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A New Aspect of Comrade Matrices by Reachability Matrices

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Abstract. In this paper, we study orthanogonal polynomials by looking at their comrade matrices and reachability matrices. First, we focus on the algebraic structure that is exhibited by comrade matrices. Then, we explain some properties of this algebraic structure which helps us to find a connection between comrade matrices and reachability matrices. In the last section, we use this connection to determine the determinant, eigenvalues, and eigenvectors of these matrices. Finally, we derive a factorization for *det* $R(A, x)$, where $R(A, x)$ is the reachability matrix for a comrade matrix A and x is a vector of indeterminates.

1. Introduction

Some recurrence relations from the general theory of orthogonal polynomials are shown in [1, 2, 4], these relations are given in terms of comrade matrices. Comrade matrices have regular form and may be used in finding the roots of a polynomial that an important problem in the numerical analysis.

As we know, Frobenius's original idea used of companion matrices to find the zeros of a polynomial or a function. It can be expressed by some limitations and conditions for the condition number and floating point arithmetic [3, 4].

Specht, Boyd and Good et al. [5, 8] used this structure for finding the roots of a polynomial in Chebyshev form, for their method of rootfinding-by-proxy, and for Chebyshev interpolation. They derived these works using the Chebyshev-Frobenius matrix, which is also known as a colleague matrix.

The colleague matrix and companion matrix are known to be special cases of comrade matrices. In this paper, we try to use their algebraic structures to

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explain a connection between comrade matrices and reachability matrices. Using this connection, we then find the determinant, eigenvalues, and eigenvectors of their algebraic structures.

This paper is organized as follows: Some necessary details about comrade matrices and orthogonal polynomials are presented in Section 2. A connection between comrade matrices and reachability matrices is introduced in Section 3. Some results of this connection are given in Section 4. Finally, a summary is given in Section 5.

2. Preliminaries

$$p(x) = x^n - c_{n-1}x^{n-1} - \ldots - c_1 x - c_0 \in \mathbb{F}[x]. \tag{2.1}$$

We can find a companion matrix

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & c_0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & c_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & c_{n-2} \\ 0 & 0 & 0 & \ldots & 0 & 1 & c_{n-1} \end{pmatrix}. \tag{2.2}$$

We limit ourselves to the case that $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, the fields of real and complex numbers respectively. If the polynomial is in generalized form, i.e. has a basis of orthogonal polynomials $\{\mathbf{q}_i(x)\}$. Then we have an analogue to the companion matrix that is called the comrade matrix. Let an orthogonal basis $\{\mathbf{q}_i(x)\}$, be defined by the standard relations

$$\mathbf{q}_0(x) = 1, \ \mathbf{q}_1(x) = a_1 x + b_1, \tag{2.3}$$

$$\mathbf{q}_i(x) = (a_i x + b_i)\mathbf{q}_{i-1}(x) - c_i \mathbf{q}_{i-2}(x), \ i = 2, 3, \ldots,$$

with $a_i > 0, \quad c_i > 0$. We write an nth degree generalized polynomial as

$$d(x) = d_n \mathbf{q}_n(x) - d_{n-1}\mathbf{q}_{n-1}(x) - \ldots - d_1 \mathbf{q}_1(x) - d_0 \mathbf{q}_0(x). \tag{2.4}$$

If we write $\mathbf{q}_i(x) = \sum_{j=0}^{i} q_{ij} x^i, \ i = 1, 2, \ldots, n$, then the leading coefficient of $\mathbf{q}_i(x)$ is $q_{ii} = a_1 a_2 \ldots a_i > 0$.

Assume, without loss of generality, that $q_{00} = 1$ and $d_n = 1$, then the corresponding relationship between $p(x)$ in (2.1) and $d(x)$ in (2.4) will be

$$\tilde{d}(x) = d(x)/(a_1 \ldots a_n) \tag{2.5}$$

$$= x^n - \tilde{d}_{n-1}x^{n-1} - \ldots - \tilde{d}_1 x - \tilde{d}_0 = p(x).$$

