EXTENSION OF PHASE-ISOMETRIES BETWEEN THE UNIT SPHERES OF ATOMIC $L_p$-SPACES FOR $p > 0$

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Abstract. In this paper, we prove that for every surjective phase-isometry between the unit spheres of real atomic $L_p$-spaces for $p > 0$, its positive homogeneous extension is a phase-isometry which is phase equivalent to a linear isometry.

1. Introduction

Let $X$ and $Y$ be real normed spaces. A mapping $f : X \to Y$ is called a phase-isometry if $f$ satisfies the functional equation
\[
\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\} \quad (x, y \in X).
\]

Let us say that a mapping $f : X \to Y$ is phase equivalent to a linear isometry if there exists a phase function $\varepsilon : X \to \{-1, 1\}$ such that $\varepsilon f$ is a linear isometry.

The notation of phase-isometry is linked to the famous Wigner’s theorem, which plays a fundamental role in quantum mechanics and in representation theory in physics. There are several equivalent formulations of Wigner’s theorem, see [1, 4, 5, 8, 10, 12] to list just some of them. The real version of Wigner’s theorem [10] says that a mapping $f : H \to K$ satisfies the functional equation
\[
|\langle f(x), f(y) \rangle| = |\langle x, y \rangle| \quad (x, y \in H)
\]
is phase equivalent to a linear isometry provided that $H$ and $K$ are real inner product spaces. This is equivalent to saying that every phase-isometry from the real inner product space $H$ into $K$ is phase equivalent to a linear isometry.

Recently, Huang and Tan [6] showed that every surjective phase-isometry between real atomic $L_p$-spaces for $p > 0$ is phase equivalent to a linear isometry, which generalizes Wigner’s theorem to real atomic $L_p$-spaces for $p > 0$.

In 1987, D. Tingley [11] proposed the following question: Let $f$ be a surjective isometry between the unit spheres $S_X$ and $S_Y$ of real normed spaces $X$ and $Y$, respectively. Is it true that $f : S_X \to S_Y$ extends to a linear isometry?
$F : X \to Y$ of the corresponding spaces? This problem is known as the Tingly’s problem or isometric extension problem. We refer the reader to the introduction of [9] for more information and recent development on this problem. The survey of Ding [3] is one of the good references for understanding the history of the problem. Let us consider the natural positive homogeneous extension $F$ of $f$, where $F$ is given by

$$F(x) = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1)

Then Tingley’s problem can be solved in positive for pairs $(X, Y)$ if and only if the natural positive homogeneous extension $F$ is a (linear) isometry. Inspired by Tingley’s problem, it is natural to ask the following question:

**Problem 1.1.** Let $f$ be a surjective phase-isometry between the unit spheres $S_X$ and $S_Y$ of real normed spaces $X$ and $Y$, respectively. Is it true that the natural positive homogeneous extension $F$ is a phase-isometry?

In this paper, we answer Problem 1.1 in positive for real atomic $L_p$-spaces for $p > 0$. That is for every phase-isometry from the unit sphere $S_{l_p(\Gamma)}$ onto $S_{l_p(\Delta)}$ of real atomic $L_p$-spaces for $p > 0$, the natural positive homogeneous extension is phase equivalent to a linear isometry, and therefore actually a phase-isometry. We also show that Problem 1.1 is solved in positive for real inner product spaces.

2. Results

We first discuss the phase-isometric extension problem on real inner product spaces and show that Problem 1.1 is solved in positive for such spaces.

**Proposition 2.1.** Let $H$ and $K$ be inner product spaces, and let $f : S_H \to S_K$ be a phase-isometry. Then the positive homogeneous extension $F$ of $f$ is a phase-isometry.

**Proof.** Since $H$ and $K$ are inner product spaces, by the polarization identity, we have

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2),$$

$$\langle f(x), f(y) \rangle = \frac{1}{4} (\|f(x) + f(y)\|^2 - \|f(x) - f(y)\|^2)$$

for all $x, y \in S_H$. By the assumption of $f$, we have $|\langle f(x), f(y) \rangle| = |\langle x, y \rangle|$ for all $x, y \in S_H$. Hence,

$$|\langle F(x), F(y) \rangle| = |\langle \|x\| f\left(\frac{x}{\|x\|}\right), \|y\| f\left(\frac{y}{\|y\|}\right) \rangle|$$

$$= \|x\| \|y\| |\langle f\left(\frac{x}{\|x\|}\right), f\left(\frac{y}{\|y\|}\right) \rangle| = |\langle x, y \rangle|$$
for all $x, y \in H$ with $x, y \neq 0$. It follows from Wigner’s Theorem that $F$ is phase equivalent to a linear isometry, and this completes the proof. \hfill \square

