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The Zero-divisor Graph of $\mathbb{Z}_n[X]\!]$

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Abstract. Let $\mathbb{Z}_n$ be the ring of integers modulo n and let $\mathbb{Z}_n[X]\!]$ be either $\mathbb{Z}_n[X]$ or $\mathbb{Z}_n[\![X]\!]$. Let $\Gamma(\mathbb{Z}_n[X]\!])$ be the zero-divisor graph of $\mathbb{Z}_n[X]\!]$. In this paper, we study some properties of $\Gamma(\mathbb{Z}_n[X]\!])$. More precisely, we completely characterize the diameter and the girth of $\Gamma(\mathbb{Z}_n[X]\!])$. We also calculate the chromatic number of $\Gamma(\mathbb{Z}_n[X]\!])$.

1. Introduction

1.1. Preliminaries

In this subsection, we review some concepts from basic graph theory. Let G be a (undirected) graph. Recall that G is *connected* if there exists a path between any two distinct vertices of G. The graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n. The graph G is a *complete bipartite graph* if G can be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. For vertices a and b in G, $d(a, b)$ denotes the length of the

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shortest path from a to b. If there is no such path, then $d(a, b)$ is defined to be ∞; and $d(a, a)$ is defined to be zero. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the supremum of $\{d(a, b) \mid a \text{ and } b \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is defined as the length of the shortest cycle in G. If G contains no cycles, then $\mathrm{g}(G)$ is defined to be ∞. A subgraph H of G is an *induced subgraph* of G if two vertices of H are adjacent in H if and only if they are adjacent in G. The *chromatic number* of G is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, and is denoted by $\chi(G)$. A *clique* C in G is a subset of the vertex set of G such that the induced subgraph of G by C is a complete graph. A *maximal clique* in G is a clique that cannot be extended by including one more adjacent vertex. The *clique number* of G, denoted by $\mathrm{cl}(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$. If $K_n \subseteq G$ for all integers $n \geq 1$, then $\mathrm{cl}(G)$ is defined to be ∞. It is easy to see that $\chi(G) \geq \mathrm{cl}(G)$.

1.2. The Zero-divisor Graph of a Commutative Ring

Let R be a commutative ring with identity and $\mathrm{Z}(R)$ the set of nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the simple graph with the vertex set $\mathrm{Z}(R)$, and for distinct $a, b \in \mathrm{Z}(R)$, a and b are adjacent if and only if $ab = 0$. Clearly, $\Gamma(R)$ is the null graph if and only if R is an integral domain.

In [6], Beck first introduced the concept of the zero-divisor graph of a commutative ring and in [3], Anderson and Naseer continued to study. In [3] and [6], all elements of R are vertices of the graph and the authors were mainly interested in graph coloring. In [2], Anderson and Livingston gave the present definition of $\Gamma(R)$ in order to emphasize the study of the interplay between graph-theoretic properties of $\Gamma(R)$ and ring-theoretic properties of R. It was shown that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ [2, Theorem 2.3]. In [8, (1.4)], Mulay proved that $\mathrm{g}(\Gamma(R)) \leq 4$. In [5], the authors studied the zero-divisor graphs of polynomial rings and power series rings.

For more on the zero-divisor graph of a commutative ring, the readers can refer to a survey article [1].

Let $\mathbb{Z}_n$ be the ring of integers modulo n and let $\mathbb{Z}_n[X]$ (resp., $\mathbb{Z}_n[\![X]\!]$) be the polynomial ring (resp., power series ring) over $\mathbb{Z}_n$. Let $\mathbb{Z}_n[X]\!]$ be either $\mathbb{Z}_n[X]$ or $\mathbb{Z}_n[\![X]\!]$. In [9], the authors studied the zero-divisor graph of $\mathbb{Z}_n$. In fact, they completely characterized the diameter, the girth and the chromatic number of $\Gamma(\mathbb{Z}_n)$. The purpose of this paper is to study some properties of the zero-divisor graph of $\mathbb{Z}_n[X]\!]$. If n is a prime number, then $\mathbb{Z}_n[X]\!]$ has no zero-divisors; so $\Gamma(\mathbb{Z}_n[X]\!])$ is the null graph. Hence in this paper, we only consider the case that n is a composite. In Section 2, we completely characterize the diameter and the girth of $\Gamma(\mathbb{Z}_n[X]\!])$. In Section 3, we calculate the chromatic number of $\Gamma(\mathbb{Z}_n[X]\!])$. Note that all figures are drawn via website http://graphonline.ru/en/.

