Bull. Korean Math. Soc. **56** (2019), No. 6, pp. 1525–1537
https://doi.org/10.4134/BKMS.b181232
pISSN: 1015-8634 / eISSN: 2234-3016

RIEMANNIAN SUBMANIFOLDS WITH CONCIRCULAR CANONICAL FIELD

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Abstract. Let $\tilde{M}$ be a Riemannian manifold equipped with a concircular vector field $\tilde{X}$ and M a submanifold (with its induced metric) of $\tilde{M}$. Denote by X the restriction of $\tilde{X}$ on M and by X^T the tangential component of X, called the canonical field of M. In this article we study submanifolds of $\tilde{M}$ whose canonical field X^T is also concircular. Several characterizations and classification results in this respect are obtained.

1. Introduction

A vector field $\tilde{X}$ on a Riemannian (or pseudo-Riemannian) manifold $\tilde{M}$ is called a *concircular vector field* if it satisfies

$$\tilde{\nabla}_{\tilde{Z}}\tilde{X} = \tilde{\varphi}\tilde{Z} \tag{1.1}$$

for any $\tilde{Z}$ tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{M}$ and $\tilde{\varphi}$ is a real-valued function on $\tilde{M}$. In particular, if $\tilde{\varphi} = 1$ (resp., $\tilde{\varphi} = 0$), then the concircular vector field $\tilde{X}$ is called a *concurrent vector field* (resp., *parallel vector field*).

The notion of concircular vector fields on a Riemannian manifold was first introduced by A. Fialkow in [17]. Concircular vector fields are also known as geodesic fields in literature since unit speed integral curves of such vector fields are geodesics. Concircular vector fields appeared naturally in the study of concircular mappings, i.e., conformal mappings preserving geodesic circles [27]. Concircular vector fields also play important roles in the theory of projective and conformal transformations. Such vector fields have interesting applications in general relativity, e.g., it was shown in [23] that trajectories of timelike concircular fields in the de Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis.

It was proved in [4] that the gradient ∇f of a function f on a Riemannian manifold is a concircular vector field if and only if the Hessian of f satisfies

Received December 19, 2018; Accepted February 27, 2019.
2010 *Mathematics Subject Classification.* 53C40, 53C42.
Key words and phrases. concircular canonical field, canonical field, conservative vector field, conformal canonical field.

Hess $f = \mu g$ for some function μ. Further, it was proved in [3] that a Lorentzian manifold is a generalized Robertson-Walker space-time if and only if it admits a timelike concircular vector field.

It is well known that the position vector field $\tilde{\mathbf{x}}$ of the Euclidean m-space $\mathbb{E}^m$ is a concurrent vector field. For a submanifold M of $\mathbb{E}^m$, the most elementary and natural vector field on M is its position vector field $\mathbf{x}$ induced from $\tilde{\mathbf{x}}$ in $\mathbb{E}^m$. The tangential component $\mathbf{x}^T$ of $\mathbf{x}$ is called the *canonical field* of the Euclidean submanifold M (cf. [9]).

In earlier articles, we have studied Euclidean submanifolds whose canonical fields are concurrent [5, 9], concircular [15], torse-forming [13], conformal [12], or incompressible [9] (see also [7,8]). Further, submanifolds with incompressible canonical fields in a Riemannian manifold equipped with a concurrent vector field have been investigated recently in [10].

In this article we study the following simple and natural problem.

Problem. *Let $\tilde{M}$ be a Riemannian manifold equipped with a concircular vector field $\tilde{X}$ and let M be a submanifold of $\tilde{M}$. When the* (*induced*) *canonical field X^T on M is also concircular?*

Since the cononical field X^T of any 1-dimensional submanifold of $\tilde{M}$ is always concircular, we shall always assume "$\dim M \geq 2$" in the Problem proposed above.