Then, the relationship between the coefficients in (2.4) and (2.5) will be

$$(a_1a_2\dots a_n)[c_0, c_1, \dots, c_{n-1}, 1] = [d_0, d_1, \dots, d_{n-1}, 1]Q_{n+1} \tag{2.6}$$

where

$$Q_{n+1} = \begin{pmatrix} 1 & 0 & & \dots & 0 \\ q_{10} & q_{11} & 0 & \dots & 0 \\ q_{20} & q_{21} & q_{22} & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ q_{n0} & q_{n1} & q_{n2} & \dots & q_{nn} \end{pmatrix}. \tag{2.7}$$

By properties associated with $d(x)$ in (2.4) without using (2.6), the comrade matrix is introduced by:

$$A = \begin{pmatrix} \frac{-b_1}{a_1} & \frac{1}{a_1} & 0 & \dots & & & \frac{d_0}{a_n} \\ \frac{c_2}{a_2} & \frac{-b_2}{a_2} & \frac{1}{a_2} & 0 & \dots & & \frac{d_1}{a_n} \\ 0 & \frac{c_3}{a_3} & \frac{-b_3}{a_3} & \frac{1}{a_3} & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \frac{d_{n-2}+c_n}{a_n} \\ 0 & \dots & & \frac{c_{n-1}}{a_{n-1}} & \frac{-b_{n-1}}{a_{n-1}} & \frac{1}{a_{n-1}} & \frac{d_{n-1}-b_n}{a_n} \end{pmatrix}. \tag{2.8}$$

When $a_i = 1,\ b_i = c_i = 0$, for all $1 \le i \le n$ then (2.8) reduces to the companion matrix from (2.2).

The colleague matrix deduced from Chebyshev polynomials in (2.3) and matrix A in (2.8).

3. Comrade Matrices and Reachability Matrices

Let $A \in \mathbb{F}^{n\times n},\ \mathbf{b} \in \mathbb{F}^n$. Then the matrix

$$R(A, \mathbf{b}) = [\mathbf{b}, A\mathbf{b}, \dots, A^{n-1}\mathbf{b}] \in \mathbb{F}^{n\times n},$$

is the reachability matrix of the pair (A, b). Also, the Krylov matrix of A and $\mathbf{b}$ is defined by $K(A, \mathbf{b}) := [\mathbf{b}\ A\mathbf{b}\ A^2\mathbf{b}\ \dots\ A^{n-1}\mathbf{b}]$, so this is a reachability matrix. Krylov matrices of C^T are Hankel matrices, see [6, 9]. Let $p(x) = x^n - 1$. Then Krylov matrices of C would be Circulant matrices.
The comrade matrix in (2.8) is nonderogatory and cyclic with the monic characteristic polynomial of (2.4), i.e.,

$$det(xI - A) = d(x)/(a_1a_2\dots a_n). \tag{3.1}$$

Also, A is similar to C associated with (2.7), i.e.

$$A = Q_nCQ_n^{-1}. \tag{3.2}$$

Easily, we can determine

$$R(C, \mathbf{b}) = Q_n^{-1} R(A, Q_n \mathbf{b}). \tag{3.3}$$

Let us define the generating polynomial of

$$\mathbf{b} = [b_0, b_1, \ldots, b_{n-1}]^T \in \mathbb{F}^{n \times 1} \textit{ by } \mathbf{b}(x) = \sum_{k=0}^{n-1} b_k x^k.$$

Then

$$R(C, \mathbf{b}) = \sum_{k=0}^{n-1} b_k R(C, \mathbf{e_k}) = \sum_{k=0}^{n-1} b_k C^k = \mathbf{b}(C) = Q_n^{-1} R(A, Q_n \mathbf{b}), \tag{3.4}$$

where $\mathbf{e_k}$ stands for the $(k+1)$th unit vector in the standard basis of $\mathbb{F}^{n\times 1}$.