2. The Diameter and the Girth of $\Gamma(\mathbb{Z}_n[\![X]\!])$

In order to give the complete characterization of the diameter of $\Gamma(\mathbb{Z}_n[\![X]\!])$, we need the following lemma.

Lemma 2.1.([4, Chapter 1, Exercise 2(iii)] and [7, Theorem 5]) *Let R be a commutative ring with identity. Then the following assertions hold.*

(1) *If $f \in Z(R[X])$, then there exists a nonzero element $r \in R$ such that $rf = 0$.*

(2) *If R is a Noetherian ring and $f \in Z(R[\![X]\!])$, then there exists a nonzero element $r \in R$ such that $rf = 0$.*

Let R be a commutative ring with identity. For a nonempty subset C of R, let $C[\![X]\!]$ be the subset of $R[\![X]\!]$ consisting of elements whose coefficients belong to C. For an element $f = \sum_{i\geq 0} a_i X^i \in R[\![X]\!]$, the *order* of f is defined to be the smallest nonnegative integer n such that $a_n \neq 0$ and is denoted by $\mathrm{ord}(f)$.

Theorem 2.2. *The following statements hold.*

(1) $\mathrm{diam}(\Gamma(\mathbb{Z}_n[\![X]\!])) = 1$ *if (and only if) $n = p^2$ for some prime p.*

(2) $\mathrm{diam}(\Gamma(\mathbb{Z}_n[\![X]\!])) = 2$ *if (and only if) $n = p^r$ for some prime p and some integer $r \geq 3$, or $n = pq$ for some distinct primes p and q.*

(3) $\mathrm{diam}(\Gamma(\mathbb{Z}_n[\![X]\!])) = 3$ *if (and only if) $n = pqr$ for some distinct primes p, q and some integer $r \geq 2$.*

Proof. Before proving the result, we note that for all integers $n \geq 2$, $\mathbb{Z}_n$ is a Noetherian ring.

(1) Suppose that $n = p^r$ for some prime p and some integer $r \geq 3$. Let f and g be two distinct elements of $Z(\mathbb{Z}_{p^2}[\![X]\!])$. Then by Lemma 2.1, f and g are elements of $Z(\mathbb{Z}_{p^2})[\![X]\!]$. Note that $Z(\mathbb{Z}_{p^2}) = \{p, 2p, \ldots, (p-1)p\}$; so the product of any two elements of $Z(\mathbb{Z}_{p^2})$ is zero. Hence $fg = 0$ in $\mathbb{Z}_{p^2}[\![X]\!]$. This indicates that $\Gamma(\mathbb{Z}_{p^2}[\![X]\!])$ is a complete graph. Thus $\mathrm{diam}(\Gamma(\mathbb{Z}_{p^2}[\![X]\!])) = 1$.

(2) Suppose that $n = p^r$, where p is a prime and r is an integer greater than or equal to 3. Let f and g be two distinct elements of $Z(\mathbb{Z}_{p^r}[\![X]\!])$. Then by Lemma 2.1, f and g are elements of $Z(\mathbb{Z}_{p^r})[\![X]\!]$. Note that $Z(\mathbb{Z}_{p^r}) = \{p, 2p, \ldots, (p^{r-1}-1)p\}$; so for all $a \in Z(\mathbb{Z}_{p^r})$, $ap^{r-1} = 0$ in $\mathbb{Z}_{p^r}$. Hence $f - p^{r-1} - g$ is a path in $\Gamma(\mathbb{Z}_{p^r}[\![X]\!])$, which implies that $\mathrm{diam}(\Gamma(\mathbb{Z}_{p^r}[\![X]\!])) \leq 2$. Note that pX is not adjacent to $(p^{r-1}-1)pX$ in $\Gamma(\mathbb{Z}_{p^r}[\![X]\!])$. Thus $\mathrm{diam}(\Gamma(\mathbb{Z}_{p^r}[\![X]\!])) = 2$.