This article is organized as follows. Basic definitions and formulas are given in Section 2. Submanifolds with concircular canonical fields are characterized in Section 3. In Section 4 we determine submanifolds of $\tilde{M}$ with $X^T = 0$ or $X^T = X$. We also determine hypersurfaces of $\tilde{M}$ with concircular canonical field $X^T \neq 0, X$. Further, we provide in this section an example of hypersurface with concircular canonical field $X^T \neq 0$, X. In the last section, using some conservative vector fields, we make a remark and observation on compact Euclidean submanifolds via an extrinsic average variational method in the calculus of variations.

2. Preliminaries

For a submanifold M of a Riemannian manifold $\tilde{M}$, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [2, 6, 11])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for any vector fields X, Y tangent to M and vector field ξ normal to M, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}$. Choose a local field of orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ in $\tilde{M}$ such

that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M.

For each normal vector ξ at $p \in M$, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A are related by

$$g(A_\xi X, Y) = \tilde{g}(h(X,Y), \xi), \tag{2.3}$$

where g and $\tilde{g}$ denote the metric of M and the metric of the ambient space $\tilde{M}$, respectively.

The mean curvature vector H is defined by

$$H = \frac{1}{n} \operatorname{trace} h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.4}$$

A submanifold M is called *totally umbilical* if its second fundamental form satisfies

$$h(X,Y) = g(X,Y)H \tag{2.5}$$

for vector fields X, Y tangent to M. In particular, it is called *totally geodesic* if $h = 0$ holds identically.

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F, respectively, and let f be a positive function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The *warped product* $B \times_f F$ is the manifold $B \times F$ equipped with the warped product metric given by

$$g = g_B + f^2 g_F. \tag{2.6}$$

The function f in (2.6) is called the *warping function* of the warped product.

For a warped product $B \times_f F$, B is called the *base* of the warped product. The leaves $B \times \{q\} = \eta^{-1}(q)$ and the *fibers* $\{p\} \times F = \pi^{-1}(p)$, $p \in B$, are Riemannian submanifolds of the warped product $B \times_f F$.

The notion of warped products plays important roles in differential geometry as well as in physics, especially in the theory of general relativity (cf. [1,6,22]).

A vector field on a Riemannian manifold (M, g) is called *conservative* if it is the gradient of some function, known as a *scalar potential*. A vector field v on (M, g) is called *conformal* if it satisfies

$$\mathcal{L}_v g = 2\mu g \tag{2.7}$$

for some function μ, where $\mathcal{L}$ denotes the Lie derivative of M.

3. Characterizations

Let $\tilde{M}$ be a Riemannian manifold equipped with a concircular vector field $\tilde{X}$ satisfying

$$\tilde{\nabla}_{\tilde{Z}} \tilde{X} = \tilde{\varphi} \tilde{Z} \tag{3.1}$$

and M a submanifold of $\tilde{M}$. Let φ and X denote the restriction of $\tilde{\varphi}$ and $\tilde{X}$ on M. Denote by X^T and X^N the tangential and normal components of X, respectively. Associated with $\tilde{X}$, we simply call X^T the *canonical field* and X^N the *canonical normal field* of M.

The main results of this section are the following.

Theorem 3.1. *Let $\tilde{M}$ be a Riemannian manifold with a concircular vector field $\tilde{X}$, and let M be a submanifold of $\tilde{M}$ and φ the restriction of $\tilde{\varphi}$ on M. Then we have*:

(1) *φX^T is a conservative vector field on M. In particular, if $\tilde{X}$ is concurrent, then the canonical field X^T is conservative.*
(2) *The canonical field of M is concircular if and only if the shape operator A_{X^N} in the direction of the canonical normal field X^N is proportional to the identity map I.*
(3) *The canonical field is conformal if and only if it is concircular.*

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concircular vector field $\tilde{X}$. Assume that M is an n-dimensional submanifold of $\tilde{M}$ and $\{e_1, \ldots, e_n\}$ is an orthonormal local frame of M.

