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Active Nonlinear Vibration Absorber for a Nonlinear System with a Time Delay Acceleration Feedback under the Internal Resonance, Sub-harmonic, Superharmonic and Principal Parametric Resonance Conditions Simultaneously

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Abstract

In this paper, dynamic analysis of a nonlinear active vibration absorber is conducted with a time delay acceleration feedback to suppress the vibration of a nonlinear single degree of freedom primary system. The primary system consisting of linear and nonlinear cubic springs, mass, and damper is subjected to the multi-harmonic hard excitation with a parametric excitation. It is proposed to reduce the vibration of the primary system and the absorber by using a lead zirconate titanate (PZT) stack actuator in series with a spring in the absorber which configures as an active vibration absorber. The method of multiple scales (MMS) is used to obtain the approximate solution of the system under the internal resonance, subharmonic, superharmonic, and principal parametric resonance conditions simultaneously. Frequency and time responses of the system are investigated considering a delay in the feedback for the various parameters of the absorber configuration and controlling force.

Keywords : Dynamic vibration absorber, Method of multiple scales, PZT actuator

1. Introduction

Dynamic vibration absorber (DVA) suppress the vibration of the primary system by shifting its resonant natural frequency. Tuned vibration absorber absorbs the vibration of the primary system i.e. the amplitude of vibration of the primary system becomes zero at its natural frequency. But the response amplitudes of both the primary system and the absorber become very large at their modal frequencies [1, 2]. Hence in these systems dampers are used to suppress the vibration. The mass ratio between the absorber and the primary system plays a significant role in the suppression of vibration, which is considered as 1:20 in most of the literature [3, 4]. The high mass ratio increases the overall structural weight of the system. However, by the use of various optimization technique, active materials and new model design, the vibration of the primary system can be reduced for a larger band of frequencies [5, 7] with minimum absorber weight. Many dynamical systems are inherently nonlinear due to prolonged use and applications [8]. Therefore the nonlinear vibration absorber and the nonlinear primary systems are more practical in nature. In such systems, the active vibration absorber is more useful than the passive one. In present work, the vibration of a nonlinear primary system with a spring-mass damper is suppressed by an active vibration absorber consisting of a PZT stack actuator in series with a spring as shown in Fig.1. The mass ratio between the absorber and the primary system is considered as 1:50. The analysis is carried out by considering time delay in the acceleration feedback of the primary system which has not been explored extensively in the literature. In the proposed model the primary system is subjected to a multi-harmonic hard excitation and a parametric excitation. The proposed system is similar to the model discussed in reference [9, 10] where the coupled pitch and roll motion of the ship was considered. The governing nonlinear equations of the system are solved by using the method of multiple scales under internal resonance, sub-harmonic, super-harmonic and parametric resonance conditions simultaneously.

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2. Mathematical Modeling

In this proposed model m_i, c_i and k_i denotes mass, damping and stiffness of the primary system and the DVA respectively for i = 1, 2. k_3 denotes the stiffness of the spring connected in series with PZT actuator of stiffness k_P^E and F_c indicates the actuating force of the combination. The terms k_{13} and k_{23} denotes cubic nonlinear stiffness in the primary system and absorber respectively. Two multi-harmonic excitation force of $F_{11}\cos(\Omega_1 t)$, $F_{21}\cos(\Omega_2 t)$ and a parametric excitation force of $x_1 F_{31}\cos(\Omega_3 t)$ are acting on the primary system. The governing equation the system is described by two ordinary coupled differential equations as given below.

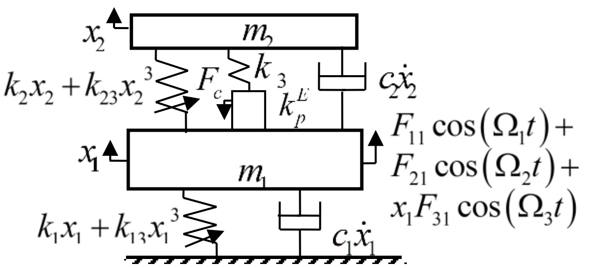

Fig. 1. Nonlinear active vibration absorber

$$m_1\ddot{x}_1 + k_1 x_1 + k_{13} x_1^3 + k_2(x_1 - x_2) + k_{23}(x_1 - x_2)^3 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) = F_{11}\cos\Omega_1 t + F_{21}\cos\Omega_2 t + x_1 F_{31}\cos\Omega_3 t - F_c \quad (1)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_{23}(x_2 - x_1)^3 + c_2(\dot{x}_2 - \dot{x}_1) = F_c \quad (2)$$

The controlling force can be written as $F_c = k_r(x_1 + \delta_0 - x_2)$ where $k_r = (k_3 k_p^E)/(k_3 + k_p^E)$. The nominal displacement of the PZT actuator δ_0 can be expressed [11] as $\delta_0 = n d_{33} V$, where n, d_{33}, k_c and τ_d are the number of wafers, dielectric charge constant, controller gain and time delay in the feedback system respectively. Considering the effect of time delay in the acceleration feedback of the primary system the voltage developed in the PZT actuator is expressed as $V = k_c(\ddot{x}_1(t - \tau_d))$. Substituting the value of F_c with $\omega_{n1} = \sqrt{k_1 / m_1}$ and $\tau = \omega_{n1} t$ Eqs. (1) and (2) are non-dimensioned. A small bookkeeping parameter ε is considered for ordering Eqs. (1) and (2) which can be expressed as

