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The p-deformed Generalized Humbert Polynomials and Their Properties

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Abstract. We introduce the p-deformation of generalized Humbert polynomials. For these polynomials, we derive the differential equation, generating function relations, Fibonacci-type representations, and recurrence relations and state the companion matrix. These properties are illustrated for certain polynomials belonging to p-deformed generalized Humbert polynomials.

1. Introduction

In 2007, Díaz and Pariguan introduced the one-parameter deformation of the classical gamma function in the form [7]:

$$\Gamma_p(z) = \int_0^\infty t^{z-1} e^{-\frac{t^p}{p}}\ dt,$$

where $z \in \mathbb{C}$, $\Re(z) > 0$ and $p > 0$. In fact, the occurrence of the product of the form $x(x+p)(x+2p)\cdots(x+(n-1)p)$ in the combinatorics of creation and annihilation operators [6, 8] and the perturbative computation of Feynman integrals [5] led them

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to generalize the gamma function in the above stated form and to generalize the Pochhammer p-symbol in the following form:

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p),$$

in which $z \in \mathbb{C},\ p \in \mathbb{R}$ and $n \in \mathbb{N}$. These generalizations lead to the following elementary properties.

$$\begin{aligned}
\Gamma_p(z+p) &= z\Gamma_p(z),\\
\Gamma_p(p) &= 1,\\
(z)_{k,p} &= \frac{\Gamma_p(z+kp)}{\Gamma_p(z)},\\
(z)_{n-k,p} &= \frac{(-1)^k (z)_{n,p}}{(p-z-np)_{k,p}},\\
(z)_{mn,p} &= m^{mn} \prod_{j=1}^{m} \left(\frac{z+jp-p}{m}\right)_{n,p}.
\end{aligned}$$

For $p > 0,\ a \in \mathbb{C}$ and $|x| < \dfrac{1}{p}$, Diaz et al. [7] demonstrated that

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!} x^n = (1-px)^{-\frac{a}{p}}. \tag{1.1}$$

This may be regarded as the p-deformed binomial series. The radius of convergence of this series can be enlarged or diminished by choosing a smaller or larger p; unlike in the classical theory of the radius of convergence of the binomial series which is fixed. This motivated us to study the p-deformation of certain Special functions, particularly, the polynomial system formed by generalized Humbert polynomials.

Our objective is to extend generalized Humbert polynomials according to Gould [9] by involving a new parameter: $p(>0)$. We call this extension a p-deformation of the polynomial. We study its properties, namely, the differential equation, generating function relations (GFRs), differential recurrence relations, and mixed relations, and we illustrate the companion matrix.

The companion matrix of the monic polynomial is defined as follows.

Definition 1.1. Let $f(x) \in \mathbb{C}[X]$ be a monic polynomial given by $f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_{k-1} x^{k-1} + x^k$. Then the $k \times k$ matrix, called the *companion matrix* of $f(x)$, is denoted and defined as follows [14, p. 39]:

$$C(f(x)) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \dots & -\delta_{k-1} \end{bmatrix}.$$

We have the following lemma [14, Proposition 1.5.14, p. 39].

Lemma 1.1. *If $f \in K[x]$ is nonconstant and $A = C(f(x))$, then $f(A) = O$, the null matrix.*

The class $\{P_n(m,x,\gamma,s,c); n = 0,1,2,\ldots\}$ of generalized Humbert polynomials is defined below [9, Eq. 5.11, p. 707].

Definition 1.2. For $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, and $x \in \mathbb{R}$,

$$P_n(m,x,\gamma,s,c) = \sum_{k=0}^{[n/m]} \binom{s-n+mk}{k} \binom{s}{n-mk} c^{s-n+mk-k} \times \gamma^k \, (-mx)^{n-mk}, \tag{1.2}$$

where $\gamma,\ c$, and s are generally unrestricted.

This class of polynomials is generated by the following relation:

$$(c - mxt + \gamma t^m)^s = \sum_{n=0}^{\infty} P_n(m,x,\gamma,s,c) t^n. \tag{1.3}$$

The substitution $s = -\nu$, $\gamma = 1$, and $c = 1$ in (1.2) result in Humbert polynomials, according to Humbert [11]:

$$\Pi^{\nu}_{n,m}(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-mx)^{n-mk}}{\Gamma(1-\nu-n+(m-1)k)(n-mk)!\ k!}. \tag{1.4}$$

Apart from the polynomials (1.4), according to Humbert [12, p. 75], *Humbert* functions also exits. They have explicit representations in a double infinite series, given by the following:

$$\Psi_1(a;b;c,d;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{r+s}\ (b)_r}{(c)_r\ (d)_s\ r!\ s!} x^r y^s,$$

$$\Psi_2(a;b,c;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{r+s}}{(b)_r\ (c)_s\ r!\ s!} x^r y^s,$$

$$\Xi_1(a,b;c;d;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_r\ (b)_s\ (c)_r}{(d)_{r+s}\ r!\ s!} x^r y^s$$

and

$$\Xi_2(a,b;c;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_r\ (b)_r}{(c)_{r+s}\ r!\ s!} x^r y^s,$$

where $|x| < 1,\ |y| < \infty$ and $c,\ d \neq 0,-1,-2,\ldots$. In recent years, these functions have been increasingly used in various fields, for example, in theoretical physics

[3, 13] and communication theory [1, 2, 18]. Moreover, for specific values of parameters and variables, their reduced forms have also been found useful, especially in connection with simplification algorithms in computer algebra systems (see [3]). Therefore, we propose the following extension of the polynomials $P_n(m, x, \gamma, s, c)$.

Definition 1.3. For $\gamma,\ s,\ c \in \mathbb{C},\ m \in \mathbb{N},\ x \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\}$, and $p > 0$,

$$P_{n,p}(m, x, \gamma, s, c) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\gamma^k c^{s-n+mk-k}}{(s+p)_{mk-k-n,p}(n-mk)!\ k!}\ (-mx)^{n-mk}, \tag{1.5}$$

in which the floor function $\lfloor r \rfloor = floor\ r$, represents the greatest integer $\leq r$.

