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Generalizations of Ramanujan's Integral Associated with Infinite Fourier Cosine Transforms in Terms of Hypergeometric Functions and its Applications

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Abstract. In this paper, we obtain an analytical solution for an unsolved definite integral $\mathbf{R}_C(m,n)$ from a 1915 paper of Srinivasa Ramanujan. We obtain our solution using the hypergeometric approach and an infinite series decomposition identity. Also, we give some generalizations of Ramanujan's integral $\mathbf{R}_C(m,n)$ defined in terms of the ordinary hypergeometric function ${}_2F_3$ with suitable convergence conditions. Moreover as applications of our result we obtain nine new infinite summation formulas associated with the hypergeometric functions ${}_0F_1$, ${}_1F_2$ and ${}_2F_3$.

1. Introduction and Preliminaries

In the literature of infinite Fourier cosine transforms (see, for example, [2, 4]) one can find analytical solutions of $\int_0^\infty \frac{x^{\upsilon-1}\cos(xy)}{\exp(bx)\pm 1}dx$ given in terms of Riemann's zeta function, the Psi (Digamma) function, hyperbolic functions and Beta functions.

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The analytical solution of the following integral of Ramanujan [7, p. 85, eq.(49)]:

$$\mathbf{R}_C(m,n)=\int_0^\infty \frac{x^m\cos(\pi nx)}{\exp(2\pi\sqrt{x})-1}dx, \tag{1.1}$$

is not known for all positive rational values of n, and non-negative integral values of m; though for three particular pairs or values of m and n the following solutions are given in [7, p.86, eq.(50)]:

$$\mathbf{R}_C(1,1/2)=\int_0^\infty \frac{x\cos(\frac{\pi x}{2})}{\exp(2\pi\sqrt{x})-1}dx=\frac{13-4\pi}{8\pi^2}, \tag{1.2}$$

$$\mathbf{R}_C(1,2)=\int_0^\infty \frac{x\cos(2\pi x)}{\exp(2\pi\sqrt{x})-1}dx=\frac{1}{64}\left(\frac{1}{2}-\frac{3}{\pi}+\frac{5}{\pi^2}\right), \tag{1.3}$$

$$\mathbf{R}_C(2,2)=\int_0^\infty \frac{x^2\cos(2\pi x)}{\exp(2\pi\sqrt{x})-1}dx=\frac{1}{256}\left(1-\frac{5}{\pi}+\frac{5}{\pi^2}\right). \tag{1.4}$$

The following theorem is proved by Ramanujan [7, p.76-77, eqs.(10 and $10'$)]:
If

$$\mathbf{R}_C(0,n)=\Phi(n)=\int_0^\infty \frac{\cos(\pi nx)}{\exp(2\pi\sqrt{x})-1}dx, \tag{1.5}$$

and

$$\Upsilon(n)=\frac{1}{2\pi n}+\int_0^\infty \frac{\sin(\pi nx)}{\exp(2\pi\sqrt{x})-1}dx, \tag{1.6}$$

then

$$\mathbf{R}_C(0,n)=\Phi(n)=\frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Upsilon\left(\frac{1}{n}\right)-\Upsilon(n), \tag{1.7}$$

and

$$\Upsilon(n)=\frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Phi\left(\frac{1}{n}\right)+\Phi(n), \tag{1.8}$$

where n is positive rational number.