Theorem 3.1. *If C be the companion matrix of $p(x)$ similar to (2.2), A be the comrade matrix of $d(x)$ similar to (2.8) and $\mathbf{a}(x) = \sum_{k=0}^{n-1} a_k x^k$ be the generating polynomial of $\mathbf{a} = [a_0, a_1, \ldots, a_{n-1}]^T \in \mathbb{F}^{n\times 1}$ then the reachability matrix of the pair $(C, \mathbf{a})$ will be similar as the generating polynomial A.*

Proof. From (3.2) and (3.4), we have

$$R(C, \mathbf{a}) = \mathbf{a}(C) = \sum_{i=0}^{n-1} a_i R(C, \mathbf{e}_i) = \sum_{i=0}^{n-1} a_i C^i = \sum_{i=0}^{n-1} a_i Q_n^{-1} A^i Q_n \tag{3.5}$$

$$= Q_n^{-1} \sum_{i=0}^{n-1} a_i A^i Q_n = Q_n^{-1} \mathbf{a}(A) Q_n. \qquad \square$$

Now, we introduce an algebraic structure of the set

$$\mathcal{S}(A) = \{Q_n^{-1} R(A, Q_n \mathbf{b}) | \mathbf{b} \in \mathbb{F}^{n\times 1}\}.$$

For any $\mathbf{a},\ \mathbf{b} \in \mathbb{F}^{n\times 1}$ and from (3.4), we derive

$$Q_n^{-1} R(A, Q_n \mathbf{a}) + Q_n^{-1} R(A, Q_n \mathbf{b}) = R(Q_n^{-1} A Q_n, \mathbf{a}) + R(Q_n^{-1} A Q_n, \mathbf{b}) \tag{3.6}$$

$$= R(C, \mathbf{a}) + R(C, \mathbf{b}) = R(C, \mathbf{a} + \mathbf{b})$$

$$= Q_n^{-1} R(A, Q_n(\mathbf{a} + \mathbf{b})).$$

From (3.3) and (3.4), we have

$$Q_n^{-1} R(A, Q_n \mathbf{a}) \mathbf{b} = R(Q_n^{-1} A Q_n, \mathbf{a}) \mathbf{b} \tag{3.7}$$

$$= R(C, \mathbf{a})\mathbf{b} = R(C, \mathbf{b})\mathbf{a} = Q_n^{-1} R(A, Q_n\mathbf{b})\mathbf{a}.$$

Also

$$Q_n^{-1} R(A, Q_n\mathbf{a}) Q_n^{-1} R(A, Q_n\mathbf{b}) = Q_n^{-1} R(A, \mathbf{b}(A) Q_n\mathbf{a}),$$

$Q_n^{-1} R(A, Q_n\mathbf{e}_0)$ is unity element of commutative ring $\mathcal{S}(A)$.

Theorem 3.2. *Let polynomial $d(x) \in \mathbb{F}[x]$ be the same as (2.4). Then*

$$\mathcal{S}(A) \simeq \mathbb{F}[x]/ \prec d(x) \succ .$$

Also, if $d(x)$ is irreducible over $\mathbb{F}$ then $\mathcal{S}(A)$ is a field.

Proof. Let $\mathcal{K}(C) = \{K(C, \mathbf{b}) | \mathbf{b} \in \mathbb{F}^{n\times 1}\}$. We can easily see, $\mathcal{S}(A)$ is isomorphism with $\mathcal{K}(C)$, then ideal $\prec d(x) \succ$ of $\mathbb{F}[x]$ is generated by $d(x)$ is similar as ideal $\prec p(x) \succ$ of $\mathbb{F}[x]$, see relations (2.5) and (3.1). Then by Theorem 2.1 and Corollary 2.2 in [6], the proof is completed. □

Theorem 3.3. *Let $d(x) \in \mathbb{F}[x]$ be the same as (2.4). A reachability matrix similar to $R(A, Q_n\mathbf{a})$ is invertible if and only if $gcd(\mathbf{a}(x), d(x)) = 1$, then*
$R(A, Q_n\mathbf{a})^{-1} = Q_n^{-1} R(A, Q_n\mathbf{b}) Q_n^{-1}$ and $\mathbf{b}$ is determined by the generating polynomial $\mathbf{b}(x)$ such that

$$d(x)q(x) + \mathbf{a}(x)\mathbf{b}(x) = 1, \quad q(x) \in \mathbb{F}[x].$$

Proof. For proof see Theorem 3.2. in this paper and Theorem 2.3 in [6]. □

4. Some Results

Let $\lambda_0, \dots, \lambda_{n-1}$ be the distinct eigenvalues of the companion matrix C in (2.2) and $\mathbf{m_0}, \dots, \mathbf{m_{n-1}}$ be the eigenvectors of C associated with the eigenvalues λ_i's. Then λ_i's will be the eigenvalues of the comrade matrix A in (2.8) with the eigenvectors

$$\mathbf{u}(\lambda_i) = [1, q_1(\lambda_i), \dots, q_{n-1}(\lambda_i)]^T, \; i = 0, 1, \dots, n-1. \tag{4.1}$$