If we put $\zeta = \frac{1}{2}\langle X, X\rangle$, then it follows that the gradient of ζ satisfies

$$\begin{aligned} \nabla\zeta &= \frac{1}{2}\sum_{i=1}^{n}(e_i\langle X, X\rangle)e_i = \sum_{i=1}^{n}\langle \tilde{\nabla}_{e_i}X, X\rangle e_i \\ &= \sum_{i=1}^{n}\varphi\langle e_i, X\rangle e_i = \varphi X^T, \end{aligned} \tag{3.2}$$

which implies that φX^T is a conservative vector field on M with scalar potential ζ. In particular, if $\tilde{X}$ is concurrent, then $\varphi = 1$. Hence the canonical field X^T is conservative. This proves statement (1).

We derive from (3.1) and the formulas of Gauss and Weingarten that

$$\begin{aligned} \varphi Z &= \tilde{\nabla}_Z\tilde{X} = \tilde{\nabla}_Z X \\ &= \nabla_Z X^T + h(X^T, Z) - A_{X^N}Z + D_Z X^N \end{aligned}$$

for any vector Z tangent to M, where φ denotes the restriction of $\tilde{\varphi}$ on M. After comparing the tangential and normal components of the last equation, we obtain the following.

Lemma 3.1. *Let $\tilde{M}$ be a Riemannian manifold with a concircular vector field $\tilde{X}$ and let M be a submanifold of $\tilde{M}$. Then we have*:

$$\nabla_Z X^T = \varphi Z + A_{X^N}Z, \tag{3.3}$$

$$D_Z X^N = -h(X^T, Z). \tag{3.4}$$

Now, assume that M has a concircular canonical field. Then we get

$$\nabla_Z X^T = \eta Z \tag{3.5}$$

for some function η. After combining (3.3) and (3.5), we find

$$A_{X^N} = (\eta - \varphi)I. \tag{3.6}$$

Conversely, if M is a submanifold whose shape operator satisfies condition (3.6) for some function λ, then, after substituting (3.6) into (3.3), we obtain

$$\nabla_Z X^T = (\varphi + \lambda)Z \tag{3.7}$$

for any vector Z tangent to M. Consequently, the canonical field of M is concircular. This proves statement (2).

To prove statement (3), let us recall that the Lie derivative on (M, g) satisfies (see, e.g. [6, Page 18] or [28])

$$(\mathcal{L}_U g)(V, W) = g(\nabla_V U, W) + g(V, \nabla_W U) \tag{3.8}$$

for any vector fields U, V, W tangent to M. After combining (3.3) and (3.8) we find

$$(\mathcal{L}_{X^T} g)(V, W) = 2\varphi g(V, W) + g(A_{X^N} V, W) + g(V, A_{X^N} W). \tag{3.9}$$

Therefore, after using (2.3) we obtain

$$(\mathcal{L}_{X^T} g)(V, W) = 2\varphi g(V, W) + 2g(h(V, W), X^N) \tag{3.10}$$

for vector fields V, W tangent to M.

Now, suppose that the canonical field X^T is a conformal vector field. Then we have

$$\mathcal{L}_{X^T} g = 2\psi g \tag{3.11}$$

for some function ψ. Thus, it follows from (3.10) and (3.11) that

$$g(h(V, W), X^N) = (\psi - \varphi)g(V, W), \tag{3.12}$$

which implies that A_{X^N} is proportional to the identity map I. Hence the canonical field of M is concircular according to statement (2).

Conversely, if the canonical field X^T is concircular, then statement (2) yields

$$g(h(X, Y), X^N) = \eta g(X, Y) \tag{3.13}$$

for some function η. So, it follows from (3.10) and (3.13) that

$$(\mathcal{L}_{X^T} g)(X, Y) = 2(\varphi + \eta)g(X, Y). \tag{3.14}$$

Therefore the canonical vector field is a conformal vector field. This proves statement (3). □

Remark 3.1. Theorem 3.1(1) is an extension of Theorem 3.1(1) of [9].

Since the position vector field of a Euclidean space is a concurrent vector field, Theorem 3.1(3) implies the following.