Suppose that $n = pq$, where p and q are distinct primes. Let $A = \{p, 2p, \ldots, (q-1)p\}$ and $B = \{q, 2q, \ldots, (p-1)q\}$. Then $A \cap B = \emptyset$ and $Z(\mathbb{Z}_{pq}) = A \cup B$. Let $f \in Z(\mathbb{Z}_{pq}[\![X]\!])$. Then by Lemma 2.1, there exists an element $r \in Z(\mathbb{Z}_{pq})$ such that $rf = 0$. Note that for any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, $a_1a_2 \neq 0$ and $b_1b_2 \neq 0$ in $\mathbb{Z}_{pq}$; so if $r \in A$ (resp., $r \in B$), then $f \in B[\![X]\!]$ (resp., $f \in A[\![X]\!]$). Therefore $Z(\mathbb{Z}_{pq}[\![X]\!]) = A[\![X]\!] \cup B[\![X]\!]$. Note that $A[\![X]\!] \cap B[\![X]\!] = \emptyset$ and for any $a \in A$ and $b \in B$, $ab = 0$. Hence $\Gamma(\mathbb{Z}_{pq}[\![X]\!])$ is a complete bipartite graph. Thus $\mathrm{diam}(\Gamma(\mathbb{Z}_{pq}[\![X]\!])) = 2$.

(3) Suppose that $n = pqr$, where p, q are distinct primes and r is an integer greater than or equal to 2. Then $pX, qX \in Z(\mathbb{Z}_{pqr}\llbracket X\rrbracket)$ with $(pX)(qX) \neq 0$ in $\mathbb{Z}_{pqr}\llbracket X\rrbracket$; so $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)) \geq 2$. Suppose to the contrary that there exists an element $f \in Z(\mathbb{Z}_{pqr}\llbracket X\rrbracket)$ such that $pX - f - qX$ is a path in $\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)$. Let a be the coefficient of $X^{\operatorname{ord}(f)}$ in f. Then $ap = 0 = aq$ in $\mathbb{Z}_{pqr}$; so a is a multiple of pqr. Therefore $a = 0$ in $\mathbb{Z}_{pqr}$. This is absurd. Hence $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)) \geq 3$. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)) = 3$ [2, Theorem 2.3]. $\square$

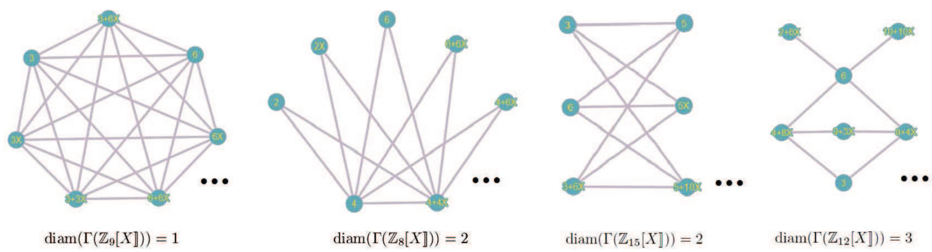

Figure 1: The diameter of some zero-divisor graphs

We next study the girth of $\Gamma(\mathbb{Z}_n\llbracket X\rrbracket)$.

Proposition 2.3. *If p is a prime and $r \geq 2$ is an integer, then* $g(\Gamma(\mathbb{Z}_{p^r}\llbracket X\rrbracket)) = 3$.