$$\ddot{x}_1 + \omega_1^2 x_1 = \varepsilon\mu\omega_2^2 x_2 - \varepsilon\alpha_{13} x_1^3 - \varepsilon^2\alpha_{23c}(x_1 - x_2)^3 - \varepsilon^2(z_1 + z_{12})\dot{x}_1 + \varepsilon^2 z_{12}\dot{x}_2 + F_1\cos(\Omega_1\tau) + F_2\cos(\Omega_2\tau) + \varepsilon x_1 F_3\cos(\Omega_3\tau) - \varepsilon^2 F_{c1}\ddot{x}_1(\tau - \tau_d) \quad (3)$$

$$\ddot{x}_2 + \omega_2^2 x_2 = \varepsilon\frac{F_{c1}\ddot{x}_1(\tau - \tau_d)}{\mu} + \omega_2^2 x_1 - \varepsilon\alpha_{23}(x_2 - x_1)^3 - \varepsilon z_2(\dot{x}_2 - \dot{x}_1) \quad (4)$$

where

$$\omega_1 = \sqrt{\left(\frac{k_1 + k_2 + k_r}{m_1\omega_{n1}^2}\right)},\ \mu = \frac{m_2}{\varepsilon m_1\omega_{n1}^2},\ \omega_2 = \sqrt{\frac{k_2 + k_r}{m_2\omega_{n1}^2}},$$

$$\alpha_{13} = \frac{k_{13}}{\varepsilon m_1\omega_{n1}^2},\ \alpha_{23c} = \frac{k_{23}}{\varepsilon^2 m_1\omega_{n1}^2},\ \alpha_{23} = \frac{k_{23}}{\varepsilon m_2\omega_{n1}^2}, z_1 = \frac{c_1}{\varepsilon^2 m_1\omega_{n1}},$$

$$z_{12} = \frac{c_2}{\varepsilon^2 m_1\omega_{n1}},\ z_2 = \frac{c_2}{\varepsilon m_2\omega_{n1}},\ F = \frac{F_{111}}{m_1\omega_{n1}^2},\ F_2 = \frac{F_{21}}{m_1\omega_{n1}^2},$$

$$F_{31} = \frac{F_3}{\varepsilon m_1\omega_{n1}^2}, F_{c1} = \frac{k_r k_c n d_{33}}{\varepsilon^2 m_1\omega_{n1}^2},\ \Omega_1 = \frac{\Omega_{11}}{\omega_{n1}},\ \Omega_2 = \frac{\Omega_{21}}{\omega_{n1}}, \Omega_3 = \frac{\Omega_{31}}{\omega_{n1}}$$

Method of multiple scales is used to obtain the approximate solution of the Eq. (3) and (4).

2.1 Approximate Solution Using MMS

Following the standard procedure of MMS [12], the perturbed solutions and time derivatives in the new time scale $T_n (n = 0, 1, 2, ...)$ are expanded as

$$x_1 = x_{10}(\tau_0, \tau_1) + \varepsilon x_{11}(\tau_0, \tau_1) \quad (5)$$

$$x_1(\tau - \tau_d) = x_{10}(\tau_0 - \tau_d, \tau_1 - \varepsilon\tau_d) + \varepsilon x_{11}(\tau_0 - \tau_d, \tau_1 - \varepsilon\tau_d) \quad (6)$$

$$x_2 = x_{20}(\tau_0, \tau_1) + \varepsilon x_{21}(\tau_0, \tau_1) \quad (7)$$

The time scales can be written as $T_n = \varepsilon^n \tau$. Time derivati ves along different time scales lead to the differential ope rators as $\frac{d}{d\tau} = D_0 + \varepsilon D_1 + ...$ and $\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + ...$ where $D_n = \partial/\partial T_n$. Substituting above equations into Eqs. (3) and (4), and collecting the coefficients of ε^n and eq uating them to zero, the following equations can be obtai ned.

$$\varepsilon^0 :\ D_0^2 x_{10} + \omega_1^2 x_{10} = F_1\cos(\Omega_1\tau_0) + F_2\cos(\Omega_2\tau_0) \quad (8a)$$

$$\varepsilon^0 : D_0^2 x_{20} + \omega_2^2 x_{20} = \omega_2^2 x_{10} \quad (8b)$$

$$\varepsilon^1 :\ D_0^2 x_{11} + \omega_1^2 x_{11} = -2D_0 D_1 x_{10} + \mu\omega_2^2 x_{20} - \alpha_{13} x_{10}^3 + x_{10} F_3\cos(\Omega_3\tau_0) \quad (8c)$$

$$\varepsilon^1 :\ D_0^2 x_{21} + \omega_2^2 x_{21} = -2D_0 D_1 x_{20} + F_{c1} D_0^2 x_{10}(\tau_0 - \tau_d)/\mu + \omega_2^2 x_{11} + z_2 D_0 x_{10} - z_2 D_0 x_{20} + \alpha_{23} x_{10}^3 - 3\alpha_{23} x_{10}^2 x_{20} + 3\alpha_{23} x_{10} x_{20}^2 - \alpha_{23} x_{20}^3 \quad (8d)$$

The solution of the Eq. (8a) can be written as

$$x_{10} = A_1\exp(i\omega_1\tau_0) + \Lambda_1\exp(i\Omega_1\tau_0) + \Lambda_2\exp(i\Omega_2\tau_0) + cc \quad (9)$$

where $\Lambda_1 = \frac{F_1}{2(\omega_1^2 - \Omega_1^2)}$ $\Lambda_2 = \frac{F_2}{2(\omega_1^2 - \Omega_2^2)}$, and 'cc' stands for the complex conjugate of the preceding terms. and $A_1(\tau_1)$ is an unknown complex function of time which will be determined later. Considering Ω_1 and Ω_2 away from ω_1 and substituting Eq. (9) into Eq. (8b) the following equation can be obtained.

