0(\varphi) < \infty.
 \end{aligned}$$

It is easy to see that

$$J_2 \leq \sum_{|wx-k|\leq w/2} |\varphi(wx-k)| = M_0(\varphi).$$

Next, we estimate I_2 .

$$\begin{aligned}
 I_2 &\leq 2\|f\|_\infty \sum_{|wx-k|>w/2} \left(\frac{w}{b_k - a_k} \int_{\frac{k+a_k}{w}}^{\frac{k+b_k}{w}} du \right) |\varphi(wx - k)| \\
 &\leq 2\|f\|_\infty \sum_{|wx-k|>w/2} |\varphi(wx - k)| \\
 &\leq 2\|f\|_\infty \sum_{|wx-k|>w/2} \frac{|wx - k|^\beta}{|wx - k|^\beta} |\varphi(wx - k)| \\
 &\leq \frac{2\|f\|_\infty}{w^\beta} \sum_{|wx-k|>w/2} |wx - k|^\beta |\varphi(wx - k)| \\
 &\leq 2^{\beta+1} \|f\|_\infty w^{-\beta} M_\beta(\varphi) < +\infty,
 \end{aligned}$$

which completes the proof. □

3. Applications to Special Kernels

In this section, we describe some particular examples of kernels φ which illustrates the previous theory. In particular, we will examine the B -splines kernel and Blackman-Harries kernel.

Combinations of B-spline functions.

First, we consider the sampling Kantorovich operators based upon the combinations of spline functions. For $h \in \mathbb{N}$, the B -spline of order h is defined as

$$B_h(x) := \chi_{[-\frac{1}{2}, \frac{1}{2}]} \star \chi_{[-\frac{1}{2}, \frac{1}{2}]} \star \chi_{[-\frac{1}{2}, \frac{1}{2}]} \star \dots \star \chi_{[-\frac{1}{2}, \frac{1}{2}]}, \text{ (} h \text{ times)}$$

where

$$\chi_{[-\frac{1}{2}, \frac{1}{2}]} = \begin{cases} 1, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

and \star denotes the convolution. The Fourier transform of the functions $B_h(x)$ is given by

$$\widehat{B}_h(w) = \left(\widehat{\chi_{[-\frac{1}{2}, \frac{1}{2}]}}(w) \right)^h = \left(\frac{\sin w/2}{w/2} \right)^h, \quad w \in \mathbb{R}, \quad h \in \mathbb{N}$$

(see [14] and [28]). Given real numbers ϵ_0, ϵ_1 with $\epsilon_0 < \epsilon_1$ we will construct the linear combination of translates of B_h , with $h \geq 2$ of type

$$\varphi(x) = a_0 B_h(x - \epsilon_0) + a_1 B_h(x - \epsilon_1).$$

The Fourier transform of φ is given by

$$\widehat{\varphi}(w) = \left(a_0 e^{-i\epsilon_0 w} + a_1 e^{-i\epsilon_1 w} \right) \widehat{B}_h(w).$$

Using the Poisson summation formula

$$(-i)^j \sum_{k=-\infty}^{\infty} \varphi(u-k)(u-k)^j \sim \sum_{k=-\infty}^{\infty} \widehat{\varphi}^{(j)}(2\pi k) e^{i2\pi k u},$$

we obtain

$$\sum_{k=-\infty}^{\infty} \varphi(u-k) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(2\pi k) e^{i2\pi k u}.$$

We have

$$\widehat{B}_h(2\pi k) = \left(\frac{\sin(\pi k)}{\pi k} \right)^h = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

and hence

$$\widehat{\varphi}(2\pi k) = \begin{cases} a_0 + a_1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0. \end{cases}$$

Thus

$$\sum_{k=-\infty}^{\infty} \varphi(u-k) = a_0 + a_1.$$

Therefore, condition (i) is satisfied if $a_0 + a_1 = 1$. Now, we show that condition (iii) is also satisfied. Again from the Poisson summation formula, we obtain

$$(-i) \sum_{k=-\infty}^{\infty} \varphi(u-k)(u-k) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}'(2\pi k) e^{i2\pi k u}.$$

Also, we have

$$\widehat{\varphi}'(w) = (-i\epsilon_0 a_0 e^{-i\epsilon_0 w} - i\epsilon_1 a_1 e^{-i\epsilon_1 w}) \widehat{B}_h(w) + (a_0 e^{-i\epsilon_0 w} + a_1 e^{-i\epsilon_1 w}) \widehat{B}'_h(w)$$

Since $\widehat{B}'_h(2\pi k) = 0, \forall k$ which implies that $\widehat{\varphi}'(2\pi k) = 0$. Thus, we have

$$\widehat{\varphi}(0) = a_0 + a_1 = 1, \quad \widehat{\varphi}'(0) = \epsilon_0 a_0 + \epsilon_1 a_1 = 0.$$

