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RIEMANN-LIOUVILLE FRACTIONAL FUNDAMENTAL THEOREM OF CALCULUS AND RIEMANN-LIOUVILLE FRACTIONAL POLYA TYPE INTEGRAL INEQUALITY AND ITS EXTENSION TO CHOQUET INTEGRAL SETTING

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Abstract. Here we present the right and left Riemann-Liouville fractional fundamental theorems of fractional calculus without any initial conditions for the first time. Then we establish a Riemann-Liouville fractional Polya type integral inequality with the help of generalised right and left Riemann-Liouville fractional derivatives. The amazing fact here is that we do not need any boundary conditions as the classical Polya integral inequality requires. We extend our Polya inequality to Choquet integral setting.

1. Introduction

We mention the following famous Polya's integral inequality, see [5], [6, p. 62], [7] and [8, p. 83].

Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\, dx.$$

In [9] Feng Qi presents the following very interesting Polya type integral inequality which generalizes the last inequality:

Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Then

$$\left| \int_a^b f(x)\, dx \right| \leq \frac{(b-a)^2}{4} M,$$

where $\frac{(b-a)^2}{4}$ is the best constant in the above inequality.

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We are greatly motivated by the above classical Polya inequalities.

2. Background

Here the background contains only original results.
We need:

Definition. Let $0 < q < 1$, $f \in C([a,b])$. The right Riemann-Liouville fractional integral is given by (see [1, p. 333])

$$ {}_tD_b^{-q} f(t) := \frac{1}{\Gamma(q)} \int_t^b (\tau - t)^{q-1} f(\tau)\, d\tau, \ \forall t \in [a,b], \tag{1}$$

where Γ is the gamma function.

The right Riemann-Liouville fractional derivative of order q is given by (see [4, p. 89])

$$ {}_tD_b^{q} f(t) := -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^b (\tau - t)^{-q} f(\tau)\, d\tau, \ \forall t \in [a,b]. \tag{2}$$

We give:

Theorem 2.1. *Let $0 < q < 1$ and $f \in C([a,b])$. Assume that ${}_tD_b^q f \in L_\infty([a,b])$. Then*

$$ {}_tD_b^{-q} \left({}_tD_b^q f(t) \right) = f(t), \ \ \forall\ t \in [a,b], \tag{3}$$

which means

$$ f(t) = \frac{1}{\Gamma(q)} \int_t^b (\tau - t)^{q-1} \left({}_\tau D_b^q f(\tau) \right) d\tau, \ \ \forall\ t \in [a,b]. \tag{4}$$

This is a kind of fundamental theorem for right Riemann-Liouville fractional calculus without any initial condition.

Proof. Since $0 < q < 1$, then $1 - q > 0$ and $q - 1 < 0$. We have that

$$ {}_tD_b^q f(t) = -\frac{d}{dt}\, {}_tD_b^{q-1} f(t) = -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^b (\tau - t)^{-q} f(\tau)\, d\tau, \ \forall t \in [a,b]. \tag{5}$$

Furthermore it holds

$$ \begin{aligned} {}_tD_b^{-q} \left({}_tD_b^q f(t) \right) &= \frac{1}{\Gamma(q)} \int_t^b (\tau - t)^{q-1} \ {}_\tau D_b^q f(\tau)\, d\tau \\ &= -\frac{d}{dt} \left\{ \frac{1}{\Gamma(q+1)} \int_t^b (\tau - t)^{q} \ {}_\tau D_b^q f(\tau)\, d\tau \right\}. \end{aligned} \tag{6}$$