Recall that every real atomic $L_p$-space for $p > 0$ is linearly isometric to $l_p(\Gamma)$ for some nonempty index set $\Gamma$, that is,

$$l_p(\Gamma) = \{ x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma : \| x \| = (\sum_{\gamma \in \Gamma} |\xi_\gamma|^p)\frac{1}{p} < \infty, \ \xi_\gamma \in \mathbb{R} \}.$$ 

The unit sphere of $l_p(\Gamma)$ is $\{ x \in l_p(\Gamma) : \| x \| = 1 \}$ and is denoted by $S_{l_p(\Gamma)}$. For every $x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in l_p(\Gamma)$, we denote the support of $x$ by $\Gamma_x$, i.e.,

$$\Gamma_x = \{ \gamma \in \Gamma : \xi_\gamma \neq 0 \}.$$ 

Then $x$ can be rewritten in the form $x = \sum_{\gamma \in \Gamma_x} \xi_\gamma e_\gamma \in l_p(\Gamma)$. For $x, y \in l_p(\Gamma)$, we use the symbol $xy = 0$ to represent $\Gamma_x \cap \Gamma_y = \emptyset$. It is well-known that $xy = 0$ if and only if $\| x + y \| = \| x - y \|$ for all $x, y \in l_2(\Gamma)$. We also need the following well-known result which can be found from [7, Corollary 2.1] (noting that Banach used it in his book [2] already). The statement is that $xy = 0$ if and only if $\| x + y \|^p + \| x - y \|^p = 2(\| x \|^p + \| y \|^p)$ for all $x, y \in l_p(\Gamma)$ with $p > 0$, $p \neq 2$. By this one can conclude the following result.

**Lemma 2.2.** Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \rightarrow S_Y$ is a phase-isometry. Then $xy = 0$ if and only if $f(x)f(y) = 0$ for all $x, y \in S_X$.

Our next lemma will show that every surjective phase-isometry between the unit spheres of real atomic $L_p$-space for $p > 0$ necessarily maps antipodal points to antipodal points. So the positive homogeneous extension is homogeneous for the negative scalars as well.

**Lemma 2.3.** Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \rightarrow S_Y$ is a surjective phase-isometry. Then $f$ is injective and $f(-x) = -f(x)$ for every $x \in S_X$. Moreover, for every $\gamma \in \Gamma$, there exists $\delta \in \Delta$ such that $f(e_\gamma) = \pm e_\delta$.

**Proof.** Let us take $x \in S_X$. Since $f$ is surjective, we can pick $y \in S_X$ such that $f(y) = -f(x)$. Notice that $f$ is a phase-isometry, we have

$$\{ \| x + y \|, \| x - y \| \} = \{ \| f(x) + f(y) \|, \| f(x) - f(y) \| \} = \{ 0, 2 \}$$

which implies that $y = \pm x$. If $y = x$, then $f(x) = f(y) = -f(x)$, which is impossible. Hence we get $y = -x$ and so $f(-x) = -f(x)$. On the other hand, suppose that $f(z) = f(x)$ for some $z \in S_X$. By the assumption of $f$, we have

$$\{ \| x + z \|, \| x - z \| \} = \{ \| f(x) + f(z) \|, \| f(x) - f(z) \| \} = \{ 2, 0 \}.$$

This means that $z = x$ and $f$ is injective.

We will prove the “moreover” part. Let $\delta$ be in the support of $f(e_\gamma)$ and pick $x \in S_X$ such that $f(x) = e_\delta$. Applying Lemma 2.2 we have

$$e_\gamma e_{\gamma'} = 0 \Rightarrow f(e_\gamma)f(e_{\gamma'}) = 0 \Rightarrow f(x)f(e_{\gamma'}) = 0 \Rightarrow xe_{\gamma'} = 0$$
for all $\gamma' \in \Gamma$ with $\gamma' \neq \gamma$. It follows that $x = \pm e_\gamma$, and so $f(e_\gamma) = \pm e_\delta$. \hfill \Box

Now we derive the representation theorem of surjective phase-isometries between the unit spheres of real atomic $L_p$-spaces for $p > 0$, $p \neq 2$.