We call these polynomials *p-deformed generalized Humbert polynomials* or *pGHPs*. When $p = 1$, it coincides with the polynomial (1.2). The particular polynomials belonging to these general p-polynomials provide an extension to the Humbert polynomials (1.4), Kinney polynomials, Pincherle polynomials, Gegenbauer polynomials, and Legendre polynomials (see [9]). For instance, the substitutions $\gamma = 1,\ c = 1$, and $s = -\nu$ in (1.5) yield p-deformed Humbert polynomials:

$$\Pi^{\nu}_{n,m,p}(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-mx)^{n-mk}}{\Gamma_p(p - \nu - np + (m-1)kp)(n-mk)!\ k!}. \tag{1.6}$$

For $p = 1$, this coincides with (1.4). If we substitute $\nu = 1/m,\ m \in \mathbb{N}$, in (1.6), then we obtain the p-deformed Kinney polynomial:

$$P_{n,p}(m, x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-mx)^{n-mk}}{\Gamma_p(p - 1/m - np + (m-1)kp)(n-mk)!\ k!}.$$

For $m = 3$ and $\nu = 1/2$, (1.6) reduces to the p-deformed Pincherle polynomial:

$$\mathcal{P}_{n,p}(x) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(-3x)^{n-3k}}{\Gamma_p(p - 1/2 - np + 2kp)(n-3k)!\ k!}.$$

The p-deformed Gegenbauer polynomial is the special cases where $m = 2$ of (1.6) which occurs in the following form:

$$C^{\nu}_{n,p}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma_p(p - \nu - np + kp)(n-2k)!\ k!}. \tag{1.7}$$

Further, if $\nu = 1/2$ then (1.7) is reduced to the p-deformed Legendre polynomial given by the following:

$$P_{n,p}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma_p(p - 1/2 - np + kp)(n-2k)!\ k!}. \tag{1.8}$$

All these polynomials reduce to their classical forms when $p = 1$ [9, p. 697].

2. Differential Equation

In this section, we derive the differential equation of the polynomial (1.5). Costa et al. [4] demonstrated that the homogeneous differential equation:

$$(1 - x^N)y^{(N)} + \sum_{k=1}^{N} A_{N-k} x^{N-k} y^{(N-k)} = 0,$$

has a polynomial solution if and only if $0 \le r < N$, $\forall r$, and $\exists\, n \ge 0$ such that n mod $N = r$, where n is a root of the recurrence relation, and $y^{(j)}$ is the j^{th} derivative of y with respect to x for $j = 1, 2, \ldots, N$.

Let the sequence $\{f_r\}_{r=0}^{n}$ be given by $f_r = f(r)$, where

$$f(r) = (n - r)\left(-s + rp + \left(\frac{n-r}{m}\right)p\right)_{m-1,p}.$$

We use the forward difference operator $'\Delta'$ and the shift operator $'E'$ which are defined as follows [10, Eq. (5.2.13), p. 178]:

$$\Delta f_t = f_{t+1} - f_t,\ \ E^k f_t = f_{t+k}.$$

The relation between Δ and E is given by [10, Eq. (5.2.14), p. 178] $\Delta = E - \mathbf{1}$, where $\mathbf{1}$ is the identity operator defined by $\mathbf{1}f = f$. In (1.5), we use the following formula:

$$(p+s)_{-n+mk-k,p} = (p+s)_{-(n-mk+k),p} = \frac{(-1)^{n-mk+k}}{(p-p-s)_{n-mk+k,p}},$$

to obtain the alternative form:

$$P_{n,p}(m, x, \gamma, s, c) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk)!\ k!}(mx)^{n-mk}. \tag{2.1}$$

The differential equation for this explicit form is derived in Theorem 2.1.

Theorem 2.1. *Let $s \in \mathbb{C}$, $p > 0$, and $m \in \mathbb{N}$. Then, the polynomial $y = P_{n,p}(m, x, \gamma, s, c)$ satisfies the following equation:*

$$\gamma c^{m-1} y^{(m)} + \sum_{r=0}^{m} a_r x^r y^{(r)} = 0, \tag{2.2}$$

where $a_r = \dfrac{m^{m-1}}{r!}\Delta^r f_0.$

Proof. Let $n = ml + q$, where $\lfloor n/m \rfloor = l$ and $0 \le q \le m-1$. The r^{th} derivative of (2.1) is given by the following:

$$D^r P_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor (n-r)/m \rfloor} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk-r)!\ k!} m^{n-mk} \times x^{n-mk-r}.$$

Hence,

$$x^r D^r P_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor (n-r)/m \rfloor} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk-r)!\ k!} \times (mx)^{n-mk}. \tag{2.3}$$

Next, we have

$$\begin{aligned} D^m P_{n,p}(m,x,\gamma,s,c) &= \sum_{k=0}^{\lfloor (n-m)/m \rfloor} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk-m)!\ k!} \times m^{n-mk}\ x^{n-mk-m} \\ &= \sum_{k=0}^{l-1} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk-m)!\ k!} m^m \times (mx)^{n-mk-m}, \end{aligned} \tag{2.4}$$

where

$$\left\lfloor \frac{n-r}{m} \right\rfloor = \begin{cases} l, & \text{if } r \le q \\ l-1, & \text{if } r > q \end{cases}.$$

Substituting the equation (2.3) and (2.4) on the left-hand side of the differential equation (2.2) and comparing the corresponding coefficients of x, we find that

$$\begin{aligned} \sum_{r=0}^{m} \binom{n-mk}{r} r! a_r &= \frac{m^m k (-s)_{n-mk+k+m-1,p}}{(-s)_{n-mk+k,p}} \\ &= m^m k(-s+np-mkp+kp)_{m-1,p}, \end{aligned} \tag{2.5}$$

where $k = 0, 1, 2, \ldots, l-1$, and

$$\sum_{r=0}^{q} \binom{n-ml}{r} r! a_r = m^m l(-s+np-mlp+lp)_{m-1,p}.$$

Because $n = ml + q \Rightarrow n - ml = q$, we have the following:

$$\sum_{r=0}^{q} \binom{q}{r} r! a_r = m^{m-1}(n-q) \left(-s + qp + \left(\frac{n-q}{m} \right) p \right)_{m-1,p}. \tag{2.6}$$

By substituting $a_r = \dfrac{m^{m-1}\Delta^r f_0}{r!}$ in (2.6), we obtain

$$\sum_{r=0}^{q}\binom{q}{r}\Delta^r f_0 \;=\; (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p},$$

that is,

$$(\mathbf{1}+\Delta)^q f_0 \;=\; (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p}. \tag{2.7}$$

Hence, $\mathbf{1}+\Delta = E$, the shift operator, in (2.7) becomes the following:

$$E^q f_0 \;=\; f(q) \;=\; (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p}.$$