Next we apply integration by parts to

$$ \begin{aligned} & -\frac{1}{\Gamma(q+1)} \int_t^b (\tau - t)^{q} \ {}_\tau D_b^q f(\tau)\, d\tau \\ = & \frac{1}{\Gamma(q+1)} \int_t^b (\tau - t)^{q} \ \frac{d}{d\tau}\ {}_\tau D_b^{q-1} f(\tau)\, d\tau \end{aligned}$$

$$(7)\quad = \frac{1}{\Gamma(q+1)}\left[(\tau-t)^q\ \ {}_{\tau}D_b^{q-1}f(\tau)\right]_t^b - \frac{1}{\Gamma(q)}\int_t^b (\tau-t)^{q-1}\ \ {}_{\tau}D_b^{q-1}f(\tau)\,d\tau$$
$$= \frac{(b-t)^q}{\Gamma(q+1)}\left[{}_tD_b^{q-1}f(t)\right]_{t=b} - {}_t\,D_b^{-1}f(t)\,.$$

Therefore we have

$$(8)\qquad -\frac{d}{dt}\left\{\frac{1}{\Gamma(q+1)}\int_t^b (\tau-t)^q\ \ {}_{\tau}D_b^q f(\tau)\,d\tau\right\}$$
$$= f(t) - \frac{(b-t)^{q-1}}{\Gamma(q)}\left[{}_tD_b^{q-1}f(t)\right]_{t=b}.$$

Consequently we find

$$(9)\qquad {}_tD_b^{-q}\left({}_tD_b^q f(t)\right) = f(t) - \left[{}_tD_b^{q-1}f(t)\right]_{t=b}\frac{(b-t)^{q-1}}{\Gamma(q)},\ 0<q<1.$$

Here by assumption $f\in C([a,b])$ and ${}_tD_b^q f(t)\in L_\infty([a,b])$, therefore ${}_tD_b^{q-1}f(t)$ is bounded at $t=b$.

We notice the following: we have

$$(10)\qquad {}_tD_b^{q-1}f(t) = \frac{1}{\Gamma(1-q)}\int_t^b (\tau-t)^{-q} f(\tau)\,d\tau,$$

and

$$\left|{}_tD_b^{q-1}f(t)\right| \le \frac{1}{\Gamma(1-q)}\int_t^b (\tau-t)^{-q}|f(\tau)|\,d\tau$$
$$\le \frac{\|f\|_{\infty,[a,b]}}{\Gamma(1-q)}\int_t^b (\tau-t)^{-q}\,d\tau$$
$$(11)\qquad = \frac{\|f\|_{\infty,[a,b]}}{\Gamma(1-q)}\frac{(b-t)^{1-q}}{(1-q)}$$
$$= \frac{\|f\|_{\infty,[a,b]}(b-t)^{1-q}}{\Gamma(2-q)}.$$

That is

$$(12)\qquad \left|{}_tD_b^{q-1}f(t)\right| \le \frac{\|f\|_{\infty,[a,b]}}{\Gamma(2-q)}(b-t)^{1-q},\ \forall t\in[a,b].$$

Hence

$$(13)\qquad \lim_{t\to b-}\left|{}_tD_b^{q-1}f(t)\right| = 0.$$

Therefore

$$\left[{}_tD_b^{q-1}f(t)\right]_{t=b} = 0.$$

We have proved that

$$(14)\qquad {}_tD_b^{-q}\left({}_tD_b^q f(t)\right) = f(t),\ \forall t\in[a,b].\qquad\square$$

We need:

Definition. Let $0 < q < 1$, $f \in C([a,b])$. The left Riemann-Liouville fractional integral is given by (see [4, p. 65])

$$ {}_aD_t^q f(t) := \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau)\, d\tau, \ \forall t \in [a,b]. \tag{15}$$

The left Riemann-Liouville fractional derivative of order q is given by (see [4, p. 68])

$$ {}_aD_t^q f(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t (t-\tau)^{-q} f(\tau)\, d\tau, \ \forall t \in [a,b]. \tag{16}$$

We give:

Theorem 2.2. *Let $0 < q < 1$ and $f \in C([a,b])$. Assume that ${}_aD_t^q f \in L_\infty([a,b])$. Then*

$$ {}_aD_t^{-q}\left({}_aD_t^q f(t)\right) = f(t), \ \ \forall\ t \in [a,b], \tag{17}$$

which means

$$ f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} \left({}_aD_\tau^q f(\tau)\right) d\tau, \ \ \forall\ t \in [a,b]. \tag{18}$$

This is a kind of fundamental theorem of left Riemann-Liouville fractional calculus without any initial condition.