$$x_{20} = A_2\exp(i\omega_2\tau_0) + \Lambda_3 A_1\exp(i\omega_1\tau_0) + \Lambda_4\exp(i\Omega_1\tau_0) + \Lambda_5\exp(i\Omega_2\tau_0) + cc \quad (10)$$

where

$$\Lambda_3 = \frac{\omega_2^{\ 2}}{\omega_2^{\ 2} - \omega_1^{\ 2}},\ \Lambda_4 = \frac{\Lambda_1 \omega_2^{\ 2}}{\left(\omega_2^{\ 2} - \Omega_1^{\ 2}\right)},\ \Lambda_5 = \frac{\Lambda_2 \omega_2^{\ 2}}{2\left(\omega_2^{\ 2} - \Omega_2^{\ 2}\right)}$$

Substituting Eqs. (9) and (10) into Eq. (8c) and (8d) respectively, the corresponding equations are reduced as

$$\begin{aligned}
&D_0^{\ 2} x_{11} + \omega_1^{\ 2} x_{11} = -2iD_1 \begin{pmatrix} \omega_1 A_1 \exp\left(i\omega_{n1}\tau_0\right) + \Omega_1 \Lambda_1 \exp\left(i\Omega_1 \tau_0\right) \\ +\Omega_2 \Lambda_2 \exp\left(i\Omega_2 \tau_0\right) \end{pmatrix} \\
&+\mu\omega_2^{\ 2} \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} - \\
&\alpha_{13}\left(A_1 \exp\left(i\omega_{n1}\tau_0\right) + \Lambda_1 \exp\left(i\Omega_1\tau_0\right) + \Lambda_2 \exp\left(i\Omega_2\tau_0\right)\right)^3 + \\
&\frac{F_3}{2} \begin{pmatrix} A_1 \exp\left(i\omega_{n1}\tau_0\right) + \Lambda_1 \exp\left(i\Omega_1\tau_0\right) \\ +\Lambda_2 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} \begin{pmatrix} \exp\left(i\Omega_3\tau\right) + \\ \exp\left(-i\Omega_3\tau\right) \end{pmatrix} + cc
\end{aligned} \tag{11}$$

$$\begin{aligned}
&D_0^2 x_{21} + \omega_2^2 x_{21} = -2D_0 D_1 \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) \\ +\Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} \\
&\frac{F_{c1}}{\mu} D_0^2 \begin{pmatrix} A_2 \exp\left(i\omega_2\left(\tau_0 - \tau\right)\right) + \Lambda_3 A_1 \exp\left(i\omega_1\left(\tau_0 - \tau\right)\right) + \\ \Lambda_4 \exp\left(i\Omega_1\left(\tau_0 - \tau\right)\right) + \Lambda_5 \exp\left(i\Omega_2\left(\tau_0 - \tau\right)\right) \end{pmatrix} + \\
&\omega_2^2 x_{11} + z_2 D_0\left(A_1 \exp\left(i\omega_1\tau_0\right) + \Lambda_1 \exp\left(i\Omega_1\tau_0\right) + \Lambda_2 \exp\left(i\Omega_2\tau_0\right)\right) \\
&-z_2 D_0 \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} \\
&+\alpha_{23}\left(A_1 \exp\left(i\omega_1\tau_0\right) + \Lambda_1 \exp\left(i\Omega_1\tau_0\right) + \Lambda_2 \exp\left(i\Omega_2\tau_0\right)\right)^3 - \\
&3\alpha_{23} \begin{pmatrix} A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_1 \exp\left(i\Omega_1\tau_0\right) + \\ \Lambda_2 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix}^2 \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} \\
&+3\alpha_{23} \begin{pmatrix} A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_1 \exp\left(i\Omega_1\tau_0\right) \\ +\Lambda_2 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix} \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix}^2 \\
&-\alpha_{23} \begin{pmatrix} A_2 \exp\left(i\omega_2\tau_0\right) + \Lambda_3 A_1 \exp\left(i\omega_1\tau_0\right) + \\ \Lambda_4 \exp\left(i\Omega_1\tau_0\right) + \Lambda_5 \exp\left(i\Omega_2\tau_0\right) \end{pmatrix}^3 + cc
\end{aligned} \tag{12}$$

2.2 Resonance Cases

The various resonance conditions are observed by comparing the natural frequencies with the forcing frequencies in the Eqs. (11) and (12), by which one may observe six different resonance conditions namely as 1) primary resonance ($\Omega_j \cong \omega_1$), 2) sub-harmonic resonance ($\Omega_2 \cong 3\omega_1$), 3) superharmonic resonance ($3\Omega_1 \cong \omega_1$), 4) principal parametric resonance ($\Omega_3 \cong 2\omega_1$), 5) internal resonance ($\omega_2 \cong m\omega_1$, $\omega_1 \cong m\omega_2$ for m =1, 3) and 6) simultaneous resonance (for any combination of above resonance cases). In this case, the natural frequency of the absorber is an integer multiple of the natural frequency of the primary system, which leads to the internal resonance condition. In present work, the combinations of the internal resonance ($\omega_2 = 3\omega_1 + \varepsilon\sigma$), subharmonic resonance ($\Omega_2 = 3\omega_1 + \varepsilon\sigma_2$) superharmonic resonance ($3\Omega_1 = \omega_1 + \varepsilon\sigma_1$) and principal parametric resonance ($\Omega_3 = 2\omega_1 + \varepsilon\sigma_3$) are considered. The terms σ, σ_1, σ_2 and σ_3 indicates the detuning parameters for internal resonance, superharmonic, subharmonic and principal parametric harmonic resonance conditions respectively. The detuning parameter represents the nearness of the external or internal frequency to the natural frequency of the system. For example, in the case of internal resonance, the detuning parameter σ represents how close the second mode natural frequency ω_2 is close to three times the first mode natural frequency. The secular terms for these resonance conditions are obtained from Eqs. (11) and (12) which can be written as