$M_C = [\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_{n-1}]$, will be modal matrix of C, $D = diag[\mathbf{a}(\lambda_0) \dots \mathbf{a}(\lambda_{n-1})]$ and $N = [\mathbf{u}(\lambda_0), \mathbf{u}(\lambda_1), \dots, \mathbf{u}(\lambda_{n-1})]$ will be the generalized Vandermonde matrix. Then from [1, 6], we obtain:

$$M_C^{-1} C M_C = D \;\; \text{and} \;\; N^{-1} A N = D. \tag{4.2}$$

Theorem 4.1. *Let A be the comrade matrix of the generalized polynomial $d(x)$ similar to (2.8) and $\mathbf{a}(x) = \sum_{k=0}^{n-1} a_k x^k$ be the generating polynomial of $\mathbf{a} = [a_0, a_1, \dots, a_{n-1}]^T \in \mathbb{F}^{n\times 1}$ and $Q_n^{-1} R(A, Q_n\mathbf{a})$ be an element of $\mathcal{S}(A)$. Then*

$\mathbf{a}(\lambda_0), \ldots, \mathbf{a}(\lambda_{n-1})$ *are the eigenvalues of* $R(A, \mathbf{a})$ *with eigenvectors* $\mathbf{u}(\lambda_0), \ldots, \mathbf{u}(\lambda_{n-1})$.

Proof. If $\mathbf{a}(x)$ will be the generating polynomial of $\mathbf{a} = [a_0, \ldots, a_{n-1}]^T$, then

$$R(C, \mathbf{a})\mathbf{m}_i = \mathbf{a}(\lambda_i)\mathbf{m}_i, \tag{4.3}$$

and

$$R(A, Q_n\mathbf{a})M_C = Q_n D M_C. \tag{4.4}$$

Namely diagonal elements of matrix:

$$Q_n D = \begin{pmatrix} \mathbf{a}(\lambda_0) & 0 & \ldots & 0 \\ q_{10}\mathbf{a}(\lambda_0) & q_{11}\mathbf{a}(\lambda_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{n-10}\mathbf{a}(\lambda_0) & q_{n-11}\mathbf{a}(\lambda_1) & \ldots & q_{nn}\mathbf{a}(\lambda_{n-1}), \end{pmatrix} \tag{4.5}$$

are eigenvalues of matrix $R(A, Q_n\mathbf{a})$, then M_C will be the modal matrix of C. We have $\mathbf{a}(\lambda_i) = \mathbf{a}^T\Lambda_i$, that $\Lambda_i = [1, \lambda_i, \ldots, \lambda_i^{n-1}]^T$. $M_{C^T} = [\Lambda_1, \ldots, \Lambda_{n-1}]$ is the modal matrix of C^T and $N = Q_n M_{C^T}$. If we write:

$$R(A, \mathbf{a})\mathbf{u}(\lambda_i) = Q_n\mathbf{a}(C)Q_n^{-1}\mathbf{u}(\lambda_i) \tag{4.6}$$

$$= Q_n\mathbf{a}(C)Q_n^{-1}Q_n M_{C^T}\mathbf{e}_i = \mathbf{a}(\lambda_i)Q_n\Lambda_i = \mathbf{a}(\lambda_i)\mathbf{u}(\lambda_i)$$

then $\mathbf{a}(\lambda_0), \ldots, \mathbf{a}(\lambda_{n-1})$ are the eigenvalues of $R(A, \mathbf{a})$ with eigenvectors $\mathbf{u}(\lambda_0), \ldots, \mathbf{u}(\lambda_{n-1})$. □

Now we derive:

$$det\ R(A, \mathbf{a}) = \prod_{i=0}^{n-1} \mathbf{a}^T\Lambda_i \ or\ det\ R(A, \mathbf{a}) = \prod_{i=0}^{n-1} \mathbf{a}(\lambda_i). \tag{4.7}$$