Proof. From [4, p. 71, (2.113)], there, when $0 < q < 1$, we get

$$ {}_aD_t^{-q}\left({}_aD_t^q f(t)\right) = f(t) - \left[{}_aD_t^{q-1} f(t)\right]_{t=a} \frac{(t-a)^{q-1}}{\Gamma(q)}. \tag{19}$$

We notice that $(q-1<0)$

$$ {}_aD_t^{q-1} f(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-\tau)^{-q} f(\tau)\, d\tau, \ \forall t \in [a,b]. \tag{20}$$

Hence it holds

$$\begin{aligned} \left|{}_aD_t^{q-1} f(t)\right| &\le \frac{1}{\Gamma(1-q)} \int_a^t (t-\tau)^{-q} |f(\tau)|\, d\tau \\ &\le \frac{\|f\|_{\infty,[a,b]}}{\Gamma(1-q)} \left(\int_a^t (t-\tau)^{-q}\, d\tau\right) \\ &= \frac{\|f\|_{\infty,[a,b]}}{\Gamma(1-q)} \frac{(t-a)^{1-q}}{(1-q)} \\ &= \frac{\|f\|_{\infty,[a,b]}\,(t-a)^{1-q}}{\Gamma(2-q)}. \end{aligned} \tag{21}$$

That is

$$ \left|{}_aD_t^{q-1} f(t)\right| \le \frac{\|f\|_{\infty,[a,b]}}{\Gamma(2-q)} (t-a)^{1-q}, \ \forall t \in [a,b]. \tag{22}$$

Hence
$$\lim_{t\to a+}\left|{}_aD_t^{q-1}f(t)\right|=0.$$
Therefore
$$\left[{}_aD_t^{q-1}f(t)\right]_{t=a}=0. \tag{23}$$
The theorem is proved. □

3. Main results

Next we present the Riemann-Liouville fractional Polya type inequality without any boundary conditions.

Theorem 3.1. *Let* $0<q<1$, $f\in C([a,b])$. *Assume that* ${}_aD_t^qf\in L_\infty\left(\left[a,\frac{a+b}{2}\right]\right)$ *and* ${}_tD_b^qf(t)\in L_\infty\left(\left[\frac{a+b}{2},b\right]\right)$. *Set*
$$N(f):=\max\left\{\|{}_aD_t^qf(t)\|_{\infty,\left[a,\frac{a+b}{2}\right]},\|{}_tD_b^qf(t)\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}. \tag{24}$$
Then
$$\int_a^b|f(t)|\,dt\le N(f)\frac{(b-a)^{q+1}}{\Gamma(q+2)\,2^q}. \tag{25}$$
Inequality (25) *is sharp, namely it is attained by*
$$\overline{f}(t):=\begin{cases}(t-a)^q, & t\in\left[a,\frac{a+b}{2}\right],\\ (b-t)^q, & t\in\left[\frac{a+b}{2},b\right],\end{cases}\quad 0<q<1. \tag{26}$$
Clearly here non-zero constant functions f are not possible.

Proof. By (18) we get that
$$\begin{aligned}|f(t)|&\le\frac{1}{\Gamma(q)}\int_a^t(t-\tau)^{q-1}\left|{}_aD_\tau^qf(\tau)\right|d\tau\\&\le\|{}_aD_t^qf(t)\|_{\infty,\left[a,\frac{a+b}{2}\right]}\frac{(t-a)^q}{\Gamma(q+1)}\end{aligned} \tag{27}$$
for any $t\in\left[a,\frac{a+b}{2}\right]$.