Solving the above linear system we get the unique solution

$$a_0 = \frac{\epsilon_1}{\epsilon_1 - \epsilon_0}, \quad a_1 = -\frac{\epsilon_0}{\epsilon_0 - \epsilon_1}.$$

Moreover it is easy to see that the support of the function φ is contained in the interval $[\epsilon_0 - \frac{h}{2}, \epsilon_1 - \frac{h}{2}]$. Since $\varphi(u - k) = 0$ if $|u - k| > r$ for r sufficiently large, we have

$$\lim_{r \rightarrow \infty} \sum_{|k-u|>r} \varphi(u - k)(k - u)^2 = 0.$$

Condition (ii) is satisfied. Finally, we verify the condition that $M^\beta(\varphi) < \infty$.

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\varphi(u - k)||k - u|^\beta &= \sum_{|k-u|<R} |\varphi(u - k)||k - u|^\beta \\ &+ \sum_{|k-u|\geq R} |\varphi(u - k)||k - u|^\beta. \end{aligned}$$

We can see that $\sup_u |\{k : |u - k| < R\}| \leq N_0$. Thus, we get

$$M_\beta(\varphi) = \sum_{k \in \mathbb{Z}} |\varphi(u - k)||k - u|^\beta < \infty.$$

A particular Blackman-Harris kernel.

Next, we consider the Blackman-Harris kernel. For every $x \in \mathbb{R}$, we define the kernel (see [8])

$$\varphi(x) \equiv H(x) = \frac{1}{2} \text{sinc}(x) + \frac{9}{32} (\text{sinc}(x + 1) + \text{sinc}(x - 1)) - \frac{1}{32} (\text{sinc}(x + 3) + \text{sinc}(x - 3)),$$

where $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$. From [24], there holds that $H(x) = O(|x|^{-5})$ as $|x| \rightarrow \infty$. In view of [12], it follows that $M_2(H)$ is finite and

$$\lim_{r \rightarrow \infty} \sum_{|k-u|>r} |H(u - k)|(u - k)^2 = 0.$$

Indeed, there exists $N_0 > 0$ such that $|H(x)| \leq M/|x|^5$ for $|x| \geq N_0$. Thus, we have for $r > N_0$

$$\sum_{|k-u|>r} |H(u - k)|(u - k)^2 \leq M \sum_{|k-u|>r} \frac{1}{|u - k|^3} \leq \frac{M}{r} \sum_{|k-u|>r} \frac{1}{|u - k|^2} \leq \frac{2M}{r} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The Fourier transform of the function $H(x)$ is given by

$$\widehat{H}(w) = \frac{1}{\sqrt{2\pi}} \lambda\left(\frac{w}{\pi}\right),$$

where $\lambda(w) = \left(\frac{1}{2} + \frac{9}{16} \cos(\pi w) - \frac{1}{16} \cos(3\pi w)\right) \chi_{[-1,1]}(w)$, χ_I is the characteristic function of the set I . From Lemma 3 in [14], we obtain

$$m_1(H) = \sum_{k=-\infty}^{\infty} H(u-k)(u-k) = 0.$$

Hence the condition (i)-(iii) are satisfied. Finally, we verify that $M^\beta(\varphi) < \infty$.

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |H(u-k)|(k-u)^\beta &= \sum_{|k-u| < R} |H(u-k)|(k-u)^\beta \\ &+ \sum_{|k-u| \geq R} |H(u-k)|(k-u)^\beta = S_1 + S_2, \text{ (say)}. \end{aligned}$$

First, we consider S_2 . There exists $N > 0$ such that $|H(x)| \leq M/|x|^5$ for $|x| \geq N$. Thus, we have for $R > N$,

$$S_2 \leq M \sum_{|k-u| \geq R} \frac{|u-k|^\beta}{|u-k|^5} \leq 2M \sum_{k=1}^{\infty} \frac{1}{k^{5-\beta}}.$$

Next, we estimate S_1 . We have $\sup_u |\{k : |u-k| < R\}| < \infty$. Thus, we obtain

$$S_1 \leq \sum_{|k-u| < R} |H(u-k)||k-u|^\beta \leq \sum_{|k-u| < R} M_0 R^\beta \leq [2R] M_0 R^\beta,$$

where $[x]$ denotes the smallest integer greater than or equal to x . Hence, we get

$$M_\beta(H) = \sum_{k \in \mathbb{Z}} |H(u-k)|(k-u)^\beta < \infty.$$

Thus, all the conditions are satisfied for the function $H(x)$.

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