Corollary 3.1. *If M is a submanifold of a Euclidean m-space $\mathbb{E}^m$, then the canonical field of M is conformal if and only if it is concircular.*

The *relative null subspace* of a Riemannian submanifold M is defined by (cf. for instance [2, page 44]):

$$\mathcal{N}_p = \{V \in T_pM : h(V, W) = 0 \ \ \forall W \in T_pM\}.$$

An easy consequence of Theorem 3.1 and Lemma 3.1 is the following.

Corollary 3.2. *Let $\tilde{M}$ be a Riemannian manifold with a concircular vector field $\tilde{X}$ and let M be a submanifold of $\tilde{M}$. Then we have*:

(1) *The canonical normal field X^N is parallel in the normal bundle, i.e., $DX^N = 0$, if and only if $X_p^T \in \mathcal{N}_p$ holds for $p \in M$.*
(2) *If X^N is nowhere zero on a hypersurface M, then the canonical field of M is concircular if and only if M is totally umbilical.*

Corollary 3.3. *Let $\tilde{M}$ be a Riemannian manifold with a concurrent vector field $\tilde{X}$ and let M be a submanifold of $\tilde{M}$. Then the canonical field of M is concurrent if and only if the shape operator $A_{X^N} = 0$.*

Further, if M is a hypersurface and X^N is nowhere zero on M, then the canonical field is concurrent if and only if M is totally geodesic.

Proof. Since $\tilde{X}$ is assumed to be concurrent, (3.3) holds with $\varphi = 1$. If X^T is also concurrent, then (3.5) holds with $\eta = 1$. Hence, $A_{X^N} = 0$ in (3.6) with $\lambda = \eta - \varphi = 0$.

Conversely, if the shape operator of M satisfies $A_{X^N} = 0$, then after substituting this into (3.3) we obtain (3.7) with $\varphi = 1$ and $\lambda = 0$, i.e., $\nabla_Z X^T = Z$ for any Z tangent to M. Hence the canonical field X^T is concurrent.

The last assertion is an easy consequence of the first assertion. □

Example 3.1. The ordinary n-sphere S^n is a non-totally geodesic hypersurface of $\mathbb{E}^{n+1}$, the position vector $\mathbf{x}$ of $\mathbb{E}^{n+1}$ is concurrent, but the canonical field $\mathbf{x}^T$ of S^n is not concurrent according to Corollary 3.3.

Example 3.2. The ordinary n-sphere S^n is a totally umbilic hyersurface of $\mathbb{E}^{n+1}$. Thus A_{X^N} satisfies (3.6) in which λ is a constant. Since the position vector field $\mathbf{x}$ of $\mathbb{E}^{n+1}$ is concurrent, the canonical field $\mathbf{x}^T$ of S^n is hence conservative by Theorem 3.1(1); concircular by Theorem 3.1(2); and conformal by Theorem 3.1(3).

4. Submanifolds with concircular canonical field

First, we provide the following.

Example 4.1. Let $\tilde{I}$ be an open interval, $f(s)$ a positive function defined on $\tilde{I}$, and $\tilde{F}$ a Riemannian manifold. Consider the warped product $\tilde{M} = \tilde{I} \times_{f(s)} \tilde{F}$ equipped with the metric $\tilde{g} = ds^2 + f^2(s) g_{\tilde{F}}$, where $g_{\tilde{F}}$ is the metric tensor of $\tilde{F}$. Then the vector field $\tilde{X} = f(s)\frac{\partial}{\partial s}$ satisfies $\tilde{\nabla}_{\tilde{V}} \tilde{X} = f'(s)\tilde{V}$ for any $\tilde{V}$ tangent to $\tilde{M}$. Therefore $\tilde{X}$ is a concircular vector field on $\tilde{M}$.

We recall the following definition (cf. for instance [2, 6, 14]).

Definition 4.1. A *slice* of $\tilde{M} = \tilde{I} \times_{f(s)} \tilde{F}$ is a hypersurface given by $S(s_0) = \{s_0\} \times \tilde{F}$, $s_0 \in \tilde{I}$. A submanifold M of $\tilde{M}$ is called *transverse* if it is contained in a *slice* $S(s_0)$, $s_0 \in \tilde{I}$. A submanifold M in $\tilde{M}$ is called an *$\mathcal{H}$-submanifold* or a *horizontal submanifold* if $\tilde{X} = f(s)\frac{\partial}{\partial s}$ is always tangent to M.