By (4) we get that
$$\begin{aligned}|f(t)|&\le\frac{1}{\Gamma(q)}\int_t^b(\tau-t)^{q-1}\left|{}_\tau D_b^qf(\tau)\right|d\tau\\&\le\|{}_tD_b^qf(t)\|_{\infty,\left[\frac{a+b}{2},b\right]}\frac{(b-t)^q}{\Gamma(q+1)}\end{aligned} \tag{28}$$
for any $t\in\left[\frac{a+b}{2},b\right]$.

Hence we get that (by (27), (28))
$$\int_a^b|f(t)|\,dt=\int_a^{\frac{a+b}{2}}|f(t)|\,dt+\int_{\frac{a+b}{2}}^b|f(t)|\,dt$$

$$\le \frac{1}{\Gamma(q+1)}\left[\left\|{}_aD_t^q f(t)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}\int_a^{\frac{a+b}{2}}(t-a)^q\,dt\right.$$

$$\left.+\left\|{}_tD_b^q f(t)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\int_{\frac{a+b}{2}}^b (b-t)^q\,dt\right] \tag{29}$$

$$=\frac{1}{\Gamma(q+1)(q+1)}\left[\left\|{}_aD_t^q f(t)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}\left(\frac{b-a}{2}\right)^{q+1}\right.$$

$$\left.+\left\|{}_tD_b^q f(t)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\left(\frac{b-a}{2}\right)^{q+1}\right]$$

$$=\frac{1}{\Gamma(q+2)}\left(\frac{b-a}{2}\right)^{q+1}\left[\left\|{}_aD_t^q f(t)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}\right.$$

$$\left.+\left\|{}_tD_b^q f(t)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right]$$

$$\le \max\left\{\left\|{}_aD_t^q f(t)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]},\left\|{}_tD_b^q f(t)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}\frac{(b-a)^{q+1}}{\Gamma(q+2)\,2^q}. \tag{30}$$

So we have proved

$$\int_a^b |f(t)|\,dt \le \max\left\{\left\|{}_aD_t^q f(t)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]},\left\|{}_tD_b^q f(t)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}\frac{(b-a)^{q+1}}{\Gamma(q+2)\,2^q}. \tag{31}$$

Inequality (31) is sharp, it is attained by

$$\overline{f}(t) := \begin{cases}(t-a)^q, & t\in\left[a,\frac{a+b}{2}\right],\\ (b-t)^q, & t\in\left[\frac{a+b}{2},b\right],\end{cases}\quad 0<q<1. \tag{32}$$

Notice that

$$\overline{f}\left(\left(\frac{a+b}{2}\right)_-\right)=\overline{f}\left(\left(\frac{a+b}{2}\right)_+\right)=\left(\frac{b-a}{2}\right)^q, \tag{33}$$

so that $\overline{f}\in C([a,b])$.

We see that (by (16))

$${}_aD_t^q\overline{f}(t)=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t (t-\tau)^{-q}(\tau-a)^q\,d\tau$$

$$=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t (t-\tau)^{(1-q)-1}(\tau-a)^{(q+1)-1}\,d\tau \quad \text{(by [11, p. 256])}$$

$$=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\frac{\Gamma(1-q)\Gamma(q+1)}{\Gamma(2)}(t-a) \tag{34}$$

$$=\Gamma(q+1)\frac{d}{dt}(t-a)=\Gamma(q+1),\ \forall t\in\left[a,\frac{a+b}{2}\right].$$

That is

$$\left\| {}_aD_t^q \overline{f}(t) \right\|_{\infty,\left[a,\frac{a+b}{2}\right]} = \Gamma(q+1). \tag{35}$$

Similarly acting (by (2))

$$\begin{aligned}
{}_tD_b^q \overline{f}(t) &= -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^b (\tau - t)^{-q} (b-\tau)^q \, d\tau \\
&= -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^b (\tau - t)^{(1-q)-1} (b-\tau)^{(q+1)-1} \, d\tau \text{ (by [11, p. 256])} \\
&= -\frac{1}{\Gamma(1-q)} \frac{d}{dt} \frac{\Gamma(q+1)\Gamma(1-q)}{\Gamma(2)} (b-t) \qquad (36) \\
&= -\frac{d}{dt} \Gamma(q+1)(b-t) \\
&= -\Gamma(q+1) \frac{d}{dt}(b-t) = \Gamma(q+1), \ \forall t \in \left[\frac{a+b}{2}, b\right].
\end{aligned}$$