For particular values of n, Ramanujan [7, p.85, eq. (48)] also showed the following:

$$\mathbf{R}_C(0,1)=\Phi(1)=\int_0^\infty \frac{\cos(\pi x)}{\exp(2\pi\sqrt{x})-1}dx=\frac{2-\sqrt{2}}{8}, \tag{1.9}$$

$$\mathbf{R}_C(0,2)=\Phi(2)=\int_0^\infty \frac{\cos(2\pi x)}{\exp(2\pi\sqrt{x})-1}dx=\frac{1}{16}, \tag{1.10}$$

$$\mathbf{R}_C(0,4)=\Phi(4)=\int_0^\infty \frac{\cos(4\pi x)}{\exp\left(2\pi\sqrt{x}\right)-1}dx=\frac{3-\sqrt{2}}{32}, \tag{1.11}$$

$$\mathbf{R}_C(0,6)=\Phi(6)=\int_0^\infty \frac{\cos(6\pi x)}{\exp\left(2\pi\sqrt{x}\right)-1}dx=\frac{13-4\sqrt{3}}{144}, \tag{1.12}$$

$$\mathbf{R}_C(0,1/2)=\Phi\left(\frac{1}{2}\right)=\int_0^\infty \frac{\cos\left(\frac{\pi x}{2}\right)}{\exp\left(2\pi\sqrt{x}\right)-1}dx=\frac{1}{4\pi}, \tag{1.13}$$

$$\mathbf{R}_C(0,2/5)=\Phi\left(\frac{2}{5}\right)=\int_0^\infty \frac{\cos\left(\frac{2\pi x}{5}\right)}{\exp\left(2\pi\sqrt{x}\right)-1}dx=\frac{8-3\sqrt{5}}{16}. \tag{1.14}$$

A natural generalization of Gauss hypergeometric series ${}_2F_1$ is the general hypergeometric series ${}_pF_q$ [9, p.42, eq.(1)] and see also [1] with p numerator parameters $\alpha_1,...,\alpha_p$ and q denominator parameters $\beta_1,...,\beta_q$. For $p,q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}=\{0,1,2,...\}$, it is defined as

$${}_pF_q\left(\begin{array}{l}\alpha_1,...,\alpha_p\ ;\\ \beta_1,...,\beta_q\ ;\end{array}\ z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n}\frac{z^n}{n!}\ , \tag{1.15}$$

where $\alpha_i\in\mathbb{C}$ for $i=1,...,p$ and $\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Here we use $\mathbb{Z}_0^-:=\{0,-1,-2,...\}$. Also for $\lambda,\upsilon\in\mathbb{C}$, Pochhammer's symbol $(\lambda)_\upsilon$,(or the shifted factorial, since $(1)_n=n!$) is defined, in general, by

$$(\lambda)_\upsilon:=\frac{\Gamma(\lambda+\upsilon)}{\Gamma(\lambda)}=\begin{cases}1, & (\upsilon=0\ ;\ \lambda\in\mathbb{C}\backslash\{0\})\\ \lambda(\lambda+1)...(\lambda+n-1), & (\upsilon=n\in\mathbb{N}\ ;\ \lambda\in\mathbb{C}).\end{cases} \tag{1.16}$$

The hypergeometric series ${}_pF_q$ in eq.(1.15) is convergent for $|z|<\infty$ if $p\le q$, and for $|z|<1$ if $p=q+1$.

Furthermore, if we set

$$\omega=\left(\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i\right), \tag{1.17}$$

it is known that the series ${}_pF_q$, with $p=q+1$, is

(i) absolutely convergent for $|z|=1$ if $Re(\omega)>0$,

(ii) conditionally convergent for $|z|=1,z\neq 1$, if $-1<Re(\omega)\le 0$.

Also the binomial function is given by

$$(1-z)^{-\alpha}={}_1F_0\left(\begin{array}{l}\alpha\ ;\\ \text{—};\end{array}\ z\right)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n\ , \tag{1.18}$$

where $|z|<1,\ \alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$.