From [4, Theorem 3.1] and its proof, we have:

Theorem 4.1. *Let $\tilde{M}$ be a Riemannian manifold equipped with a nowhere zero concircular vector field $\tilde{X}$. Then M is locally a warped product $\tilde{I} \times_{f(s)} \tilde{F}$, where $f(s) = |\tilde{X}|$ and $\tilde{F}$ is a Riemannian manifold. Moreover, we have $\tilde{X} = f(s)\frac{\partial}{\partial s}$.*

Now, we provide examples of submanifolds of $\tilde{M}$ with $X = X^T$.

Example 4.2 (Transverse submanifold)**.** Let $\tilde{M}$ denote the warped product $\tilde{I} \times_{f(s)} \tilde{F}$, I an open subinterval of $\tilde{I}$ and F a submanifold of $\tilde{F}$. Then $M = I \times_{f(s)} F$ is a Riemannian submanifold of $\tilde{M} = \tilde{I} \times_{f(s)} \tilde{F}$. Consider the concircular vector field $\tilde{X} = f(s)\frac{\partial}{\partial s}$ of $\tilde{M}$. Then $X = \tilde{X}|_M$ is a tangent vector field of $M = I \times_{f(s)} F$. So, we get $X = X^T$ and $X^N = 0$. Hence we have $A_{X^N} = \lambda I$ with $\lambda = 0$. Therefore $X = X^T$ is a concircular vector field on M according to Theorem 3.1(2). In particular, if M a hypersurface, then M is totally geodesic in $\tilde{M}$.

The next result describes all submanifolds of $\tilde{I} \times_{f(s)} \tilde{F}$ satisfying the condition $X^T = X$.

Proposition 4.1. *Let $\tilde{M}$ be the warped product $\tilde{I} \times_{f(s)} \tilde{F}$ and let M be a submanifold of $\tilde{M}$. If the restriction X of $\tilde{X} = f(s)\frac{\partial}{\partial s}$ on M is a tangent vector field of M, then M is locally given by Example 4.2. Moreover, the canonical field X^T of M is concircular.*

Proof. Let $\tilde{M}$ be the warped product $\tilde{I} \times_{f(s)} \tilde{F}$ and let M be a submanifold of $\tilde{M}$ such that $X = \tilde{X}|_M$ satisfies $X = X^T$. Then we have

$$\varphi Z = \tilde{\nabla}_Z X = \nabla_Z X^T + h(X^T, Z) \tag{4.1}$$

for any $Z \in TM$. Thus we find

$$\nabla_Z X^T = \varphi Z, \quad h(X^T, Z) = 0, \tag{4.2}$$

which shows that the canonical filed X^T of M is concircular. Hence, by Theorem 4.1, M is locally a warped product $I \times_{\beta(s)} F$ with $\beta(s) = |X^T|$. Since $X^T = X$, we may assume that I is an open subinterval of $\tilde{I}$. Also, it follows from $X^T = X$ that we have $\beta = f = |X^T|$ for M. Moreover, because the second factor F of $I \times_{\beta(s)} F$ is orthogonal to $\tilde{I}$, F can be regarded a submanifold of $\tilde{F}$. Consequently, M is locally given by Example 4.2. □

Next, we provide examples of submanifolds satisfying $X^T = 0$.