$$\begin{aligned}
&2i\omega_1 D_1 A_1 = \mu\omega_2^{\ 2}\Lambda_3 A_1 - \alpha_{13} \begin{pmatrix} \Lambda_1^{\ 3} \exp\left(i\sigma_1\tau_1\right) + 3A_1^{\ 2}\bar{A}_1^{\ 2} + 6A_1\Lambda_1^{\ 2} \\ +6A_1\Lambda_2^{\ 2} + 3\bar{A}_1^{\ 2}\Lambda_2 \exp\left(i\sigma_2\tau_1\right) \end{pmatrix} \\
&\qquad + \frac{\bar{A}_1 F_3 \exp\left(i\sigma_3\tau_1\right)}{2} + \frac{\Lambda_2 F_3 \exp i\left(\sigma_2 - \sigma_3\right)\tau_1}{2}
\end{aligned} \tag{13}$$

$$\begin{aligned}
&2i\omega_2 D_1 A_2 = -\frac{F_{c1}}{\mu}\Omega_2^{\ 2}\Lambda_2 \exp\left(i\left(\left(\sigma_2 - \sigma\right)\tau_1 - \Omega_2\tau_d\right)\right) + \\
&iz_2\Omega_2 \exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right)\left(\Lambda_2 - \Lambda_5\right) - iz_2 A_2\omega_2 + \\
&\omega_2^2\left(\frac{\mu\omega_2^{\ 2} A_2}{\omega_1^{\ 2} - \omega_2^{\ 2}}\right) + \frac{\mu\omega_2^{\ 4}\Lambda_5}{\omega_1^{\ 2} - \Omega_2^{\ 2}} \exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right) + \\
&\omega_2^2\left(\frac{3\Lambda_2^{\ 3} + 6\Lambda_1^{\ 2}\Lambda_2 + 6A_1\bar{A}_1\Lambda_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}}\right)\exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right) + \\
&\frac{\omega_2^2 A_1^{\ 3}\exp\left(-i\sigma\tau_1\right)}{\omega_1^{\ 2} - \left(3\omega_1\right)^2} + 6\alpha_{23}A_1\bar{A}_1\Lambda_2 \exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right) + \\
&6\alpha_{23}A_1\bar{A}_1 \begin{pmatrix} -2\Lambda_2\Lambda_3 - \Lambda_5 + \Lambda_2\Lambda_3^{\ 2} \\ +2\Lambda_3\Lambda_5 - \Lambda_3^{\ 2}\Lambda_5 \end{pmatrix} \exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right) + \\
&3\alpha_{23}\exp\left(i\left(\sigma_2 - \sigma\right)\tau_1\right) \begin{pmatrix} 2A_2\bar{A}_2\left(\Lambda_2 - \Lambda_5\right) + \Lambda_2^{\ 3} + \\ 2\Lambda_1^{\ 2}\Lambda_2 - 2\Lambda_1^{\ 2}\Lambda_5 - 3\Lambda_2^{\ 2}\Lambda_5 \\ -4\Lambda_1\Lambda_2\Lambda_4 + 6\Lambda_2\Lambda_4^{\ 2} + \\ 3\Lambda_2\Lambda_5^{\ 2} + 4\Lambda_1\Lambda_4\Lambda_5 \\ -\Lambda_5^{\ 3} - 2\Lambda_4^{\ 2}\Lambda_5 \end{pmatrix} + \\
&3\alpha_{23}A_2 \begin{pmatrix} 4A_1\bar{A}_1\Lambda_3 - 2\left(\Lambda_1^{\ 2} + \Lambda_2^{\ 2}\right) - 2A_1\bar{A}_1 + 4\Lambda_1\Lambda_4 \\ +4\Lambda_2\Lambda_5 - 2\left(\Lambda_4^{\ 2} + \Lambda_5^{\ 2}\right) - A_2\bar{A}_2 - 2A_1\bar{A}_1\Lambda_3^{\ 2} \end{pmatrix} + \\
&\alpha_{23} \begin{pmatrix} A_1^{\ 3}\exp\left(-i\sigma\tau_1\right)\left(1 - 3\Lambda_3 + 3\Lambda_3^{\ 2} - \Lambda_3^{\ 3}\right) + \\ A_2^{\ 2}\exp\left(-i\left(\sigma_2 - \sigma\right)\tau_1\right)\left(3\Lambda_2 - 3\Lambda_5\right) + \\ \bar{A}_2\exp\left(\left(2\left(\sigma_2 - \sigma\right)\right)\tau_1\right)\left(2\Lambda_2\Lambda_5 - 3\Lambda_2^{\ 2} - 3\Lambda_5^{\ 2}\right) \end{pmatrix}
\end{aligned} \tag{14}$$

The amplitudes A_1 and A_2 may be expressed in polar form as $A_1 = \frac{1}{2}a_1 e^{i\beta_1}$ and $A_2 = \frac{1}{2}a_2 e^{i\beta_2}$. Here a_1, a_2, β_1 and β_2 are the real numbers representing amplitude and phase of the response respectively. Separating the real and imaginary parts from the Eq. (13) and (14) the final reduced autonomous equations are expressed as

$$\dot{a}_1\omega_1 = -\alpha_{13}\Lambda_1^{\ 3}\sin\gamma_1 - \frac{3}{4}\alpha_{13}a_1^{\ 2}\Lambda_2\sin 3\gamma_1 + \frac{a_1F_3\sin 2\gamma_1}{4} + \frac{\Lambda_2F_3\sin\gamma_1}{2} \tag{15}$$