The Fox-Wright function ${}_p\Psi_q$ from [11, 12] is given by

$$(1.19)\quad {}_p\Psi_q\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{array}\ z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+kA_1)...\Gamma(\alpha_p+kA_p)}{\Gamma(\beta_1+kB_1)...\Gamma(\beta_q+kB_q)}\frac{z^k}{k!},$$

$$=\frac{\Gamma(\alpha_1)...\Gamma(\alpha_p)}{\Gamma(\beta_1)...\Gamma(\beta_q)}\sum_{k=0}^{\infty}\frac{(\alpha_1)_{kA_1}...(\alpha_p)_{kA_p}}{(\beta_1)_{kB_1}...(\beta_q)_{kB_q}}\frac{z^k}{k!},$$

$$(1.20)\quad =\frac{\Gamma(\alpha_1)...\Gamma(\alpha_p)}{\Gamma(\beta_1)...\Gamma(\beta_q)}{}_p\Psi_q^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{array}\ z\right],$$

$$(1.21)\quad =\frac{1}{2\pi\rho}\int_L\frac{\Gamma(\zeta)\prod_{i=1}^{p}\Gamma(\alpha_i-A_i\zeta)}{\prod_{j=1}^{q}\Gamma(\beta_j-B_j\zeta)}(-z)^{-\zeta}d\zeta\ ,$$

where we have $\rho=\sqrt{-1}$, and $z,\alpha_i,\beta_j\in\mathbb{C}$ everywhere; and $A_i,B_j\in\mathbb{R}\setminus 0$ except in the case of (1.21) where we have $A_i,B_j\in\mathbb{R}_+=(0,+\infty)$ In eq. (1.19), the parameters α_i,β_j and coefficients A_i,B_j are adjusted in such a way that the product of the Gamma functions in numerator and denominator should be well defined.

Suppose:

$$(1.22)\quad \Delta^*=\left(\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\right),$$

$$(1.23)\quad \delta^*=\left(\prod_{i=1}^{p}|A_i|^{-A_i}\right)\left(\prod_{j=1}^{q}|B_j|^{B_j}\right),$$

$$(1.24)\quad \mu^*=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i+\left(\frac{p-q}{2}\right),$$

and

$$(1.25)\quad \sigma^*=(1+A_1+...+A_p)-(B_1+...+B_q)=1-\Delta^*.$$

Then we have the following convergence conditions of (1.19) and (1.21):

Case(1): When the contour (L) is a left loop beginning and ending at $-\infty$, then ${}_p\Psi_q[\cdot]$, given by (1.19) or (1.21), converges under any of the following conditions.

i) $\Delta^*>-1$, and $0<|z|<\infty$.

ii) $\Delta^*=-1$ and $0<|z|<\delta^*$.

iii) $\Delta^*=-1$, $|z|=\delta^*$, and $Re(\mu^*)>\frac{1}{2}$.

Case(2): When the contour (L) is a right loop beginning and ending at $+\infty$, then ${}_p\Psi_q[\cdot]$, given by (1.19)or (1.21), converges under the following conditions.

iv) $\Delta^* < -1$, and $0 < |z| < \infty$.

v) $\Delta^* = -1$, and $|z| > \delta^*$.

vi) $\Delta^* = -1$, $|z| = \delta^*$, and $Re(\mu^*) > \frac{1}{2}$.

Case(3): When contour (L) is starts at $\gamma - i\infty$ and ends at $\gamma + i\infty$ where $\gamma \in \mathbb{R}$, then ${}_p\Psi_q[\cdot]$ is also convergent under the following conditions.

vii) $\sigma^* > 0$, $|arg(-z)| < \frac{\pi}{2}\sigma^*$, and $0 < |z| < \infty$.

viii) $\sigma^* = 0$, $arg(-z) = 0$, $0 < |z| < \infty$ and $-\gamma\Delta^* + Re(\mu^*) > \frac{1}{2} + \gamma$.

ix) $\gamma = 0$, $\sigma^* = 0, arg(-z) = 0$, $0 < |z| < \infty$, and $Re(\mu^*) > \frac{1}{2}$.