For $k=0,1,2,\ldots,l-1$, (2.5) can be written in the following form:

$$\sum_{r=0}^{m}\binom{n-mk}{r}\Delta^r f_0 \;=\; f_{n-mk} \;=\; mk(-s+np-mkp+kp)_{m-1,p}. \tag{2.8}$$

As $t\mapsto f(t)$ is a polynomial of degree m, the equation (2.8) is a forward difference formula for f at the point $t=n-mk$. Thus, the proof is completed for the choice:

$$a_r = \frac{m^{m-1}\Delta^r f_0}{r!} = \frac{m^{m-1}}{r!}\Delta^r\left(n\left(\frac{np-ms}{m}\right)_{m-1,p}\right). \qquad \square$$

2.1. Particular Cases

We illustrate special instances of the differential equation (2.2). In particular, the equations of the Pincherle, Gegenbauer, and Legendre polynomials. We choose $r=0,1,2$ and 3 to obtain the following:

$$a_0 \;=\; m^{m-1}\Delta^0 f_0 \;=\; m^{m-1}n\left(\frac{np-ms}{m}\right)_{m-1,p}, \tag{2.9}$$

$$\begin{aligned} a_1 \;&=\; m^{m-1}\Delta f_0 \;=\; m^{m-1}(E-1)f(0) \;=\; m^{m-1}(f(1)-f(0)) \\ &=\; m^{m-1}\Bigg[(n-1)\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p} \\ &\qquad\qquad -n\left(\frac{np-ms}{m}\right)_{m-1,p}\Bigg], \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
(2.11)\quad a_2 &= \frac{m^{m-1}\Delta^2 f_0}{2!} = \frac{m^{m-1}}{2!}(E-1)^2 f_0 = \frac{m^{m-1}}{2!}(E^2-2E+1)f(0) \\
&= \frac{m^{m-1}}{2!}(f(2)-2f(1)+f(0)) \\
&= \frac{m^{m-1}}{2!}\left[(n-2)\left(\frac{(n-2)p+m(-s+2p)}{m}\right)_{m-1,p} - 2(n-1)\right. \\
&\quad \left. \times\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p} + n\left(\frac{np-ms}{m}\right)_{m-1,p}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(2.12)\quad a_3 &= \frac{m^{m-1}\Delta^3 f_0}{3!} = \frac{m^{m-1}}{3!}\left[f(3)-3f(2)+3f(1)-f(0)\right] \\
&= \frac{m^{m-1}}{3!}\left[(n-3)\left(\frac{(n-3)p+m(-s+3p)}{m}\right)_{m-1,p} - 3(n-2)\right. \\
&\quad \times\left(\frac{(n-2)p+m(-s+2p)}{m}\right)_{m-1,p} + 3(n-1) \\
&\quad \left. \times\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p} - n\left(\frac{np-ms}{m}\right)_{m-1,p}\right].
\end{aligned}
$$

By choosing $m = 3$ in (2.9), (2.10), (2.11) and (2.12), we obtain the following:

$$
\begin{aligned}
(2.13)\quad a_0 &= 3^2 n\left(\frac{np-3s}{3}\right)_{2,p} = n(np-3s)(np-3(s-p)), \\
(2.14)\quad a_1 &= 3^2\left[(n-1)\left(\frac{(n-1)p+3(-s+p)}{3}\right)_{2,p} - (n)\left(\frac{np-3s}{3}\right)_{2,p}\right] \\
&= 3np(np-2s+p)-(3s-2p)(3s-5p), \\
(2.15)\quad a_2 &= 12ps-18p^2, \\
(2.16)\quad a_3 &= -4p^2.
\end{aligned}
$$

Further, after entering $m = 3,\ \gamma = 1,\ c = 1$, and $s = -\lambda$ into the differential equation (2.2), from the particular values (2.13), (2.14), (2.15) and (2.16), we arrive at the differential equation of the p-deformed Pincherle polynomial. From the following general form:

$$y^{(3)} + \sum_{r=0}^{3} a_r x^r y^{(r)} = 0,$$

that is,

$$(1+a_3x^3)y^{(3)} + a_0 y + a_1 x y^{(1)} + a_2 x^2 y^{(2)} = 0,$$

we obtain the following equation:

$$\begin{aligned}&\left(1-4p^2x^3\right)y^{(3)}-6\left(2p\lambda+3p^2\right)x^2y^{(2)}+[3np(np+2\lambda+p)\\&-(3\lambda+2p)(3\lambda+5p)]\,xy^{(1)}+[n(np+3\lambda)(np+3(\lambda+p))]\,y \;=\; 0.\end{aligned}$$

The choice $p=1$ yields the differential equation of the Pincherle polynomial according to Humbert [11, p. 23].

Next, to obtain the equation for the p-deformed Gegenbauer polynomial, we enter $m=2,\gamma=1,c=1$, and $s=-\nu$ into (2.2) to obtain

$$y^{(2)}+\sum_{r=0}^{2}a_rx^ry^{(r)}=0,$$

or equivalently,

$$(1+a_2x^2)y''+a_1xy'+a_0y \;=\; 0.$$

From (2.9), (2.10) and (2.11), we have the following:

$$\begin{aligned}a_0 &= n\left(np-2s\right),\\ a_1 &= 2\left[(n-1)\left(\frac{(n-1)p+2(-s+p)}{2}\right)_{1,p}-(n)\left(\frac{np-2s}{2}\right)_{1,p}\right]\\ &= 2\left[(n-1)\left(\frac{(n-1)p+2(-s+p)}{2}\right)-(n)\left(\frac{np-2s}{2}\right)\right] \;=\; 2s-p,\\ a_2 &= -p.\end{aligned}$$

With a_0,a_1, and a_2, the above equation takes the precise form:

$$(1-px^2)y''+n(np+2\nu)y-(2\nu+p)xy' \;=\; 0,$$

where $y=C^{\nu}_{n,p}(x)$ is given by (1.7). When $p=1$, this reduces to the differential equation of the Gegenbauer polynomial [17, Eq. (1.4), p. 279]. The well-known special case $\nu=1/2$ of this equation is the following differential equation:

$$(1-px^2)P''_{n,p}(x)-(1+p)xP'_{n,p}(x)+n(np+1)P_{n,p}(x) \;=\; 0$$

of the p-deformed Legendre polynomial (1.8). In addition, for $p=1$, this reduces to the differential equation of the Legendre polynomial $P_n(x)$ (cf. [17, Eq. (5), p. 161]).