That is

$$\left\| {}_tD_b^q \overline{f}(t) \right\|_{\infty,\left[\frac{a+b}{2},b\right]} = \Gamma(q+1). \tag{37}$$

We have found that

$$\max\left\{ \left\| {}_aD_t^q \overline{f}(t) \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| {}_tD_b^q \overline{f}(t) \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\} = \Gamma(q+1). \tag{38}$$

Therefore the right hand side of (31) for $\overline{f}$ becomes

$$\frac{\Gamma(q+1)}{\Gamma(q+2)} \frac{(b-a)^{q+1}}{2^q} = \frac{(b-a)^{q+1}}{(q+1)\,2^q}. \tag{39}$$

But we notice that

$$\begin{aligned}
\int_a^b \left|\overline{f}(t)\right| dt &= \int_a^b \overline{f}(t)\, dt \\
&= \int_a^{\frac{a+b}{2}} (t-a)^q \, dt + \int_{\frac{a+b}{2}}^b (b-t)^q \, dt \\
&= \frac{1}{(q+1)} \left[\left(\frac{b-a}{2}\right)^{q+1} + \left(\frac{b-a}{2}\right)^{q+1} \right] \\
&= \frac{1}{(q+1)} \left(2 \left(\frac{b-a}{2}\right)^{q+1} \right) = \frac{(b-a)^{q+1}}{(q+1)\,2^q}. \qquad (40)
\end{aligned}$$

By (39) and (40), inequality (31) is attained by $\overline{f}$, that is (25) is sharp. □

In the next assume that $(X, \mathcal{F})$ is a measurable space and $(\mathbb{R}^+)$ $\mathbb{R}$ is the set of all (nonnegative) real numbers.

We recall some concepts and some elementary results of capacity and the Choquet integral [2,3].

Definition. A set function $\mu : \mathcal{F} \to \mathbb{R}^+$ is called a non-additive measure (or capacity) if it satisfies

(1) $\mu(\varnothing) = 0$;

(2) $\mu(A) \leq \mu(B)$ for any $A \subseteq B$ and $A, B \in \mathcal{F}$.

The non-additive measure μ is called concave if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \tag{41}$$

for all $A, B \in \mathcal{F}$. In the literature the concave non-additive measure is known as submodular or 2-alternating non-additive measure. If the above inequality is reverse, μ is called convex. Similarly, convexity is called supermodularity or 2-monotonicity, too.

First note that the Lebesgue measure λ for an interval $[a, b]$ is defined by $\lambda([a, b]) = b - a$, and that given a distortion function m, which is increasing (or non-decreasing) and such that $m(0) = 0$, the measure $\mu(A) = m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m by $\mu = \mu_m$. It is known that μ_m is concave (convex) if m is a concave (convex) function.

The family of all the nonnegative, measurable function

$$f : (X, \mathcal{F}) \to \left(\mathbb{R}^+, \mathcal{B}\left(\mathbb{R}^+\right)\right)$$

is denoted as L_∞^+, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field of $\mathbb{R}^+$. The concept of the integral with respect to a non-additive measure was introduced by Choquet [2].

Definition. Let $f \in L_\infty^+$. The Choquet integral of f with respect to non-additive measure μ on $A \in \mathcal{F}$ is defined by

$$(C) \int_A f d\mu := \int_0^\infty \mu(\{x : f(x) \geq t\} \cap A)\, dt, \tag{42}$$

where the integral on the right-hand side is a Riemann integral.

Instead of $(C) \int_X f d\mu$, we shall write $(C) \int f d\mu$. If $(C) \int f d\mu < \infty$, we say that f is Choquet integrable and we write

$$L_C^1(\mu) = \left\{ f : (C) \int f d\mu < \infty \right\}.$$

The next lemma summarizes the basic properties of Choquet integrals [3].