The infinite Fourier cosine transform of $g(x)$ over the interval $(0, \infty)$ is defined by

$$F_C\{g(x); y\} = \int_0^\infty g(x)\cos(xy)dx = G_C(y), \qquad (y > 0). \tag{1.26}$$

It follows that we have $g(x) = F_C^{-1}[G_C(y); x] = \dfrac{2}{\pi}\displaystyle\int_0^\infty G_C(y)\cos(xy)dy$.

Note that some authors add an extra factor of $\sqrt{\frac{2}{\pi}}$ in their definition of $\mathcal{F}_C\{g(x); y\}$. If $b > 0$ and $0 < Re(s) < 1$, then the Mellin-transform of $\cos(bx)$ or infinite Fourier Cosine transform of x^{s-1} [3, p.42, eqs.(5.2)] is given by

$$\int_0^\infty x^{s-1}\cos(bx)dx = \frac{\Gamma(s)\cos(\frac{\pi s}{2})}{b^s}. \tag{1.27}$$

If $Re(\mu) > -1$, $0 < \xi < 1$, $a > 0$ and $y > 0$, then we can prove the following integral using Maclaurin's expansion of $\exp(-ax^\xi)$ and integrating termwise with the help of the result (1.27):

(1.28)

$$\int_0^\infty x^\mu \exp(-ax^\xi)\cos(xy)dx = -y^{-\mu-1}\sum_{\ell=0}^\infty \left(-\frac{a}{y^\xi}\right)^\ell \frac{1}{\ell!}\Gamma(\mu+1+\xi\ell)\,\sin\left\{\frac{\pi}{2}(\mu+\xi\ell)\right\}.$$

An infinite series decomposition identity [8, p.193,eq.(8)] is given by

$$\sum_{\ell=0}^\infty \Omega(\ell) = \sum_{j=0}^{N-1}\left\{\sum_{\ell=0}^\infty \Omega(N\ell + j)\right\}, \tag{1.29}$$

where N is an arbitrary positive integer. Put $N = 4$ in the above eq. (1.29), we get

$$\sum_{\ell=0}^\infty \Omega(\ell) = \sum_{j=0}^{3}\left\{\sum_{\ell=0}^\infty \Omega(4\ell + j)\right\}, \tag{1.30}$$

$$(1.31)\qquad = \sum_{\ell=0}^{\infty}\Omega(4\ell)+\sum_{\ell=0}^{\infty}\Omega(4\ell+1)+\sum_{\ell=0}^{\infty}\Omega(4\ell+2)+\sum_{\ell=0}^{\infty}\Omega(4\ell+3),$$

provided that all involved infinite series are absolutely convergent.
For every positive integer m [9, p.22, eq.(26)], we have

$$(1.32)\qquad (\lambda)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n \qquad ;m\in\mathbb{N},\ n\in\mathbb{N}_0.$$

From the above result (1.32) with $\lambda = mz$, it can be proved that

$$(1.33)\qquad \Gamma(mz) = (2\pi)^{\frac{(1-m)}{2}} m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right),\quad mz\in\mathbb{C}\backslash\mathbb{Z}_0^-$$

Equation (1.33) is known as the Gauss-Legendre multiplication theorem for the Gamma function. Elementary trigonometric functions [9, p.44, eq.(9) and eq.(10)] are given by

$$(1.34)\qquad \cos\ z = {}_0F_1\left(\begin{array}{c}\text{——};\\ \frac{1}{2};\end{array}\ \frac{-z^2}{4}\right),$$

$$(1.35)\qquad \sin\ z = z\ {}_0F_1\left(\begin{array}{c}\text{——};\\ \frac{3}{2};\end{array}\ \frac{-z^2}{4}\right).$$

The Lommel function [9, p.44, eq.(13)] is given by

$$(1.36)\qquad s_{\mu,\upsilon}(z) = \frac{z^{\mu+1}}{(\mu-\upsilon+1)(\mu+\upsilon+1)}\,{}_1F_2\left(\begin{array}{c}1;\\ \frac{\mu-\upsilon+3}{2},\frac{\mu+\upsilon+3}{2};\end{array}\ \frac{-z^2}{4}\right),$$

where $\mu\pm\upsilon\in\mathbb{C}\setminus\{-1,-3,-5,-7,...\}$.