Proof. It suffices to note that $p^{r-1} - p^{r-1}X - p^{r-1}X^2 - p^{r-1}$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{p^r}\llbracket X\rrbracket)$. $\square$

Proposition 2.4. *If p and q are distinct primes, then* $g(\Gamma(\mathbb{Z}_{pq}\llbracket X\rrbracket)) = 4$.

Proof. Note that by the proof of Theorem 2.2(2), $\Gamma(\mathbb{Z}_{pq}\llbracket X\rrbracket)$ is a complete bipartite graph; so $\Gamma(\mathbb{Z}_{pq}\llbracket X\rrbracket)$ does not have a cycle of length 3. Let A and B be as in the proof of Theorem 2.2(2). Then for any $f \in A[X]$ and $g \in B[X]$, $pX - g - f - qX - pX$ is a cycle of length 4 in $\Gamma(\mathbb{Z}_{pq}\llbracket X\rrbracket)$. Thus $g(\Gamma(\mathbb{Z}_{pq}\llbracket X\rrbracket)) = 4$. $\square$

Lemma 2.5. *If* $g(\Gamma(\mathbb{Z}_n\llbracket X\rrbracket)) = 3$, *then* $g(\Gamma(\mathbb{Z}_{mn}\llbracket X\rrbracket)) = 3$ *for all integers $m \geq 1$.*

Proof. Let m be any positive integer. Note that if $f - g - h - f$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_n\llbracket X\rrbracket)$, then $mf - mg - mh - mf$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{mn}\llbracket X\rrbracket)$. Thus $g(\Gamma(\mathbb{Z}_{mn}\llbracket X\rrbracket)) = 3$. $\square$

Lemma 2.6. *Let p and q be distinct primes. Then* $g(\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)) = 3$ *for any prime r.*

Proof. If $r = p$, then $p - pq - pqX - p$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{p^2q}\llbracket X\rrbracket)$. If $r = q$, then $q - pq - pqX - q$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{pq^2}\llbracket X\rrbracket)$. Suppose that $r \neq p$ and $r \neq q$. Then $pq - qr - pr - pq$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)$. Thus $g(\Gamma(\mathbb{Z}_{pqr}\llbracket X\rrbracket)) = 3$ for any prime r. $\square$

Proposition 2.7. *Let p and q be distinct primes. Then* $\mathrm{g}(\Gamma(\mathbb{Z}_{pqr}[X]\!]))=3$ *for any integer $r \geq 2$.*

Proof. Note that r is a multiple of some prime. Thus the result follows directly from Lemmas 2.5 and 2.6. □

By Propositions 2.3, 2.4 and 2.7, we can completely characterize the girth of $\Gamma(\mathbb{Z}_n[X]\!])$ as follows:

Theorem 2.8. *The following statements hold.*

(1) $\mathrm{g}(\Gamma(\mathbb{Z}_n[X]\!]))=3$ *if* (*and only if*) *each of the following conditions holds.*

   (a) *$n = p^r$ for some prime p and integer $r \geq 2$.*

   (b) *$n = pqr$ for some distinct primes p, q and integer $r \geq 2$.*

(2) $\mathrm{g}(\Gamma(\mathbb{Z}_n[X]\!]))=4$ *if* (*and only if*) *$n = pq$ for some distinct primes p and q.*

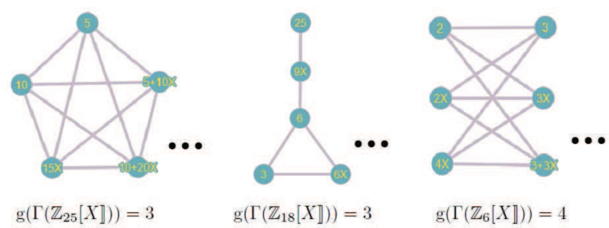

Figure 2: The girth of some zero-divisor graphs

3. The Chromatic Number of $\Gamma(\mathbb{Z}_n[X]\!])$

In this section, we calculate the chromatic number of $\Gamma(\mathbb{Z}_n[X]\!])$. Clearly, if there exists a clique in a graph, then the chromatic number of the graph is greater than or equal to the size of the clique. Hence in order to find the chromatic number of $\Gamma(\mathbb{Z}_n[X]\!])$, we investigate to find a (maximal) clique of $\Gamma(\mathbb{Z}_n[X]\!])$.