Example 4.3 (Horizontal submanifold)**.** Let $\tilde{M}$ denote the warped product manifold $\tilde{I} \times_{f(s)} \tilde{F}$ and let F be a submanifold of $\tilde{F}$ and $s_0 \in \tilde{I}$. Clearly, the fiber $M = \{s_0\} \times_{f(s)} F$ is a submanifold of $\tilde{M} = \tilde{I} \times_{f(s)} \tilde{F}$. Since $\tilde{X} = f(s)\frac{\partial}{\partial s}$ is normal to M, we have $X^T = 0$ and $X = X^N$, which shows that M has concircular canonical field X^T trivially. Thus, it follows from Lemma 3.1 that $A_{X^N} = -\varphi I$ and $D_Z X^N = 0$ hold identically. Consequently, $M = \{s_0\} \times_{f(s)} F$ is an *extrinsic sphere*, i.e., it is a totally umbilical submanifold with nonzero parallel mean curvature vector (cf. [21]). In particular, this shows that each fiber $\{s_0\} \times_{f(s)} \tilde{F}$ of $\tilde{I} \times_{f(s)} \tilde{F}$ is a totally umbilical hypersurface of $\tilde{M}$ with constant mean curvature.

Now, we describe all submanifolds of $\tilde{I} \times_{f(s)} \tilde{F}$ satisfying $X^T = 0$.

Proposition 4.2. *Let $\tilde{M}$ be the warped product $\tilde{I} \times_{f(s)} \tilde{F}$ and let M be a submanifold of $\tilde{M}$. If the restriction X of $\tilde{X} = f(s)\frac{\partial}{\partial s}$ on M is normal to M, then M is locally given by Example 4.3.*

Proof. Let $\tilde{M}$ denote the warped product $\tilde{I} \times_{f(s)} \tilde{F}$ and let M be a submanifold of $\tilde{M}$. If the restriction X of $\tilde{X} = f(s)\frac{\partial}{\partial s}$ on M is always normal to M, then each tangent vector of M is orthogonal to $\frac{\partial}{\partial s}$. But this can happen only when M lies in a fiber $\{s_0\} \times_{f(s)} \tilde{F}$ for some real number $s_0 \in I$. □

The next result follows easily from Theorem 3.1(2), Propositions 4.1 and 4.2 and Definition 4.1.

Proposition 4.3. *Let $\tilde{M}$ be the warped product $\tilde{I} \times_{f(s)} \tilde{F}$ and M a hypersurface of $\tilde{M}$. If M has concircular canonical field $X^T \neq 0, X$, then it is a totally umbilical hypersurface which is neither transversal or horizontal.*

Finally, we provide an example of totally umbilical hypersurface of a warped product $\tilde{I} \times_{f(s)} \tilde{F}$ which is neither transversal or horizontal.

Example 4.4. Let $f = \operatorname{sech} s$, $s \in \tilde{I} = (-2, 2)$, and let $(x_1, \ldots, x_n)$ be a Euclidean coordinate system on $\mathbb{E}^n$. If we put $\tilde{M} = \tilde{I} \times \mathbb{E}^n$, then the metric tensor of $\tilde{M}$ is given by

$$\tilde{g} = ds^2 + \operatorname{sech}^2(s) \sum_{i=1}^{n} dx_i^2. \tag{4.3}$$

Put $I = (-1, 1)$. Let us consider the immersion:

$$\psi : I \times \mathbb{R}^{n-1} \to \tilde{M} = \tilde{I} \times_{f(s)} \mathbb{E}^n$$

defined by

$$\psi(t, u_2, \ldots, u_n) = \big(t, \sinh t, u_2, \ldots, u_n\big) \in \tilde{I} \times \mathbb{E}^n \tag{4.4}$$

for any $t \in (-1,1)$ and $(u_2,\dots,u_n) \in \mathbb{R}^{n-1}$. Then we derive from (4.4) that

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= (1, \cosh t, 0, \dots, 0), \\ \frac{\partial \psi}{\partial u_j} &= (0, \dots, 0, \overbrace{1}^{(j+1)\text{-th}}, 0, \dots, 0), \ 2 \le j \le n. \end{aligned} \tag{4.5}$$

By applying (4.3) and (4.5), it is easy to verify that the metric tensor g on $I \times \mathbb{R}^{n-1}$ induced from the immersion ψ is given by

$$g = 2dt^2 + \operatorname{sech}^2 t \sum_{i=2}^{n} du_i^2. \tag{4.6}$$

Let M denote the Riemannian manifold $I \times \mathbb{R}^{n-1}$ equipped with the induced metric g via the immersion ψ. Then it follows from (4.3) and (4.5) that

$$\xi = \frac{1}{\sqrt{2}}(1, -\cosh t, 0, \dots, 0), \tag{4.7}$$

is a unit normal vector field of M in $\tilde{M}$.