**Theorem 2.4.** Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$, $p \neq 2$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then for every $x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in S_X$, we have $f(x) = \sum_{\gamma \in \Gamma} \eta_\gamma f(e_\gamma)$, where $|\xi_\gamma| = |\eta_\gamma|$ for all $\gamma \in \Gamma$.

**Proof.** Let $x$ be in $S_X$ and write $x = \sum_{\gamma \in \Gamma_x} \xi_\gamma e_\gamma$, where $\sum_{\gamma \in \Gamma_x} |\xi_\gamma|^p = 1$ and $\xi_\gamma \neq 0$ for all $\gamma \in \Gamma_x$. According to Lemma 2.3, we can set

$$M := \{ \delta \in \Delta : f(e_\gamma) = \pm e_\delta, \forall \gamma \in \Gamma_x \}.$$ 

Choose $y \in S_X$ such that $f(y) = e_\delta$ for some $\delta \in \Delta \setminus M$. Applying Lemma 2.2, we have

$$f(e_\gamma) f(y) = 0 \Rightarrow e_\gamma y = 0 \Rightarrow xy = 0 \Rightarrow f(x) f(y) = f(x) e_\delta = 0$$

for all $\gamma \in \Gamma_x$. Thus we can write $f(x) = \sum_{\gamma \in \Gamma_x} \eta_\gamma f(e_\gamma)$, where $\sum_{\gamma \in \Gamma_x} |\eta_\gamma|^p = 1$. By the assumption of $f$,

$$\|f(x) + f(e_\gamma)\|^p + \|f(x) - f(e_\gamma)\|^p = \|x + e_\gamma\|^p + \|x - e_\gamma\|^p = 1 - |\xi_\gamma|^p + |\xi_\gamma + 1|^p + 1 - |\xi_\gamma|^p + |\xi_\gamma - 1|^p = |1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p + 2.$$ 

On the other hand,

$$\|f(x) + f(e_\gamma)\|^p + \|f(x) - f(e_\gamma)\|^p = 1 - |\eta_\gamma|^p + |\eta_\gamma + 1|^p + 1 - |\eta_\gamma|^p + |\eta_\gamma - 1|^p = |1 + \eta_\gamma|^p + |1 - \eta_\gamma|^p - 2|\eta_\gamma|^p + 2.$$ 

It follows that

$$|1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p = |1 + \eta_\gamma|^p + |1 - \eta_\gamma|^p - 2|\eta_\gamma|^p.$$ 

Notice that the function $\varphi(t) = (1 + t)^p + (1 - t)^p - 2t^p$ is strictly decreasing (increasing) on $[0, 1]$ for $0 < p < 2$ ($p > 2$) (Here, we need the fact that $(s + r)^p < s^p + r^p$ for $0 < p < 1$ and $(s + r)^p > s^p + r^p$ for $p > 1$ whenever $s, r > 0$). Consequently, we obtain $|\xi_\gamma| = |\eta_\gamma|$ for all $\gamma \in \Gamma_x$. \hfill \Box

Our next results are devoted to determining the behaviour of surjective phase-isometries between the unit spheres of real atomic $L_p$-spaces for $p > 0$, $p \neq 2$ on vectors which are linear combinations of two zero-product norm-one vectors.
Lemma 2.5. Let \( X = l_p(\Gamma) \) and \( Y = l_p(\Delta) \) for \( p > 0, p \neq 2 \). Suppose that \( f : S_X \to S_Y \) is a surjective phase-isometry. Let \( x, y \in S_X \) with \( xy = 0 \) and \( \lambda \in \mathbb{R} \). Then there exist two real numbers \( \alpha, \beta \) with \( |\alpha| = |\beta| = 1 \) such that

\[
\|x + \lambda y\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) = \alpha f(x) + \beta \lambda f(y).
\]