If S be the companion matrix of x^n and M_{C^T} be the modal matrix of C^T, then $M_C = R(S^T, \mathbf{w})M_{C^T}$, that $\mathbf{w} = [-c_1, \ldots, -c_{n-2}, -c_{n-1}, 1]^T$.
Let $y = Q_n\mathbf{a}$, easily we can see

$$R(A, y)M_C = Q_n R(C, \mathbf{a})R(S^T, \mathbf{v})Q_n^{-1}N. \tag{4.8}$$

Also, we obtain

$$det\ R(A, y) = det\ Q_n\ det\ R(C, \mathbf{a}), \tag{4.9}$$

we know that

$$det\ Q_n = q_{11}q_{22}\ldots q_{n-1n-1} \ \ and\ \ det\ R(C, \mathbf{a}) = \prod_{i=0}^{n-1} \mathbf{a}^T\Lambda_i.$$

Then

$$\det R(A,y)=\prod_{i=0}^{n-1} q_{ii}\mathbf{a}^T\Lambda_i. \tag{4.10}$$

From $\det R(S^T,\mathbf{w})=(-1)^{\lfloor \frac{n}{2}\rfloor}$ and relation (13) in [6], we have:

$$\det R(A^T,\mathbf{a})=(-1)^{\lfloor \frac{n}{2}\rfloor}\prod_{i=0}^{n-1} q_{ii}\mathbf{a}^T\mathbf{m}_i. \tag{4.11}$$

Let $Z=Q_n/\det Q_n$ from (3.5), we have:

$$R(A,\mathbf{a})=Z^{-1}Q_n^{-1}Z\mathbf{a}(A)Q_n=Z^{-1}(Z\mathbf{a})(C). \tag{4.12}$$

Then $\det R(A,\mathbf{a})=\det Z\mathbf{a}(C)$.

Now we derive the following theorem:

Theorem 4.2. *Let A be a comrade matrix with distinct eigenvalues and characteristic polynomial $\tilde{d}(x)$ in (2.5) that*

$$\tilde{d}(x)=d_1(x)^{m_1}\dots d_r(x)^{m_r}, \tag{4.13}$$

be the prime factorization of $\tilde{d}(x)$ and $Z^{-1}AZ=C$, (let C be the companion matrix similar as (2.2) *and $Z=Q_n/\det Q_n$). Set $f_j(x)=\det\,(Zx)(C_{d_j})$, $\quad j=1,\dots,r$. Then $f_j(x)=\det\,(Zx)(A_{d_j})$ and*

$$\det R(A,x)=(f_1(x))^{m_1}\dots(f_r(x))^{m_r} \tag{4.14}$$

that $f_1(x),\dots,f_r(x)$ are irreducible and homogeneous of degree $l_1,\dots,l_r$, respectively.

Proof.

$$(Zx)(C_{d_j})=\mathbf{b}(C_{d_j})=\sum_{i=0}^{n-1} b_iC_{d_j}^i=\sum_{i=0}^{n-1} b_iQ_{d_j}^{-1}A_{d_j}^iQ_{d_j}$$

$$=Q_{d_j}^{-1}\sum_{i=0}^{n-1} b_iA_{d_j}^iQ_{d_j}=Q_{d_j}^{-1}\mathbf{b}(A_{d_j})Q_{d_j}=Q_{d_j}^{-1}(Zx)(A_{d_j})Q_{d_j},$$

then $f_j(x) = det\ (Zx)(C_{d_j}) = det\ (Zx)(A_{d_j}), \quad j = 1, \ldots, r$. Now let

$$C(d_j, m_j) = \begin{pmatrix} C_{d_j} & I_{l_j} & 0 & . & 0 \\ 0 & C_{d_j} & I_{l_j} & . & 0 \\ . & . & . & . & . \\ . & . & . & C_{d_j} & I_{l_j} \\ . & . & . & . & C_{d_j} \end{pmatrix}_{m_j l_j \times m_j l_j}$$