Lemma 3.1. *If $r \geq 2$ is an integer, $n = p_1 \cdots p_r$ for distinct primes $p_1, \ldots, p_r$, and $C = \{\frac{n}{p_i} \mid i = 1, \ldots, r\}$, then C is a maximal clique of $\Gamma(\mathbb{Z}_n[X]\!])$.*

Proof. Note that the product of any two distinct members of C is zero in $\mathbb{Z}_n[X]\!]$; so C is a clique. Suppose to the contrary that there exists an element $f \in \mathrm{Z}(\mathbb{Z}_n[X]\!]) \setminus C$ such that $cf = 0$ in $\mathbb{Z}_n[X]\!]$ for all $c \in C$. Let a be the coefficient of $X^{\mathrm{ord}(f)}$ in f. Then $ca = 0$ in $\mathbb{Z}_n$; so a is a multiple of p_i for all $i = 1, \ldots, r$. Hence a is a multiple of n, *i.e.*, $a = 0$ in $\mathbb{Z}_n$. This is a contradiction. Thus C is a maximal clique of $\Gamma(\mathbb{Z}_n[X]\!])$. □

Proposition 3.2. *If $r \geq 2$ is an integer and $n = p_1 \cdots p_r$ for distinct primes $p_1, \ldots, p_r$, then $\chi(\Gamma(\mathbb{Z}_n[X]\!])) = r$.*

Proof. Let $C = \{\frac{n}{p_i} \mid i = 1, \ldots, r\}$. Then by Lemma 3.1, C is a maximal clique of $\Gamma(\mathbb{Z}_n[\![X]\!])$. For each $i = 1, \ldots, r$, let $\bar{i}$ be the color of $\frac{n}{p_i}$. Clearly, $Z(\mathbb{Z}_n[\![X]\!]) \setminus C$ is a nonempty set. For each $f \in Z(\mathbb{Z}_n[\![X]\!]) \setminus C$, let $S_f = \{c \in C \mid f$ and c are not adjacent$\}$. Then by Lemma 3.1, C is a maximal clique of $\Gamma(\mathbb{Z}_n[\![X]\!])$; so S_f is a nonempty set. Hence we can find the smallest integer $k \in \{1, \ldots, r\}$ such that $\frac{n}{p_k} \in S_f$. In this case, we color f with $\bar{k}$.

To complete the proof, it remains to check that any two vertices of $\Gamma(\mathbb{Z}_n[\![X]\!])$ with the same color cannot be adjacent. Fix an element $k \in \{1, \ldots, r\}$ and let f and g be distinct vertices of $\Gamma(\mathbb{Z}_n[\![X]\!])$ with the same color $\bar{k}$. Since C is a clique, f and g cannot belong to C at the same time. Suppose that $f \in C$ and $g \in Z(\mathbb{Z}_n[\![X]\!]) \setminus C$. Then $f = \frac{n}{p_k}$; so by the coloring of g, f and g are not adjacent. Suppose that $f, g \in Z(\mathbb{Z}_n[\![X]\!]) \setminus C$ and write $f = \sum_{i \geq 0} a_i X^i$ and $g = \sum_{i \geq 0} b_i X^i$. Then by the coloring of f and g, f and g are not adjacent to $\frac{n}{p_k}$; so $\frac{n}{p_k} f \neq 0$ and $\frac{n}{p_k} g \neq 0$ in $\mathbb{Z}_n[\![X]\!]$. Let α be the smallest nonnegative integer such that $\frac{n}{p_k} a_\alpha \neq 0$ in $\mathbb{Z}_n$ and let β be the smallest nonnegative integer such that $\frac{n}{p_k} b_\beta \neq 0$ in $\mathbb{Z}_n$. Then $a_0, \ldots, a_{\alpha-1}, b_0, \ldots, b_{\beta-1}$ are divided by p_k and a_α, b_β are not divided by p_k; so the coefficient of $X^{\alpha+\beta}$ in fg is not divided by p_k. Therefore $fg \neq 0$ in $\mathbb{Z}_n[\![X]\!]$. Hence f and g are not adjacent.