$$a_1\dot{\gamma}_1\omega_1 = a_1\omega_1\sigma_1 + \frac{\mu\omega_2^{\ 2}\Lambda_3a_1}{2} - \alpha_{13}\left(\Lambda_1^{\ 3}\cos\gamma_1 + 3a_1\Lambda_1^{\ 2} + 3a_1\Lambda_2^{\ 2}\right) - \frac{3\alpha_{13}a_1^{\ 2}}{8} - \frac{3\alpha_{13}a_1^{\ 2}\Lambda_2\cos 3\gamma_1}{4} + \frac{a_1F_3\cos 2\gamma_1}{4} + \frac{\Lambda_2F_3\cos\gamma_1}{2} \tag{16}$$

$$\begin{aligned}\dot{a}_2\omega_2 = &-\frac{F_{c1}}{\mu}\Omega_2^{\ 2}\Lambda_2\sin\left(\gamma_2 - \Omega_2\tau_d\right) + z_2\Omega_2\Lambda_2\cos\gamma_2\left(\Lambda_2 - \Lambda_5\right) - \\ &z_2\omega_2\frac{a_2}{2} + \mu\omega_2^{\ 4}\frac{\Lambda_5\sin\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}} - \frac{3\omega_2^2\alpha_{13}\sin\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}}\begin{pmatrix}\Lambda_2^{\ 3} + 2\Lambda_1^{\ 2}\Lambda_2 \\ +0.5a_1^{\ 2}\Lambda_2\end{pmatrix} - \\ &\frac{\alpha_{13}a_1^{\ 3}\omega_2^2\sin\left(\gamma_2 - 3\gamma_1\right)}{-64\omega_1^{\ 2}} - \frac{2\omega_2^2\Omega_2\Lambda_2\sigma_2\cos\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}} + \\ &1.5\alpha_{23}a_1^{\ 2}\left(\Lambda_2 - 2\Lambda_2\Lambda_3 - \Lambda_5 + \Lambda_2\Lambda_3^{\ 2} + 2\Lambda_3\Lambda_5 - \Lambda_3^{\ 2}\Lambda_5\right)\sin\gamma_2 + \\ &3\alpha_{23}\sin\gamma_2\begin{pmatrix}0.5a_2^{\ 2}\left(\Lambda_2 - \Lambda_5\right) + \Lambda_2^{\ 3} + 2\Lambda_1^{\ 2}\Lambda_2 - 2\Lambda_1^{\ 2}\Lambda_5 - \\ 3\Lambda_2^{\ 2}\Lambda_5 - 4\Lambda_1\Lambda_2\Lambda_4 + 2\Lambda_2\Lambda_4^{\ 2} + 3\Lambda_2\Lambda_5^{\ 2} + \\ 4\Lambda_1\Lambda_4\Lambda_5 - \Lambda_5^{\ 3} - 2\Lambda_4^{\ 2}\Lambda_5\end{pmatrix} + \\ &\frac{\alpha_{23}}{8}\left(a_1^{\ 3}\sin\left(\gamma_2 - 3\gamma_1\right)\begin{pmatrix}1 - 3\Lambda_3 + \\ 3\Lambda_3^{\ 2} - \Lambda_3^{\ 3}\end{pmatrix} - 6a_2^{\ 2}\sin\gamma_2\left(\Lambda_2 - \Lambda_5\right)\right) + \\ &3\alpha_{23}a_2\Lambda_2\Lambda_5\sin\left(2\gamma_2\right) - 1.5\alpha_{23}a_2\sin\left(2\gamma_2\right)\left(\Lambda_2^{\ 2} + \Lambda_5^{\ 2}\right)\end{aligned} \tag{17}$$

$$\begin{aligned}a_2\omega_2\dot{\gamma}_2 = &a_2\omega_2\left(\sigma_2 - \sigma\right) - \frac{F_{c1}}{\mu}\Omega_2^{\ 2}\Lambda_2\cos\left(\gamma_2 - \Omega_2\tau_d\right) - \\ &z_2\Omega_2\sin\gamma_2\left(\Lambda_2 - \Lambda_5\right) + \frac{\mu\omega_2^{\ 4}\Lambda_5\cos\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}} + \frac{\mu\omega_2^{\ 4}a_2}{2\left(\omega_1^{\ 2} - \omega_2^{\ 2}\right)} + \\ &\frac{2\omega_2^2\Omega_2\Lambda_2\sigma_2\sin\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}} - \frac{3\omega_2^2\alpha_{13}\cos\gamma_2}{\omega_1^{\ 2} - \Omega_2^{\ 2}}\begin{pmatrix}\Lambda_2^{\ 3} + 2\Lambda_1^{\ 2}\Lambda_2 \\ +0.5a_1^{\ 2}\Lambda_2\end{pmatrix} - \\ &\frac{\omega_2^2\alpha_{13}a_1^{\ 3}\cos\left(\gamma_2 - 3\gamma_1\right)}{8\left(\omega_1^{\ 2} - \left(3\omega_1\right)^2\right)} + 1.5\alpha_{23}a_1^{\ 2}\left(\Lambda_2 - 2\Lambda_2\Lambda_3 - \Lambda_5\right)\cos\gamma_2 + \\ &1.5\alpha_{23}\cos\gamma_2\left(a_1^{\ 2}\left(\Lambda_2\Lambda_3^{\ 2} + 2\Lambda_3\Lambda_5 - \Lambda_3^{\ 2}\Lambda_5\right) + a_2^{\ 2}\left(\Lambda_2 - \Lambda_5\right)\right) + \\ &3\alpha_{23}\cos\gamma_2\begin{pmatrix}\Lambda_2^{\ 3} + 2\Lambda_1^{\ 2}\Lambda_2 - 2\Lambda_1^{\ 2}\Lambda_5 - 3\Lambda_2^{\ 2}\Lambda_5 - 4\Lambda_1\Lambda_2\Lambda_4 + \\ 2\Lambda_2\Lambda_4^{\ 2} + 3\Lambda_2\Lambda_5^{\ 2} + 4\Lambda_1\Lambda_4\Lambda_5 - \Lambda_5^{\ 3} - 2\Lambda_4^{\ 2}\Lambda_5\end{pmatrix} + \\ &3\alpha_{23}a_2\begin{pmatrix}0.25a_1^{\ 2}\left(2\Lambda_3 - \Lambda_3^{\ 2} - 1\right) + 2\Lambda_1\Lambda_4 + 2\Lambda_2\Lambda_5 - \\ \Lambda_1^{\ 2} - \Lambda_2^{\ 2} - \Lambda_4^{\ 2} - \Lambda_5^{\ 2} - 0.125a_2^{\ 2}\end{pmatrix} + \\ &\frac{a_1^{\ 3}}{8}\cos\left(\gamma_2 - 3\gamma_1\right)\left(1 - 3\Lambda_3 + 3\Lambda_3^{\ 2} - \Lambda_3^{\ 3}\right) + 0.75\Lambda_2a_2^{\ 2}\cos\gamma_2 - \\ &a_2\left(0.75\Lambda_5a_2^{\ 2}\cos\gamma_2 - 1.5\cos 2\gamma_2\left(\Lambda_2^{\ 2} + \Lambda_5^{\ 2}\right) + 3\Lambda_2\Lambda_5\cos 2\gamma_2\right)\end{aligned} \tag{18}$$