3. Generating Function Relations

We derive GFR of the *pGHP* (1.5). This is accomplished with the help of the p-deformed version of the identity [16, Ex. 212 and 216, p. 146]:

$$\frac{(1+z)^{a+1}}{1-zb}=\sum_{n=0}^{\infty}\binom{a+bn+n}{n}w^n,$$

where $a, b \in \mathbb{C}$ and $w = z(1+z)^{-b-1}$. It is given as explained in Theorem 3.1.

Theorem 3.1. *For $p > 0$, $a, b \in \mathbb{C}$, and* $w = \dfrac{z}{(1+z)^{b+1}}$.

$$\frac{(1+z)^{a/p+1}}{1-zb} = \sum_{n=0}^{\infty} \frac{\Gamma_p(a+bnp+np+p)}{\Gamma_p(a+bnp+p)\ n!} \frac{w^n}{p^n}. \tag{3.1}$$

Proof. We use the Lagrange's series [19, Eq. (3), p. 354]:

$$\frac{f(z)}{1-wg'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[f(z)(g(z))^n\right]_{z=z_0}, \quad D = \frac{d}{dz},$$

where $w = \dfrac{z-z_0}{g(z)}$.

To derive (3.1), we take $z_0 = 0$, $f(z) = (1+z)^{a/p}$, $g(z) = (1+z)^{b+1}$, and $w = z(1+z)^{-b-1}$ in the left-hand side of Lagrange's series gives the following:

$$\frac{f(z)}{1-wg'(z)} = \frac{(1+z)^{a/p}}{1-w(b+1)(1+z)^b} = \frac{(1+z)^{a/p}}{1-\dfrac{z}{(1+z)}(b+1)} = \frac{(1+z)^{a/p+1}}{1-zb}.$$

The same substitution on the right-hand side of the Lagrange's series gives the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[f(z)(g(z))^n\right]_{z=z_0} &= \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[(1+z)^{\frac{a}{p}+bn+n}\right]_{z=0} \\
&= \sum_{n=0}^{\infty} \frac{w^n}{n!} \left(\frac{a}{p}+bn+n\right)\left(\frac{a}{p}+bn+n-1\right) \\
&\quad \times \cdots \left(\frac{a}{p}+bn+n-n+1\right) \\
&= \sum_{n=0}^{\infty} \frac{w^n(-1)^n}{p^n\ n!}(-a-bnp-np)_{n,p} \\
&= \sum_{n=0}^{\infty} \frac{w^n}{p^n\ n!}(a+bnp+p)_{n,p} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma_p(a+bnp+np+p)}{\Gamma_p(a+bnp+p)\ n!} \frac{w^n}{p^n}.
\end{aligned}$$

This completes the proof. □

We define the function as follows:

$$R(A_n, \alpha, \gamma, r, m, p) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(-\alpha+mrk+p)}{\Gamma_p(-\alpha+mrk-kp+p)\ k!} \gamma^k p^{-k} A_{n-mk}, \tag{3.2}$$

which is required in deriving the following GFR.

Theorem 3.2. *For* $m \in \mathbb{N},\ w = t(1+\gamma w^m)^{-\beta/p},\ p > 0,\ G(z) = \sum_{n=0}^{\infty} A_n z^n$ *where* $A_0 \neq 0$,

$$\sum_{n=0}^{\infty} R(A_n, \alpha+\beta n, \gamma, r, m, p)t^n = \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} G\left[\frac{w}{(1+\gamma w^m)^{r/p}}\right], \tag{3.3}$$

where $\{A_n\}$ *is an arbitrary sequence such that* $\sum_{i=0}^{\infty} |A_i| < \infty$ *and other parameters are generally unrestricted.*

Proof. We begin with the following:

$$\begin{aligned}
&\sum_{n=0}^{\infty} R(A_n, \alpha+\beta n, \gamma, r, m, p)t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(p-\alpha-\beta n-nr+mrk)}{\Gamma_p(p-\alpha-\beta n-nr+mrk-kp)\ k!} \gamma^k p^{-k} A_{n-mk} t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma_p(p-\alpha-\beta n-\beta mk-nr)}{\Gamma_p(p-\alpha-\beta n-\beta mk-nr-kp)\ k!} \gamma^k p^{-k} A_n t^{n+mk} \\
&= \sum_{n=0}^{\infty} A_n t^n \sum_{k=0}^{\infty} \frac{\Gamma_p(p-\alpha-\beta n-\beta mk-nr)}{\Gamma_p(p-\alpha-\beta n-\beta mk-nr-kp)\ k!} \gamma^k p^{-k} t^{mk}.
\end{aligned}$$

In view of the sum in (3.1), the inner series simplifies to the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} R(A_n, \alpha+\beta n, \gamma, r, m, p)t^n &= \sum_{n=0}^{\infty} A_n\ t^n \left[\frac{(1+v)^{(-\alpha-\beta n-nr)/p+1}}{1+\left(\frac{\beta m}{p}+1\right)v}\right] \\
&= \left[\frac{(1+v)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)v}\right] \sum_{n=0}^{\infty} A_n \frac{\left(t(1+v)^{-\beta/p}\right)^n}{(1+v)^{nr/p}},
\end{aligned}$$

where $v = \gamma t^m (1+v)^{-\beta m/p}$. If we replace v with γw^m, then $w = t(1+\gamma w^m)^{-\beta/p}$ and, consequently, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} R(A_n, \alpha+\beta n, \gamma, r, m, p)t^n &= \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} \sum_{n=0}^{\infty} A_n \frac{\left(t(1+\gamma w^m)^{-\beta/p}\right)^n}{(1+\gamma w^m)^{nr/p}} \\
&= \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} \sum_{n=0}^{\infty} A_n \left[\frac{w}{(1+\gamma w^m)^{r/p}}\right]^n.
\end{aligned}$$