Thus we conclude that $\chi(\Gamma(\mathbb{Z}_n[\![X]\!])) = r$. □

We denote the set of nonnegative integers by $\mathbb{N}_0$.

Lemma 3.3. *If n is a multiple of the square of a prime, then $\Gamma(\mathbb{Z}_n[\![X]\!])$ has an infinite clique.*

Proof. Suppose that n is a multiple of the square of a prime.

Case 1. $n = p_1^{2a_1} \cdots p_r^{2a_r}$ for distinct primes $p_1, \ldots, p_r$ and positive integers $a_1, \ldots, a_r$. In this case, let $C = \{\sqrt{n} X^m \mid m \in \mathbb{N}_0\}$. Then the product of any two elements of C is zero in $\mathbb{Z}_n[\![X]\!]$; so C is an infinite clique of $\Gamma(\mathbb{Z}_n[\![X]\!])$.

Case 2. $n = p_1^{2a_1} \cdots p_r^{2a_r} q_1^{2b_1+1} \cdots q_s^{2b_s+1}$ for distinct primes $p_1, \ldots, p_r, q_1, \ldots, q_s$ and nonnegative integers $a_1, \ldots, a_r, b_1, \ldots, b_s$, not all zero. In this case, let $k = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1+1} \cdots q_s^{b_s+1}$ and let $C = \{kX^m \mid m \in \mathbb{N}_0\}$. Then the product of any two elements of C is zero in $\mathbb{Z}_n[\![X]\!]$; so C is an infinite clique of $\Gamma(\mathbb{Z}_n[\![X]\!])$.

By Cases 1 and 2, $\Gamma(\mathbb{Z}_n[\![X]\!])$ has an infinite clique. □

Proposition 3.4. *Let n be a multiple of the square of a prime. Then $\chi(\Gamma(\mathbb{Z}_n[\![X]\!])) = \infty$.*

Proof. By Lemma 3.3, $\Gamma(\mathbb{Z}_n[\![X]\!])$ has an infinite clique; so $\mathrm{cl}(\Gamma(\mathbb{Z}_n[\![X]\!])) = \infty$. Thus $\chi(\Gamma(\mathbb{Z}_n[\![X]\!])) = \infty$. □

By Propositions 3.2 and 3.4, we obtain the main result in this section.

Theorem 3.5. *The following statements hold.*

(1) $\chi(\Gamma(\mathbb{Z}_n[\![X]\!])) = r$ *if (and only if) $n = p_1 \cdots p_r$ for some distinct primes $p_1, \ldots, p_r$.*

(2) $\chi(\Gamma(\mathbb{Z}_n[\![X]\!])) = \infty$ *if (and only if) n is a multiple of the square of some prime.*

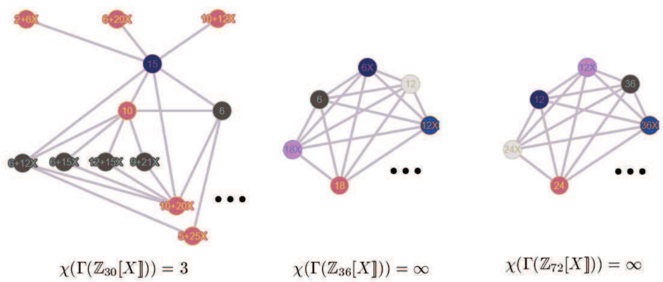

Figure 3: The coloring of some zero-divisor graphs

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