After a direct computation, we know that the second fundamental form of M satisfies

$$h(X,Y) = \frac{\tanh t}{\sqrt{2}} \langle X, Y \rangle \xi.$$

Therefore, we conclude that M is totally umbilical in $\tilde{M}$ with non-constant mean curvature given by $(\tanh t)/\sqrt{2}$. Consequently, according to Examples 4.2 and 4.3, M is a totally umbilical hypersurface which is neither transversal or horizontal in $\tilde{M} = \tilde{I} \times \mathbb{E}^n$.

5. A remark and observation on conservative fields

The position vector field $\widetilde{\mathbf{x}}$ of a Euclidean space $\mathbb{E}^m$ is a concurrent vector field. Let M be a compact n-manifold isometrically immersed in $\mathbb{E}^m$ with second fundamental form h. Denote by $\mathbf{x}$ the restriction of $\widetilde{\mathbf{x}}$ on M. Then the tangential component $\mathbf{x}^T$ of $\mathbf{x}$ on M is a conservative vector field with scalar potential $\zeta = \frac{1}{2}\langle \mathbf{x}, \mathbf{x} \rangle$ (cf. Theorem 3.1(1) and (3.2) or [9, Theorem 3.1(1)]). In view of this, we make the following remark and observation.

Let $\{v_1, \dots, v_n, v_{n+1}, \dots, v_m\}$ be an orthonormal basis of $\mathbb{E}^m$. As v_ℓ, $1 \le \ell \le m$, can be identified with a parallel and concircular vector field in $\mathbb{E}^m$ (i.e., $\tilde{\nabla}_{\tilde{Z}} v_\ell = 0$ with $\tilde{\varphi} = 0$ in (1.1) for every $\tilde{Z}$ tangent to $\mathbb{E}^m$), this gives rise to a set of conservative vector fields

$$\{v_1^T, \dots, v_n^T, v_{n+1}^T, \dots, v_m^T\} \quad \text{on } M,$$

where $v_1^T, \dots, v_m^T$ are defined by $v_\ell^T = \nabla \langle v_\ell, \mathbf{x} \rangle$.

Clearly, each v_ℓ^T generates a flow or a one-parameter group of diffeomorphisms $\varphi_t^{v_\ell^T} : M \to M$. Further, given a smooth map $u : N \to M$ between two

compact Riemannian manifolds, we can deform u in v_ℓ^T direction to obtain the variation $u_t = \varphi_t^{v_\ell^T} \circ u$ of u with the initial condition $u_0 = u$.

Let us consider the energy of $\varphi_t^{v_\ell^T} \circ u$:

$$E(\varphi_t^{v_\ell^T} \circ u) = \int_N \sum_{i=1}^k \langle d(\varphi_t^{v_\ell^T} \circ u)e_i, d(\varphi_t^{v_\ell^T} \circ u)e_i\rangle dV_N,$$

where $d(\varphi_t^{v_\ell^T} \circ u)$ is the differential of $\varphi_t^{v_\ell^T} \circ u$, $\{e_1, \dots, e_k\}$ is a local orthonormal frame field on N, and dV_N is the volume element of N. Thus, to each direction v_ℓ^T, the energy $E(\varphi_t^{v_\ell^T}(u))$ via the variation $\varphi_t^{v_\ell^T} \circ u$ is a smooth real valued function of t, and there corresponds to its rate of change of the energy in that direction v_ℓ^T to the second order, i.e., $\frac{d^2}{dt^2} E(\varphi_t^{v_\ell^T}(u))\big|_{t=0}$. Therefore, to the set of the m vector fields $\{v_1^T, \dots, v_n^T, v_{n+1}^T, \dots, v_m^T\}$ on M, there correspond to the set of m real numbers, i.e., m second variations given by