This completes the proof of GFR (3.3). □

The replacement of γ with $\gamma p/c$, and the substitutions $\alpha = -s$, $r = p$, and

$$A_n = (-m)^n \, c^{s-n} \frac{\Gamma_p(p+s)}{\Gamma_p(p+s-np)\, n!}\, x^n$$

in (3.3), yield the GFR of the $pGHP$ in the following form:

$$\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s+\beta n,c)t^n = \frac{(1+\gamma p w^m/c)^{(p+s)/p}}{1+\frac{\gamma p w^m}{c}\left(\frac{\beta m}{p}+1\right)} G\left[\frac{w}{\left(1+\frac{\gamma p w^m}{c}\right)}\right], \tag{3.4}$$

where $w = t\left(1+\frac{\gamma p w^m}{c}\right)^{-\beta/p}$ and

$$G(u) = \sum_{n=0}^{\infty} (c)^{s-n} \frac{\Gamma_p(p+s)}{\Gamma_p(p+s-np)n!}(-mxu)^n.$$

We note that $\beta = 0 \Leftrightarrow w = t$; and hence, using the p-binomial series (1.1), we find the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c)t^n &= (1+\gamma p t^m/c)^{s/p}\, G\left[\frac{t}{\left(1+\frac{\gamma p t^m}{c}\right)}\right] \\
&= c^{(1-1/p)s}\,(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-m)^n \Gamma_p(p+s)}{\Gamma_p(p+s-np)n!} \\
&\quad \times \left(\frac{xt}{c+\gamma p t^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-m)^n}{(p+s)_{-n,p}n!} \\
&\quad \times \left(\frac{xt}{c+\gamma p t^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-s)_{n,p}}{n!} \\
&\quad \times \left(\frac{mxt}{c+\gamma p t^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \left(1-\frac{pmxt}{c+\gamma p t^m}\right)^{s/p} \\
&= c^{(1-1/p)s}\,(c+\gamma p t^m - pmxt)^{s/p}.
\end{aligned} \tag{3.5}$$

This generalizes the GFR (1.3).

The GFR of the p-deformed Humbert polynomials occurs as a special case of (3.4) with the substitutions $\gamma = 1,\ c = 1,$ and $s = -\mu$ given by

$$\sum_{n=0}^{\infty} \Pi_{n,m,p}^{\mu+\beta n}(x)t^n = \frac{(1+pw^m)^{(p-\mu)/p}}{1+(\beta m+p)\,w^m} G\left[\frac{w}{(1+pw^m)}\right],$$

where $w = t\,(1+pw^m)^{-\beta/p}$ and

$$G(z) = \sum_{n=0}^{\infty} \frac{\Gamma_p(p-\mu)}{\Gamma_p(p-\mu-np)n!}(-mxz)^n.$$

The case $\beta = 0$ yields the following GFR:

$$\sum_{n=0}^{\infty} \Pi_{n,m,p}^{\mu}(x)t^n = (1+pt^m - pmxt)^{-\mu/p}. \tag{3.6}$$

This extends the GFR according to Humbert [11, p. 24] (also see [15, Eq. (1.15), p. 5] with $p = 1$).

The GFR of the p-deformed Kinney polynomial occurs with $\gamma = 1,\ c = 1,$ and $s = -1/m$ from (3.4) given by

$$\sum_{n=0}^{\infty} P_{n,p}(m,\beta n,x)t^n = \frac{(1+pw^m)^{(p-1/m)/p}}{1+(\beta m+p)\,w^m} G\left[\frac{w}{(1+pw^m)}\right],$$

where $w = t\,(1+pw^m)^{-\beta/p}$ and

$$G(z) = \sum_{n=0}^{\infty} \frac{\Gamma_p(p-\frac{1}{m})}{\Gamma_p(p-\frac{1}{m}-np)n!}(-mxz)^n.$$

The GFR of the p-deformed Pincherle polynomial is obtained by substituting $m = 3,\ \gamma = 1,\ c = 1,$ and $s = -\lambda$ in (3.4), and it is given by the following:

$$\sum_{n=0}^{\infty} \mathcal{P}_{n,p}^{\lambda+\beta n}(x)t^n = \frac{(1+pw^3)^{(p-\lambda)/p}}{1+\left(\frac{\beta 3}{p}+1\right)pw^3} G\left[\frac{w}{(1+pw^3)}\right],$$

where $w = t\left(1+pw^3\right)^{-\beta/p}$ and

$$G(z) = \sum_{n=0}^{\infty} \frac{\Gamma_p(p-\lambda)}{\Gamma_p(p-\lambda-np)n!}(-3xz)^n.$$

Similarly, taking $m = 2$, $\gamma = 1$, $c = 1$, and $s = -\nu$ in (3.4), we obtain

$$\sum_{n=0}^{\infty} C_{n,p}^{\nu+\beta n}(x)t^n = \frac{(1+pw^2)^{(p-\nu)/p}}{1+\left(\frac{\beta 2}{p}+1\right)pw^2} G\left[\frac{w}{(1+pw^2)}\right], \tag{3.7}$$

where $w = t\left(1+pw^2\right)^{-\beta/p}$ and

$$G(z) \;=\; \sum_{n=0}^{\infty} \frac{\Gamma_p(p-\nu)}{\Gamma_p(p-\nu-np)\; n!}(-2xz)^n,$$

which is the GFR of the p-deformed Gegenbauer polynomial. Furthermore, entering $\nu = 1/2$ into (3.7), we obtain the GFR of the p-deformed Legendre polynomial or briefly pLP given by

$$\sum_{n=0}^{\infty} P_{n,p}(x)t^n \;=\; \frac{(1+pw^2)^{(2p-1)/2p}}{1+\left(\frac{\beta 2}{p}+1\right)pw^2} G\left[\frac{w}{(1+pw^2)}\right],$$

with $w = t\left(1+pw^2\right)^{-\beta/p}$ and

$$G(z) \;=\; \sum_{n=0}^{\infty} \frac{\Gamma_p(p-\frac{1}{2})}{\Gamma_p(p-\frac{1}{2}-np)\; n!}(-2xz)^n.$$

3.1. Fibonacci-type Polynomials

We provide a computation formula of Fibonacci-type polynomials of order n (cf. [15, Theorem 2.2, p. 6]) in Theorem using (3.5).