Lemma 3.2. *Assume that $f, g \in L_C^1(\mu)$.*

(1) $(C) \int 1_A d\mu = \mu(A)$, $A \in \mathcal{F}$.

(2) (*Positive homogeneity*) *For all $\lambda \in \mathbb{R}^+$, we have $(C) \int \lambda f d\mu = \lambda \cdot (C) \int f d\mu$.*

(3) (*Translation invariance*) *For all $c \in \mathbb{R}$, we have $(C) \int (f + c) d\mu = (C) \int f d\mu + c$.*

(4) (*Monotonicity in the integrand*) *If $f \leq g$, then we have*

$$(C) \int f d\mu \leq (C) \int g d\mu.$$

(*Monotonicity in the set function*) *If* $\mu \leq \nu$, *then we have* $(C)\int f d\mu \leq (C)\int f d\nu$.

(5) (*Subadditivity*) *If* μ *is concave, then*

$$(C)\int (f+g)\,d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

(*Superadditivity*) *If* μ *is convex, then*

$$(C)\int (f+g)\,d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

(6) (*Comonotonic additivity*) *If* f *and* g *are comonotonic, then*

$$(C)\int (f+g)\,d\mu = (C)\int f d\mu + (C)\int g d\mu,$$

where we say that f *and* g *are comonotonic, if for any* $x, x' \in X$, *then*

$$(f(x) - f(x'))\,(g(x) - g(x')) \geq 0.$$

We next mention the amazing result from [10], which permits us to compute the Choquet integral when the non-additive measure is a distorted Lebesgue measure.

Theorem 3.3. *Let* f *be a nonnegative and measurable function on* $\mathbb{R}^+$ *and* $\mu = \mu_m$ *be a distorted Lebesgue measure. Assume that* $m(x)$ *and* $f(x)$ *are both continuous and* $m(x)$ *is differentiable. When* f *is an increasing (non-decreasing) function on* $\mathbb{R}^+$, *the Choquet integral of* f *with respect to* μ_m *on* $[0,t]$ *is represented as*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(t-x)\,f(x)\,dx, \tag{43}$$

however, when f *is a decreasing (non-increasing) function on* $\mathbb{R}^+$, *the Choquet integral of* f *is*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(x)\,f(x)\,dx. \tag{44}$$

We make:

Remark 3.4. From now on we assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone continuous function, and $\mu = \mu_m$, i.e., $\mu(A) = m(\lambda(A))$, denotes a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing (non-decreasing) and continuously differentiable.

By Theorem 3.3 and mean value theorem for integrals we get:
i) If f is an increasing (non-decreasing) function on $\mathbb{R}^+$, we have

$$(C)\int_{[0,t^*]} f d\mu_m \overset{(43)}{=} \int_0^{t^*} m'(t^*-x)\,f(x)\,dx$$

$$= m'(t^* - \xi) \int_0^{t^*} f(x)\, dx, \text{ where } \xi \in (0, t^*), \ t^* > 0. \tag{45}$$

ii) If f is a decreasing (non-increasing) function on $\mathbb{R}^+$, we have

$$(C) \int_{[0,t^*]} f d\mu_m \overset{(44)}{=} \int_0^{t^*} m'(x) f(x)\, dx = m'(\xi) \int_0^{t^*} f(x)\, dx, \tag{46}$$

where $\xi \in (0, t^*)$, $t^* > 0$.

We denote by

$$\gamma(t^*, \xi) := \begin{cases} m'(t^* - \xi), \text{ when } f \text{ is increasing (non-decreasing)} \\ m'(\xi), \text{ when } f \text{ is decreasing (non-increasing),} \end{cases} \tag{47}$$

for some $\xi \in (0, t^*)$ per case, $t^* > 0$.

We give the following Choquet-fractional-Polya inequality without any boundary conditions.