$$\left\{ \frac{d^2}{dt^2} E(\varphi_t^{v_1^T}(u))\big|_{t=0}, \dots, \frac{d^2}{dt^2} E(\varphi_t^{v_n^T}(u))\big|_{t=0}, \right.$$
$$\left. \frac{d^2}{dt^2} E(\varphi_t^{v_{n+1}^T}(u))\big|_{t=0}, \dots, \frac{d^2}{dt^2} E(\varphi_t^{v_m^T}(u))\big|_{t=0} \right\}$$

and their average or sum: $\sum_{\ell=1}^m \frac{d^2}{dt^2} E(\varphi_t^{v_\ell^T}(u))\big|_{t=0}$.

Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a local orthonormal frame field on M. It was shown by Howard and Wei in [19] that if the map u is non-constant and the second fundamental form h of M in $\mathbb{E}^m$ satisfies

$$\sum_{j=1}^n \left\{ 2\langle h(X, \varepsilon_j), h(X, \varepsilon_j)\rangle - \langle h(X, X), h(\varepsilon_j, \varepsilon_j)\rangle \right\} < 0 \tag{5.1}$$

for each non-zero tangent vector X to M at any point in M, then the average variation, or the sum satisfies

$$\begin{aligned} &\sum_{\ell=1}^m \frac{d^2}{dt^2} E(\varphi_t^{v_\ell^T}(u))\big|_{t=0} \\ &= \int_N \sum_{i=1}^k \sum_{j=1}^n \left\{ 2\langle h(du(e_i), \varepsilon_j), h(du(e_i), \varepsilon_j)\rangle \right. \\ &\quad \left. - \langle h(du(e_i), du(e_i)), h(\varepsilon_j, \varepsilon_j)\rangle \right\} \\ &< 0, \end{aligned} \tag{5.2}$$

by applying (5.1) in which $X = du(e_i)$ and summing it from $i = 1$ to k. Hence one of the terms must be < 0, or the sum would be nonnegative, a contradiction,

i.e.,

$$\frac{d^2}{dt^2}E(\varphi_t^{v_\ell^T}(u))\big|_{t=0} < 0 \quad \text{for some} \quad 1 \le \ell \le m. \tag{5.3}$$

This means that along one of the directions, v_ℓ^T, the variation decreases the energy of u, and hence u is not stable. Notice that if we only compute the second variation of the energy along *any* single direction v_ℓ^T, we do not know the sign of $\frac{d^2}{dt^2}E(\varphi_t^{v_\ell^T}(u))\big|_{t=0}$ to begin with, because of some troublesome terms. But average the result $\sum_{\ell=1}^m \frac{d^2}{dt^2}E(\varphi_t^{v_\ell^T}(u))\big|_{t=0}$ over the set of variation vector fields, then the troublesome terms are cancelled, we get (5.2), and we know the sign of the average is negative, under the above extrinsic condition (5.1), and hence we know the sign of the second variation of the energy along *some* single direction v_ℓ^T in the end, indirectly as in (5.3).

We call a manifold M satisfying (5.1) "*superstrongly unstable* (SSU) *manifold*". The foregoing shows that, in particular a compact SSU manifold cannot be the target of *any* nonconstant stable harmonic maps from *any* compact manifold. Similarly, one can show that a compact SSU manifold cannot be the domain of *any* nonconstant stable harmonic map into *any* compact manifold. In fact, these conservative "distinguished" vector fields $\{v_1^T, \dots, v_n^T, v_{n+1}^T, \dots, v_m^T\}$ on $M \hookrightarrow \mathbb{E}^m$ "universally" decrease the energy, mass, p-energy, Yang-Mills, Φ-energy functionals, etc under appropriate conditions on the second fundamental form of the isometric immersion of M in $\mathbb{E}^m$ via this extrinsic average variational method in the calculus of variations (cf. [18,20,24–26]). It is in contrast to an average method in PDE that was applied by Chen and Wei in [16] to obtain sharp growth estimates for warping functions in multiply warped product manifolds.

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