where

$$\gamma_1 = \sigma_1\tau_1 - \beta_1,\ \gamma_2 = \left(\sigma_2 - \sigma\right)\tau_1 - \beta_2,\ \sigma_2 = 3\sigma_1,\ \sigma_3 = 2\sigma_1.$$

The steady-state equations of the system are obtained by submitting $\dot{a}_1 = \dot{\gamma}_1 = \dot{a}_2 = \dot{\gamma}_2 = 0$. Further, the stability of the obtained steady-state solution can be obtained by finding the eigenvalues of the Jacobian matrix. In the following section, the physical system parameters, the numerical results and discussion will be carried out.

3. Results and discussions

In this section, considering the mass of the absorber by 50 times less than the primary system, frequency and time responses are carried out. The primary system parameters are the same as those considered in Habib et. al [3]. The mass, stiffness and damping coefficient of the primary system is 1 kg, 1 N/m and 0.002 Ns/m respectively. The absorber stiffness and damping coefficient are considered as 0.02 N/m and 0.012 Ns/m, respectively. The external excitation forces F_{11}, F_{21} and F_{31} on the primary system are considered as 0.4 N, 0.38 N and 0.04 N respectively. The piezoelectric properties were referred from Mallik and Chatterjee [11]. It may be noted that the natural frequencies of the primary system ω_1 and the absorber ω_2 depend upon the equivalent stiffness of PZT actuator and spring stiffness k_3. It can be observed that while the variation of the primary system ω_1 occurs small from 1 rad/s to 1.18 rad/s the absorber ω_2 changes largely from 1 rad/s to 3 rad/s by changing the stiffness k_r from 0 to 0.176 N/m. The nonlinear stiffness in the primary system and absorber varies from 4 % to 10% of the linear stiffness in both the primary system and the absorber. The frequency responses of the primary system and the absorber are obtained from the steady-state Eq. (15) to (18) using Newton's method. The non-dimensional frequency response of the primary system and the absorber are shown in Fig. 2 for various values of a detuning parameter σ. The stable and unstable branches are respectively represented by black and red colour. In Figs. 2(a) and (b) the plots are obtained for F_{c1} = 0.8, α_{13} = 0.8, α_{23} =4 z_2 = 6 at σ = 0. It is observed from Fig. 2(a) that the frequency response amplitude of the primary system attains higher amplitude when the detuning parameter σ_1 is in the range of -1.4 □ -0.5. The maximum amplitude of 0.4 is observed at σ_1 = -1.2. The instability in the frequency response with both stable and unstable solutions are observed for σ_1 in the range of -1.1 □ -0.6. For all other frequencies of operation i.e, when σ_1 is away from -1.4 □ -0.5, the amplitude of the frequency response is 0.02. The frequency response of the absorber shown in Fig. 2(b) attains a value of 2 at

σ_1=0 and the system has muti stable solution for σ_1 in the range of 0 □ 0.5. Hence the system may be operated at a frequency corresponding to the detuning parameter σ_1= -1.2 □ -2. It is also observed from the Figures that at σ_1 = -1 the amplitude of the primary system is high whereas at the same frequency the absorber indicates the minimum amplitude. The primary system indicates the minimum amplitude for σ_1 value in the range of -0.5 □ 2 whereas, at the same frequency, the absorber response is high. Therefore the amplitude responses of both the primary system and the absorber are found to be minimum for σ_1 in the range of -4 □ -1.5 frequency of operation. In Figs. 2(c) and (d) the frequency response of the system is studied for the passive case (F_{c1}=0) by considering the primary system as linear i.e. (α_{13}=0) and the absorber as nonlinear (α_{23}=4). The frequency response of the primary system shown in Fig. 2(c) is observed to be that of a linear system with instability when σ_1 is in the range of -1.1 □ -0.95. For the absorber (Fig. 2(d)) two instability region developed for σ_1 in the range of -.5 □ -0.1 and from 0.04 □ 0.14. respectively. It can be observed from Figs. 2 (a) and (c) that the instability region increases proportionally to cubic nonlinear stiffness of the primary system whereas the amplitude of the primary system is comparatively low at the high cubic nonlinear stiffness of the primary system in the stable frequency of operation. The nonlinear stiffness in the primary system produces low sensitivity in the frequency response and the amplitude gradually increases with increasing σ_1 value. In Figs. 2(e) and (f) the detuning parameter of internal resonance σ and the excitation force F_{21} are considered as zero. It is observed from Fig. 2(e) that the amplitude response of the primary system shows more hardening effect than from Fig. 2(a) with a low amplitude at σ_1 = -1. Considering the effect of σ on the frequency response of the primary system it is observed from Figs. 2(a) and 2(e) that with increasing σ the amplitude of the resonating response is shifted to the positive axis with the high response in the range of -1.4 □ -0.5. From Fig. 2(f) one can observe both stable and unstable solutions for σ_1 value increased from -0.5 with more hardening effect in the response curve. The amplitude response of the absorber shown in Fig. 2(f) gradually increases to a maximum value of 3.2 for σ_1 = 2. In Figs. 2(g) and 2(h) the detuning parameter σ is maximum value of 3.2 for σ_1 = 2. In Figs. 2(g) and 2(h) the detuning parameter σ is considered as -4 with the same excitation force F_{21} as indicated in Fig. 2(a). From Fig. 2(g) it is observed that only stable solutions are developed for σ_1 in the range of -0.5 □ -0.4 otherwise at all other values of σ_1 both stable and unstable solutions are developed. The frequency response of the absorber is shown in Fig. 2(h) where the multiple solutions consisting of both stable and unstable branches can be observed for any value of σ_1. From Fig. 2 it is observed that the frequency responses of the primary system and the absorber shows more unstable branches for σ not equal to zero. It is also shown stable solutions with a small amplitude of 0.02 for σ=0 and σ_1 in the range of -2 to -1. The time response of