Theorem 3.3. *For the pGHP defined by (1.5),*

$$P_{n,p}(m,x,\gamma,s_1+s_2,c) = \sum_{k=0}^{n} P_{n-k,p}(m,x,\gamma,s_1,c)\; P_{k,p}(m,x,\gamma,s_2,c). \tag{3.8}$$

Proof. Replacing s with s_1+s_2 in GFR (3.5) provides the following result:

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s_1+s_2,c)t^n \;&=\; (c)^{(1-1/p)(s_1+s_2)}\left(c+\gamma p t^m - pmxt\right)^{s_1+s_2/p} \\
&=\; (c)^{(1-1/p)s_1}\left(c+\gamma p t^m - pmxt\right)^{s_1/p} \\
&\quad\; \times (c)^{(1-1/p)s_2}\left(c+\gamma p t^m - pmxt\right)^{s_2/p} \\
&=\; \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s_1,c)t^n \sum_{k=0}^{\infty} P_{k,p}(m,x,\gamma,s_2,c)t^k \\
&=\; \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,p}(m,x,\gamma,s_1,c)P_{k,p}(m,x,\gamma,s_2,c)t^{n+k} \\
&=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{n-k,p}(m,x,\gamma,s_1,c)P_{k,p}(m,x,\gamma,s_2,c)t^n.
\end{aligned}$$

Comparing the coefficients of t^n, this yields (3.8). $\square$

The GFR (3.6) leads to the computation formula of Fibonacci-type polynomials of order n, stated as Corollary 3.4.

Corollary 3.4. *In the usual notations and meaning,*

$$\Pi_{n,m,p}^{\mu_1+\mu_2}(x) = \sum_{k=0}^{n} \Pi_{n-k,m,p}^{\mu_1}(x)\Pi_{k,m,p}^{\mu_2}(x).$$

4. Recurrence Relations

In this section, the differential recurrence relations and mixed relations of the $pGHP$ are derived.
First, we denote $(c+\gamma pt^m - pmxt)^{s/p}$ as $A(t;m,x,\gamma,s,c,p)$ and rewrite (3.5) in the following form:

$$A(t;m,x,\gamma,s,c,p) = c^{(1/p-1)s}\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c)t^n. \tag{4.1}$$

Then, with $D_x = d/dx$, we have

$$\begin{aligned} D_x(A(t;m,x,\gamma,s,c,p)) &= D_x\left((c+\gamma pt^m - pmxt)^{s/p}\right) \\ &= -mts\,(c+\gamma pt^m - pmxt)^{s/p-1}. \end{aligned}$$

Setting $sq=-p$, $q\in\mathbb{N}$, this results in the following:

$$\begin{aligned} D_x\left(A(t;m,x,\gamma,-p/q,c,p)\right) &= \frac{mtp}{q}\left(c+\gamma pt^m - pmxt\right)^{s/p+sq/p} \\ &= \frac{mtp}{q}A(t;m,x,\gamma,-p/q,c,p)^{1+q}. \end{aligned}$$

The successive differentiation yields the following:

$$\begin{aligned} D_x^2(A(t;m,x,\gamma,-p/q,c,p)) &= D_x\left(\frac{mtp}{q}A(t;m,x,\gamma,-p/q,c,p)^{1+q}\right) \\ &= \frac{mtp}{q}D_x\left((A(t;m,x,\gamma,-p/q,c,p))^{1+q}\right) \\ &= \frac{(mtp)(1+q)}{q}\left(A(t;m,x,\gamma,-p/q,c,p)\right)^q \\ &\quad \times D_x\left(A(t;m,x,\gamma,-p/q,c,p)\right) \\ &= \frac{(mtp)^2(1+q)}{q^2} \\ &\quad \times\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+2q}, \end{aligned} \tag{4.2}$$

$$D_x^3(A(t;m,x,\gamma,-p/q,c,p)) = \left(\frac{mtp}{q}\right)^3 (1+q)(1+2q)\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+3q},$$

and in general,

$$D_x^j(A(t;m,x,\gamma,-p/q,c,p)) = \left(\frac{mtp}{q}\right)^j \left\{\prod_{i=0}^{j-1}(1+iq)\right\}\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+jq}.$$

Next, taking the j^{th} derivative with respect to x in (4.1) yields the following:

$$\begin{aligned}
&D_x^j A(t;m,x,\gamma,s,c,p)\\
=\ & c^{(1/p-1)s}\sum_{n=0}^{\infty} t^n D_x^j P_{n,p}(m,x,\gamma,s,c)\\
=\ & c^{(1/p-1)s}\sum_{n=0}^{\infty} t^n \sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k c^{s-n+mk-k}\frac{(-s)_{n-mk+k,p}}{k!\,(n-mk)!}m^{n-mk}D_x^j\, x^{n-mk}\\
=\ & c^{(1/p-1)s}\sum_{n=j}^{\infty} t^n \sum_{k=0}^{\lfloor (n-j)/m\rfloor}(-\gamma)^k c^{s-n+mk-k}\frac{(-s)_{n-mk+k,p}}{k!\,(n-mk-j)!}m^{n-mk}x^{n-mk-j}\\
=\ & c^{(1/p-1)s}\sum_{n=0}^{\infty} t^{n+j} \sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!\,(n-mk)!}m^j(mx)^{n-mk}.
\end{aligned}$$

However, because,

$$\begin{aligned}
(4.3)\qquad & D_x^j P_{n+j,p}(m,x,\gamma,s,c)\\
=\ & \sum_{k=0}^{\lfloor (n+j)/m\rfloor}(-1)^k\gamma^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!\,(n+j-mk)!}\ m^{n+j-mk}D_x^j(x)^{n+j-mk}\\
=\ & \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\gamma^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!\,(n-mk)!}m^{n+j-mk}x^{n-mk},
\end{aligned}$$

from (4.3), we have the following:

$$(4.4)\quad D_x^j A(t;m,x,\gamma,s,c,p) \ =\ c^{(1/p-1)s}\sum_{n=0}^{\infty} t^{n+j} D_x^j P_{n+j,p}(m,x,\gamma,s,c).$$

Inserting $s = -p/q$ into (4.4) and employing (4.3), we obtain

$$\begin{aligned}
& c^{(1/p-1)s}\sum_{n=0}^{\infty} t^n D_x^j P_{n+j,p}(m,x,\gamma,-p/q,c)\\
=\ & \left(\frac{mp}{q}\right)^j \left\{\prod_{i=0}^{j-1}(1+iq)\right\}\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+jq}.
\end{aligned}$$