Proof. Suppose that \( x = \sum_{\gamma \in \Gamma_x} \xi_{\gamma} \), \( y = \sum_{\gamma \in \Gamma_y} \eta_{\gamma} \), and that \( 0 \neq \lambda \in \mathbb{R} \). By Theorem 2.4 we can write

\[
f(x) = \sum_{\gamma \in \Gamma_x} \xi'_{\gamma} f(e_{\gamma}), \quad f(y) = \sum_{\gamma \in \Gamma_y} \eta'_{\gamma} f(e_{\gamma}),
\]

\[
\|x + \lambda y\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) = \sum_{\gamma \in \Gamma_x} \xi''_{\gamma} f(e_{\gamma}) + \lambda \sum_{\gamma \in \Gamma_y} \eta''_{\gamma} f(e_{\gamma}),
\]

where \(|\xi'_{\gamma}| = |\xi''_{\gamma}| = |\xi|\) and \(|\eta'_{\gamma}| = |\eta''_{\gamma}| = |\eta|\) for all \( \gamma \in \Gamma_x \cup \Gamma_y \). To simplify the writing, we take \( A = \frac{1}{\|x + \lambda y\|} = \frac{1}{(1 + \|\lambda\|^p)^{\frac{1}{p}}} \). Since \( f \) is a phase-isometry,

\[
\{(A + 1)^p + (A|\lambda|)^p, (1 - A)^p + (A|\lambda|)^p\}
\]

\[
= \left\{ \left\|\frac{x + \lambda y}{\|x + \lambda y\|} + x\right\|^p, \left\|\frac{x + \lambda y}{\|x + \lambda y\|} - x\right\|^p \right\}
\]

\[
= \left\{ \left\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) + f(x)\right\|^p, \left\| f\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) - f(x)\right\|^p \right\}
\]

\[
= \left\{ \sum_{\gamma \in \Gamma_x} |A\xi''_{\gamma} + \xi'_{\gamma}|^p + (A|\lambda|)^p, \sum_{\gamma \in \Gamma_x} |A\xi''_{\gamma} - \xi'_{\gamma}|^p + (A|\lambda|)^p \right\}.
\]

This shows that

\[
(A + 1)^p \in \left\{ \sum_{\gamma \in \Gamma_x} |A\xi''_{\gamma} + \xi'_{\gamma}|^p, \sum_{\gamma \in \Gamma_x} |A\xi''_{\gamma} - \xi'_{\gamma}|^p \right\}.
\]

Notice that

\[
\sum_{\gamma \in \Gamma_x} |A\xi''_{\gamma} \pm \xi'_{\gamma}|^p \leq \sum_{\gamma \in \Gamma_x} (|A\xi''_{\gamma}| + |\xi'_{\gamma}|)^p = (A + 1)^p.
\]

Then we obtain \( \xi''_{\gamma} = \xi'_{\gamma} \) for all \( \gamma \in \Gamma_x \), or \( \xi''_{\gamma} = -\xi'_{\gamma} \) for all \( \gamma \in \Gamma_x \). It follows that \( \sum_{\gamma \in \Gamma_x} \xi''_{\gamma} e_{\gamma} = \pm f(x) \). Similar argument yields \( \sum_{\gamma \in \Gamma_y} \eta''_{\gamma} e_{\gamma} = \pm f(y) \). The proof is complete. \( \Box \)

In [13] Wang proved that for every surjective isometry between unit spheres of real atomic \( L_p \)-spaces for \( p > 0, p \neq 2 \), its natural positive homogeneous extension is a linear isometry on the whole space. By this result, we are now ready to present main result of this paper.
Theorem 2.6. Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$ for $p > 0$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then the positive extension $F$ of $f$ is phase equivalent to a linear isometry.

Proof. Proposition 2.1 proves the case $p = 2$. We need only consider the case $p > 0, p \neq 2$. Set $Z := \{x \in X : xe_{\gamma_0} = 0\}$ and $W := \{w \in Y : wf(e_{\gamma_0}) = 0\}$ for some $\gamma_0 \in \Gamma$. It is not hard to check that $S_X = \{z + \lambda e_{\gamma_0} : z \in S_Z, \lambda \in \mathbb{R}\} \cup \{\pm e_{\gamma_0}\}$. By Lemma 2.5 we can write
\[
\|z + \lambda e_{\gamma_0}\|f(z) = \alpha(z, \lambda)f(z) + \beta(z, \lambda)\lambda f(e_{\gamma_0}),
\]
for all $z \in S_Z$ and $\lambda \in \mathbb{R}$. Define a mapping $g : S_X \to S_Y$ as follows:
\[
g(e_{\gamma_0}) = f(e_{\gamma_0}), \quad g(-e_{\gamma_0}) = -f(e_{\gamma_0}), \quad g(z) = \alpha(z, 1)\beta(z, 1)f(z),
\]
\[
\|z + \lambda e_{\gamma_0}\|g(z) = \alpha(z, \lambda)\beta(z, \lambda)f(z) + \lambda f(e_{\gamma_0})
\]
for all $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Then $g$ is a phase-isometry, which is phase equivalent to $f$. Since $f(S_Z) = S_W$ by Theorem 2.4, we deduce that $g(S_Z) \subset S_W$.