Theorem 3.5. *Let $0 < q < 1$, $f = f|_{[0,t^*]}$, $t^* > 0$, be continuous and all considered as in Remark 3.4. Assume further that ${}_0D_t^q f(t) \in L_\infty\left(\left[0, \frac{t^*}{2}\right]\right)$ and ${}_tD_{t^*}^q f(t) \in L_\infty\left(\left[\frac{t^*}{2}, t^*\right]\right)$. Set*

$$\overline{N}(f)(t^*) := \max\left\{ \left\| {}_0D_t^q f(t) \right\|_{\infty, \left[0, \frac{t^*}{2}\right]}, \left\| {}_tD_{t^*}^q f(t) \right\|_{\infty, \left[\frac{t^*}{2}, t^*\right]} \right\}, \ t^* > 0. \tag{48}$$

Then

$$(C) \int_{[0,t^*]} f d\mu_m \leq \gamma(t^*, \xi)\, \overline{N}(f)(t^*) \frac{(t^*)^{q+1}}{\Gamma(q+2)\, 2^q}, \quad t^* > 0. \tag{49}$$

Clearly here f can not be a non-zero constant.

Proof. By Theorem 3.1 and earlier comments. □

We give some examples for m.

Remark 3.6. i) If $m(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m(t) \geq 0$, $m'(t) = \frac{1}{(1+t)^2} > 0$, and m is increasing. Then $\gamma(t^*, \xi) \leq 1$.

ii) If $m(t) = 1 - e^{-t} \geq 0$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m'(t) = e^{-t} > 0$, and m is increasing. Then $\gamma(t^*, \xi) \leq 1$.

iii) If $m(t) = e^t - 1 \geq 0$, $t \in \mathbb{R}^+$, $m(0) = 0$, $m'(t) = e^t > 0$, and m is increasing. Then $\gamma(t^*, \xi) \leq e^{t^*}$.

iv) If $m(t) = \sin t$, for $t \in \left[0, \frac{\pi}{2}\right]$, we get $m(0) = 0$, $m'(t) = \cos t \geq 0$, and m is increasing. Then $\gamma(t^*, \xi) \leq 1$.

References

[1] G. A. Anastassiou, *Intelligent mathematics: computational analysis*, Intelligent Systems Reference Library, **5**, Springer-Verlag, Berlin, 2011. `https://doi.org/10.1007/978-3-642-17098-0`

[2] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier, Grenoble **5** (1953), 131–295.

[3] D. Denneberg, *Non-additive measure and integral*, Theory and Decision Library. Series B: Mathematical and Statistical Methods, **27**, Kluwer Academic Publishers Group, Dordrecht, 1994. https://doi.org/10.1007/978-94-017-2434-0

[4] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, **198**, Academic Press, Inc., San Diego, CA, 1999.

[5] G. Pólya, *Ein Mittelwertsatz für Funktionen mehrerer Veränderlichen*, Tohoku Math. J. **19** (1921), 1–3.

[6] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis. Band I*, Dritte berichtigte Auflage. Die Grundlehren der Mathematischen Wissenschaften, Band 19, Springer-Verlag, Berlin, 1964.

[7] ———, *Problems and theorems in analysis. Vol. I*, translated from the German by D. Aeppli, Springer-Verlag, New York, 1972.

[8] ———, *Problems and theorems in analysis. Vol. II*, revised and enlarged translation by C. E. Billigheimer of the fourth German edition, springer Study Edition, Springer-Verlag, New York, 1976.

[9] F. Qi, *Pólya type integral inequalities: origin, variants, proofs, refinements, generalizations, equivalences, and applications*, Math. Inequal. Appl. **18** (2015), no. 1, 1–38. https://doi.org/10.7153/mia-18-01

[10] M. Sugeno, *A note on derivatives of functions with respect to fuzzy measures*, Fuzzy Sets and Systems **222** (2013), 1–17. https://doi.org/10.1016/j.fss.2012.11.003

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996. https://doi.org/10.1017/CBO9780511608759

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