Replacing $A(t;m,x,\gamma,-p/q,c,p)$ with its series expansion from (4.1), it becomes

$$\begin{aligned}&c^{(1/p-1)s}\sum_{n=0}^{\infty}t^nD_x^jP_{n+j,p}(m,x,\gamma,-p/q,c)\\ =\;&\left(\frac{mp}{q}\right)^j\left\{\prod_{i=0}^{j-1}(1+iq)\right\}\left(c^{(1/p-1)s}\sum_{n=0}^{\infty}P_{n,p}(m,x,\gamma,-p/q,c)t^n\right)^{1+jq},\end{aligned}$$

that is,

$$\begin{aligned}&\sum_{n=0}^{\infty}t^nD_x^jP_{n+j,p}(m,x,\gamma,-p/q,c)\\ =\;&\left(\frac{mp}{q}\right)^j\ c^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\}\left(\sum_{n=0}^{\infty}P_{n,p}(m,x,\gamma,-p/q,c)t^n\right)^{1+jq}\\ =\;&\left(\frac{mp}{q}\right)^j\ c^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\}\sum_{n=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{1+jq}=n}P_{i_1,p}P_{i_2,p}\cdots P_{i_{1+jq},p}\ t^n,\end{aligned}$$

where $q\in\mathbb{N}$ and $P_{i_r,p}=P_{i_r,p}(m,x,\gamma,-p/q,c)$. Comparing the coefficients of t^n, this yields the following:

$$\begin{aligned}&D_x^jP_{n+j,p}(m,x,\gamma,-p/q,c)\\ =\;&\left(\frac{mp}{q}\right)^j c^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\}\sum_{i_1+i_2+\cdots+i_{1+jq}=n}P_{i_1,p}P_{i_2,p}\cdots P_{i_{1+jq},p}.\end{aligned}$$

This provides the p-deformed version of the result according to Gould [9, Eq. (3.4), p. 702](cf. with $p=1$). Further, multiplying (4.3) by t^n and then taking the summation from $n=0$ to ∞, produces the following:

$$\begin{aligned}\sum_{n=0}^{\infty}D_x^jP_{n+j,p}(m,x,\gamma,s,c)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^kc^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!\ (n-mk)!}\\ &\quad\times m^{n+j-mk}x^{n-mk}t^n\\ &= \sum_{n=0}^{\infty}c^{jp-j}(-s)_{j,p}m^j\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^kc^{s-jp-n+mk-k}\\ &\quad\times\frac{(-s+jp)_{n-mk+k,p}}{k!(n-mk)!}\ (mx)^{n-mk}t^n\\ &= \sum_{n=0}^{\infty}c^{(p-1)j}(-s)_{j,p}\ m^jP_{n,p}(m,x,\gamma,s-jp,c)\ t^n\\ &= \sum_{n=0}^{\infty}c^{(p-1)j}(-s)_{j,p}\ m^jP_{n,p}(m,x,\gamma,s-jp,c)\ t^n.\end{aligned}$$

From this, it follows that

$$D_x^j P_{n+j,p}(m,x,\gamma,s,c) = c^{(p-1)j}(-s)_{j,p}\ m^j P_{n,p}(m,x,\gamma,s-jp,c). \tag{4.5}$$

This generalizes the formula given by Gould [9, Eq. (3.5), p. 702](cf. with $p=1$). In addition, setting $j=1$, $m=2$, $\gamma=1$, $c=1$, and $s=-\nu$ and replacing n with $n-1$ in (4.5) yields

$$D_x C_{n,p}^{\nu}(x) = 2\nu C_{n-1,p}^{\nu}(x),$$

involving the p-Gegenbauer polynomial. This generalizes the familiar formula of Whittaker et al. [20, (III), p. 330](cf. with $p=1$). For $\nu = 1/2$, this further reduces to the following:

$$D_x P_{n,p}(x) = P_{n-1,p}(x),$$

involving the p-Legendre polynomial.

For the recurrence relations, we first obtain the following:

$$\begin{aligned} & (c+\gamma p t^m - pmxt)\, t D_t A(t;m,x,\gamma,s,c,p) \\ = \quad & (c+\gamma p t^m - pmxt)\, t D_t \left(c+\gamma p t^m - pmxt\right)^{s/p} \\ = \quad & (c+\gamma p t^m - pmxt)\, t \frac{s}{p} \left(c+\gamma p t^m - pmxt\right)^{s/p-1} \left(\gamma pmt^{m-1} - pmx\right) \\ = \quad & (-ms)(xt-\gamma t^m) A(t;m,x,\gamma,s,c,p). \end{aligned}$$

From (4.1), we have the following:

$$\begin{aligned} & (c+\gamma p t^m - pmxt)\, t\ D_t \left(c^{(1/p-1)s} \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c) t^n \right) \\ = \quad & (-ms)(xt-\gamma t^m)\ c^{(1/p-1)s} \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c) t^n. \end{aligned}$$

Simplifying this and abbreviating $P_{n,p}(m,x,\gamma,s,c)$ by $\mathfrak{P}_{n,p}(x)$, we obtain

$$\begin{aligned} (c+\gamma p t^m - pmxt) \sum_{n=0}^{\infty} n\mathfrak{P} P_{n,p}(x) t^n \ &= \ -ms(xt-\gamma t^m) \sum_{n=0}^{\infty} \mathfrak{P}_{n,p}(x) t^n \\ \Rightarrow \sum_{n=0}^{\infty} cn\mathfrak{P}_{n,p}(x)t^n + \gamma p t^m \sum_{n=0}^{\infty} n\mathfrak{P}_{n,p}(x)t^n &- pmxt \sum_{n=0}^{\infty} n\mathfrak{P}_{n,p}(x)t^n \\ = \ -msxt \sum_{n=0}^{\infty} \mathfrak{P}_{n,p}(x)t^n &+ ms\gamma t^m \sum_{n=0}^{\infty} \mathfrak{P}_{n,p}(x)t^n \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \sum_{n=0}^{\infty} cn\mathfrak{P}_{n,p}(x)t^n + \gamma p\sum_{n=0}^{\infty} n\mathfrak{P}_{n,p}(x)t^{n+m} - pmx\sum_{n=0}^{\infty} n\mathfrak{P}_{n,p}(x)t^{n+1} \\
&\qquad = \ -msx\sum_{n=0}^{\infty}\mathfrak{P}_{n,p}(x)t^{n+1} + ms\gamma\sum_{n=0}^{\infty}\mathfrak{P}_{n,p}(x)t^{n+m} \\
&\Rightarrow \sum_{n=0}^{\infty} cn\mathfrak{P}_{n,p}(x)t^n + \gamma p\sum_{n=m}^{\infty}(n-m)\mathfrak{P}_{n-m,p}(x)t^n - pmx\sum_{n=1}^{\infty}(n-1)\mathfrak{P}_{n-1,p}(x)t^n \\
&\qquad = \ -msx\sum_{n=0}^{\infty}\mathfrak{P}_{n-1,p}(x)t^n + ms\gamma\sum_{n=0}^{\infty}\mathfrak{P}_{n-m,p}(x)t^n \\
&\Rightarrow \sum_{n=m}^{\infty}\left(cn\mathfrak{P}_{n,p}(x) + mx(s-np+p)\mathfrak{P}_{n-1,p}(x) + \gamma(np-mp-ms)\mathfrak{P}_{n-m,p}(x)\right)t^n \\
&\quad = 0,
\end{aligned}
$$