$$= \begin{pmatrix} Q_{d_j}^{-1} A_{d_j} Q_{d_j} & I_{l_j} & 0 & . & 0 \\ 0 & Q_{d_j}^{-1} A_{d_j} Q_{d_j} & I_{l_j} & . & 0 \\ . & . & . & . & . \\ . & . & . & Q_{d_j}^{-1} A_{d_j} Q_{d_j} & I_{l_j} \\ . & . & . & . & Q_{d_j}^{-1} A_{d_j} Q_{d_j} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} Q_{d_j}^{-1} & 0 & . & 0 \\ 0 & Q_{d_j}^{-1} & . & 0 \\ . & . & . & . \\ . & . & Q_{d_j}^{-1} & 0 \\ . & . & . & Q_{d_j}^{-1} \end{pmatrix}}_{\mathbf{Q}_{d_j}^{-1}} \underbrace{\begin{pmatrix} A_{d_j} & I_{l_j} & . & 0 \\ 0 & A_{d_j} & I_{l_j} & . \\ . & . & . & . \\ . & . & A_{d_j} & I_{l_j} \\ . & . & . & A_{d_j} \end{pmatrix}}_{\mathbf{A}_{d_j}}$$

$$\underbrace{\begin{pmatrix} Q_{d_j} & 0 & . & 0 \\ 0 & Q_{d_j} & . & 0 \\ . & . & . & . \\ . & . & Q_{d_j} & 0 \\ . & . & . & Q_{d_j} \end{pmatrix}}_{\mathbf{Q}_{d_j}}.$$

Then for some $T \in \mathbb{F}^{n \times n}$:

$$\begin{aligned} TCT^{-1} &= diag\,(C(d_1, m_1), \ldots, C(d_r, m_r)) \\ &= diag\left(\mathbf{Q}_{d_1}^{-1} \mathbf{A}_{d_1} \mathbf{Q}_{d_1}, \ldots, \mathbf{Q}_{d_r}^{-1} \mathbf{A}_{d_r} \mathbf{Q}_{d_r}\right) \\ &= \underbrace{\begin{pmatrix} \mathbf{Q}_{d_1}^{-1} & 0 & . & 0 \\ 0 & \mathbf{Q}_2^{-1} & . & 0 \\ . & . & . & . \\ . & . & . & \mathbf{Q}_r^{-1} \end{pmatrix}}_{\mathbf{Q}^{-1}} \underbrace{\begin{pmatrix} \mathbf{A}_{d_1} & 0 & . & 0 \\ 0 & \mathbf{A}_{d_2} & . & 0 \\ . & . & . & . \\ . & . & . & \mathbf{A}_{d_r} \end{pmatrix}}_{\mathbf{A}} \\ &\quad \underbrace{\begin{pmatrix} \mathbf{Q}_{d_1} & 0 & . & 0 \\ 0 & \mathbf{Q}_{d2} & . & 0 \\ . & . & . & . \\ . & . & . & \mathbf{Q}_{d_r} \end{pmatrix}}_{\mathbf{Q}}, \end{aligned} \tag{4.15}$$

will be the rational canonical form (the Frobenius canonical form) of C.
If $\hat{\mathbf{b}} \in \mathbb{F}^n$ then

$$R(C, \hat{\mathbf{b}}) = \hat{\mathbf{b}}(C) = T^{-1}\hat{\mathbf{b}}\,[diag\,(C(d_1, m_1), \ldots, C(d_r, m_r))]\,T$$

$$= T^{-1}\mathbf{Q}^{-1}\hat{\mathbf{b}}\,[diag\,(\mathbf{A}_{d_1}, \ldots, \mathbf{A}_{d_r})]\,\mathbf{Q}T,$$

and

$$det\ \hat{\mathbf{b}}(C) = \Big(det\ \hat{\mathbf{b}}(\mathbf{A}_{d_1})\Big)^{m_1} \ldots \Big(det\ \hat{\mathbf{b}}(\mathbf{A}_{d_r})\Big)^{m_r}. \tag{4.16}$$

Then proof is completed by Theorem 3.2 in [7]. □

5. Summary

We try to show a connection between comrade matrices and reachability matrices. For this work, we introduce an algebraic structure and some properties of this structure. Then, using this structure, we find the determinant, eigenvalues, and eigenvectors of the reachability matrices. Also, we derive a factorization of the polynomial $det\ R(A, x)$, for a comrade matrix A and a vector of indeterminates x. We can refer to Chebyshev polynomials and the colleague matrices for an experimental investigation of this connection.

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