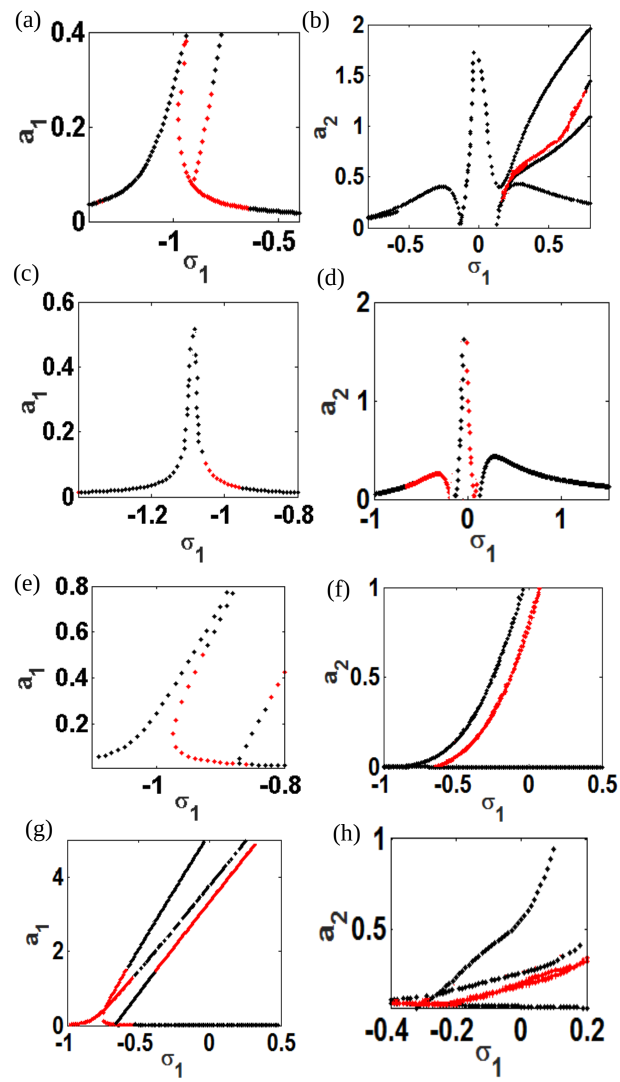

Fig. 2. Frequency response of the primary System and the absorber (a, b) for F_{c1} = 0.8, α_{13} = 0.8, α_{23} =8, z_2 = 6 at σ = 0 (c, d) for F_{c1} = 0, α_{13} = 0, $\tilde{\alpha}_{23}$ =0.4, z_2 = 6 at σ = 0 (e, f) for F_{21} = 0, F_{c1} = 0.8, α_{13} = 0.5, α_{23} =4, z_2 = 6 at σ = 0 (g, h) for F_{21} = 0.38, F_{c1} = 0.8, α_{13} = 0.5, α_{23} =4, z_2 = 6 at σ = -4.

the vibration absorber and the primary system are obtained by solving Eqs. (15) to (18) using the fourth-order Runge Kutta method (ode45 in MATLAB) for the system with a constant delay of 0.6 as shown in Fig. 3. In Fig. 3 all the system parameters are considered as the same as those in Fig. 2 thereby only the time response at a particular value of σ_1 is shown. In Figs. 3(a) and (b) the time responses of the primary system and the absorber are shown at σ_1 = -1. The periodic response for both the primary system and the absorber can be observed where the amplitude of the primary system varies from 0.08 to 0.42 at a time period of 100 non-dimensional time as shown in Fig. 3(a). The time response of the absorber also shows a periodic response whose amplitude oscillates from 0.01 to 0.0122 for a time period of 100 as shown in Fig. 3(b). In Fig. 2(a) at σ_1 = -1 the solution switches from stable to unstable ones and in Fig. 3 (a) at same σ_1 value periodic response is shown, so one can observe Hopf bifurcation point at σ_1 equal to 1. In Figs. 3(c) and (d) the amplitude responses of the primary system and the absorber are shown at σ_1 = -1.1. The amplitude response in Fig. 3(c) shows the same periodic one as shown in Fig. 3(a) except a large amplitude. In Fig. 3(d) a quasiperiodic response is observed where the amplitude of the absorber oscillates from 0.02069 to 0.02074. In Figs. 3(e) and (f) the periodic responses for both the primary system and the absorber are observed at σ_1 = -0.95. The periodic response is also observed in Fig. 3(e) with a maximum amplitude of 0.8 at the time period of 750 nondimensional value. The absorber response is shown in Fig. 3(f) is also periodic with the time period of 700 nondimensional value with maximum