because $n \geq m \geq 1$. The recurrence relation is given as follows:

$$
cn\mathfrak{P}_{n,p}(x) + mx(s-np+p)\mathfrak{P}_{n-1,p}(x) + \gamma(np-mp-ms)\mathfrak{P}_{n-m,p}(x) \ = \ 0. \tag{4.6}
$$

This identity provides the p-deformed version of a recurrence relation derived by Gould [9, Eq. (2.3), p. 700](cf. with $p = 1$). Taking derivative of (3.5) with respect to x, we obtain

$$
\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n = -smt(c)^{(1-1/p)s}\left(c+\gamma pt^m - pmxt\right)^{s/p-1}, \tag{4.7}
$$

and differentiating (3.5) with respect to t and using (4.7), we find

$$
\begin{aligned}
& \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c)nt^{n-1} \\
= \ & \frac{s}{p}\ c^{(1-1/p)s}(\gamma pmt^{m-1} - pmx)\left(c+\gamma pt^m - pmxt\right)^{s/p-1} \\
= \ & -s\ c^{(1-1/p)s}\frac{(\gamma mt^{m-1} - mx)}{smt(c)^{(1-1/p)s}}\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n \\
= \ & \frac{(x-\gamma t^{m-1})}{t}\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n.
\end{aligned}
$$

Thus, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} nP_{n,p}(m,x,\gamma,s,c)t^n \\
= \quad & x\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n - \gamma t^{m-1}\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n \\
= \quad & x\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n - \gamma \sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^{n+m-1} \\
= \quad & x\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n - \gamma \sum_{n=m-1}^{\infty} D_x P_{n-m+1,p}(m,x,\gamma,s,c)t^n.
\end{aligned}$$

After equating the coefficients of t^n on both sides for $n \geq m-1$, we obtain

$$nP_{n,p}(m,x,\gamma,s,c) \quad = \quad xD_xP_{n,p}(m,x,\gamma,s,c) - \gamma D_x P_{n-m+1,p}(m,x,\gamma,s,c).$$

This provides a p-deformation of the differential recurrence relation according to Gould [9, Eq. (2.5), p. 700]. The other recurrence relations involving the *pGHP* may be obtained similarly.

5. Companion Matrix

Let $\widetilde{P}_{n,p}(m,x,\gamma,s,c)$ be the monic polynomial obtained from (2.1) and it is defined by the following:

$$\widetilde{P}_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{N} \delta_k \ x^{n-mk}, \tag{5.1}$$

where

$$\delta_k = \frac{(-1)^{k+mk}(-n)_{mk}\gamma^k c^{mk-k}(-s+np)_{-mk+k,p}}{m^{mk}k!}.$$

Then, with this $\delta_{k,p}$, $C\left(\widetilde{P}_{n,p}(m,x,\gamma,s,c)\right)$ assumes the form stated in Definition 1.1. The eigen values of this matrix are thus precisely the zeros of $\widetilde{P}_{n,p}(m,x,\gamma,s,c)$ (see [14, p. 39]).

Iillustation 5.1. To illustrate the companion matrix of the above monic polynomial, we take $n=3, p=2$, and $m=2$ into (5.1) to obtain
$\widetilde{P}_{3,2}(2,x,\gamma,s,c) = \sum_{k=0}^{1} \delta_k \ x^{3-2k} = \delta_0 x^3 + \delta_1 x = \delta_0 x^3 + 0x^2 + \delta_1 x + 0,$
where $\delta_0 = 1$ and $\delta_1 = \frac{3}{2}\left(\frac{\gamma c}{s-4}\right)$.

Thus, the companion matrix is as follows:

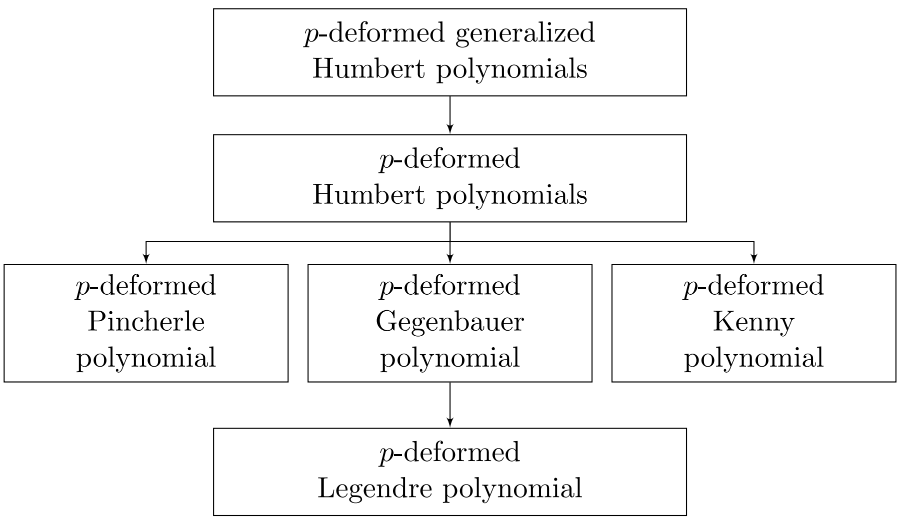
$$C\left(\widetilde{P}_{3,2}(2,x,\gamma,s,c)\right)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & \delta_1 & 0\end{bmatrix}$$

where the eigen values are determined from the following determinant equation:

$$\begin{vmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ 0 & \delta_1 & -\lambda\end{vmatrix}=0.$$

From this, we obtain the eigen values $\lambda = 0$, $\sqrt{\delta_1}$, and $-\sqrt{\delta_1}$, that satisfy the equation $\widetilde{P}_{3,2}(2,x,\gamma,s,c)=0$.

POLYNOMIAL REDUCIBILITY



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