Next, we will show that $g|S_Z : S_Z \to S_W$ is a surjective isometry. Let us take $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Set $A := \frac{1}{\|z + e_{\gamma_0}\|}$ and $B := \frac{1}{\|z + \lambda e_{\gamma_0}\|}$. Since $g$ is a phase-isometry,
\[
\{|A + B|^p + |A + B\lambda|^p, |A - B|^p + |A - B\lambda|^p\}
\]
\[
= \left\{ \left\| \frac{z + e_{\gamma_0}}{\|z + e_{\gamma_0}\|} + \frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|} \right\|^p, \left\| \frac{z + e_{\gamma_0}}{\|z + e_{\gamma_0}\|} - \frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|} \right\|^p \right\}
\]
\[
= \left\{ \left\| g\left(z + e_{\gamma_0}\right) + g\left(z + \lambda e_{\gamma_0}\right)\right\|^p, \left\| g\left(z + e_{\gamma_0}\right) - g\left(z + \lambda e_{\gamma_0}\right)\right\|^p \right\}
\]
\[
= \{|A\alpha(z, 1)\beta(z, 1) + B\alpha(z, \lambda)\beta(z, \lambda)|^p + |A + B\lambda|^p, |A\alpha(z, 1)\beta(z, 1) - B\alpha(z, \lambda)\beta(z, \lambda)|^p + |A - B\lambda|^p\}.
\]
If $\alpha(z, 1)\beta(z, 1) = -\alpha(z, \lambda)\beta(z, \lambda)$, then
\[
\{|A - B|^p + |A + B\lambda|^p, |A + B|^p + |A - B\lambda|^p\}
\]
\[
= \{|A + B|^p + |A + B\lambda|^p, |A - B|^p + |A - B\lambda|^p\}.
\]
This leads to a contradiction for $\lambda \neq 0$. It follows that $\alpha(z, 1)\beta(z, 1) = \alpha(z, \lambda)\beta(z, \lambda)$, and hence
\[
\|z + \lambda e_{\gamma_0}\|g\left(\frac{z + \lambda e_{\gamma_0}}{\|z + \lambda e_{\gamma_0}\|}\right) = g(z) + \lambda g(e_{\gamma_0})
\]
for all $z \in S_Z$ and $\lambda \in \mathbb{R}$. Let $z_1, z_2$ be in $S_Z$ and $\lambda > \|z_1 - z_2\|/2$. Clearly,
\[
\frac{1}{1 + \lambda^2}\left\{\|g(z_1) + g(z_2)\|^p + (2\lambda)^p, \|g(z_1) - g(z_2)\|^p\right\}
\]
This implies that $f$ is phase equivalent to $g$ for all $z \in G$. It follows that $G$ is phase equivalent to a linear isometry. Since the natural positive homogeneous extension $G$ of $g$ is phase equivalent to $f$, it is suffices to showing that $G : X \to Y$ is a linear isometry. By Lemma 2.3, we have $f(e_{\gamma_0}) = \pm e_{\delta_0}$ for some $\delta_0 \in \Delta$. It is easily verified that $Z$ and $W$ are linearly isometric to $l_p(\Gamma \setminus \{\gamma_0\})$ and $l_p(\Delta \setminus \{\delta_0\})$ respectively. From Wang’s result [13], the restriction of $G$ to $Z$ is a linear isometry. Set $x := \frac{z}{\|z\|} + \frac{\lambda e_{\gamma_0}}{\|z\|}$ for some $0 \neq z \in Z$ and $\lambda \in \mathbb{R}$. It follows that

$$G(z + \lambda e_{\gamma_0}) = \|z\|\|g\left(\frac{x}{\|x\|}\right)\| = \|z\|\left(\| g\left(\frac{z}{\|z\|}\right) + \lambda g(e_{\gamma_0})\|_{z}\right) = G(z) + \lambda g(e_{\gamma_0}).$$

This shows that $G : X \to Y$ is a linear isometry, which completes the proof. □

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