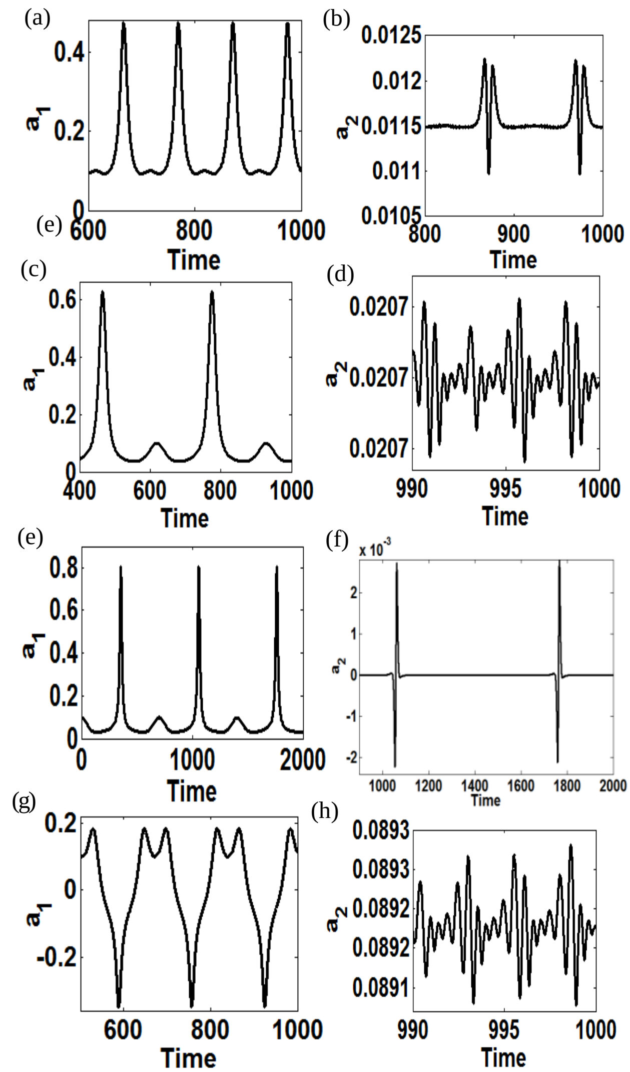

Fig. 3. Time response of the primary system and the absorber (a, b) for F_{c1} = 0.8, α_{13} = 0.8, α_{23} =8, z_2 = 6 at σ = 0 and σ_1 = -1 (c, d) for F_{c1} = 0, α_{13} = 0, α_{23} =4, z_2 = 6 at σ = 0 and σ_1 = -1.1 (e, f) for F_{21} = 0, F_{c1} = 0.8, α_{13} = 0.5, α_{23} =4 z_2 = 6 at σ = 0 and σ_1 = -0.95 (g, h) for F_{21} = 0.38, F_{c1} = 0.8, α_{13} = 0.5, α_{23} =4, z_2 = 6 at σ = -4 and σ_1 = -0.3.

and minimum amplitudes of 0.0268 and zero respectively. In the Figs. 3(g) and (h) time response of the primary system and the absorber are shown at σ_1 = -0.3. Fig. 3(g) shows periodic response such that with the time period of 160 non-dimensional value the amplitude of the system varies from -0.34 to 0.18. For the absorber as shown in Fig. 3 (h) quasiperiodic response is observed. The corresponding frequency response obtained by Newton's method shown in Fig.2 is compared with the results from the fourth-order

Runge Kutta method shown in Fig.3. two results are close to each other at the particular value of σ_1.

4. Conclusions

In this paper, the performance of an active nonlinear vibration absorber attached to a nonlinear single degree of freedom primary system, and subjected to a hard multi-harmonic and a parametric excitation is studied. The active force by the PZT actuator is developed from a combination of controlling force and a spring attached in series, thereby the frequency of the absorber can be changed actively. The analysis is carried out by a time delay acceleration feedback of the primary system using the method of multiple scales and fourth-order Runge Kuta method under internal resonance, parametric, subharmonic and superharmonic resonance conditions simultaneously for a mass ratio of 1:50 between the absorber and the primary system. The frequency response obtained by Newton's method is found to be in good agreement with the results from the fourth-order Runge Kutta method.

From the analysis, it is inferred that for controlling the vibration of the primary system subjected to hard multi excitations and parametric excitation with the absorber of mass ratio 1:50, the excitation frequencies of external forcing plays a significant role which makes the response amplitude of the system high and unstable. In the proposed model as the frequency of the absorber is dependent upon the equivalent stiffness of the PZT actuator and the spring attached in series one can tune the frequency of the absorber actively. The cubic nonlinear stiffness in the primary system reduces the amplitude of the primary system than that without considering a nonlinearity in the stiffness at the resonating frequency of operation. The effect of the controlling force on the suppression of the vibration in the primary system is not observed whereas in the absorber the vibration is minimized. Here the controlling force is considered of ε^2 order thereby the force developed by the actuator is so small to reduce the vibration of the primary system which is excited by ε^0 order of excitation. The hard excitation on the primary system produces a high amplitude and instability range for the mass ratio of 50, therefore the external forcing of the order of ε or ε^2 is more effective for studying the effect of actuating force on the primary system.

In the proposed model as the spring is connected in series with the PZT actuator, the actuating force can be controlled more easily without much dependency upon voltage supply. The proposed model is more economical and also designed as failsafe as the absorber can be operated both passively and actively to protect the system from severe vibration.

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