As we have mentioned, no general analytic solution is known for $\mathbf{R}_C(m,n)$. Motivated by the work done in [10, 5] our aim in this paper is to give an analytical solution of Ramanujan's integral in terms of ordinary hypergeometric functions.

Here in this paper, we have generalized Ramanujan's integral $\mathbf{R}_C(m,n)$ in the following forms, where $\{\Theta(k)\}_{k=0}^{\infty}$ is a bounded sequence, and obtain analytical solution for them:

(i) $\mathbf{I}_C^*(\upsilon,b,c,\lambda,y) = \sum_{k=0}^{\infty}\left[\frac{\Theta(k)}{k!}\int_0^{\infty} x^{\upsilon-1}e^{-(\lambda b+ck)\sqrt{x}}\cos(xy)dx\right],$

(ii) $\mathbf{J}_C(\upsilon,b,c,\lambda,y) = \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}\,{}_r\Psi_s\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_r,A_r);\\ (\beta_1,B_1),...,(\beta_s,B_s);\end{array}\ e^{-c\sqrt{x}}\right]\cos(xy)dx,$

(iii) $\mathbf{K}_C(\upsilon,b,c,\lambda,y) = \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}\,{}_rF_s\left(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;\ e^{-c\sqrt{x}}\right)\cos(xy)dx,$

(iv) $\mathbf{I}_C(\upsilon, b, \lambda, y) = \int_0^\infty x^{\upsilon-1}\{\exp(b\sqrt{x})-1\}^{-\lambda}\cos(xy)dx,$

Moreover, we show, in Sections 3-6, how the main general theorem given below can be applied to obtain new interesting results by suitably adjusting the parameters and variables.

2. Main General Theorem on Infinite Fourier Cosine Transform

Suppose $\{\Theta(k)\}_{k=0}^\infty$ is a bounded sequence of arbitrary real and complex numbers, and and $Re(\upsilon), c, y,$ are positive and λ and b are both positive or both negative, then

(2.1)
$$\mathbf{I}_C^*(\upsilon, b, c, \lambda, y) = \sum_{k=0}^\infty \left[\frac{\Theta(k)}{k!}\int_0^\infty x^{\upsilon-1}e^{-(\lambda b+ck)\sqrt{x}}\cos(xy)dx\right],$$

(2.2)
$$= y^{-\upsilon}\sum_{k=0}^\infty \left[\frac{\Theta(k)}{k!}\sum_{\ell=0}^\infty \frac{(-1)^\ell(\lambda b+ck)^\ell\Gamma\left(\upsilon+\frac{\ell}{2}\right)}{y^{\frac{\ell}{2}}\ \ell!}\cos\left(\frac{\upsilon\pi}{2}+\frac{\ell\pi}{4}\right)\right],$$

Now replacing ℓ by $4\ell+j$, after simplification we get

(2.3)
$$\mathbf{I}_C^*(\upsilon, b, c, \lambda, y) = y^{-\upsilon}\sum_{k=0}^\infty \left[\frac{\Theta(k)}{k!}\sum_{j=0}^3 \frac{(-1)^j(\lambda b+ck)^j\ \Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\ j!}\cos\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right)\right.$$
$$\left.\times\ {}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right],$$

(2.4)
$$= y^{-\upsilon}\sum_{k=0}^\infty \left[\frac{\Theta(k)}{k!}\sum_{j=0}^3 \frac{(-1)^j\Gamma\left(\upsilon+\frac{j}{2}\right)}{j!}\cos\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right)\left(\frac{\lambda b}{\sqrt{y}}\right)^j\times\right.$$
$$\left.\times\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^j{}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right],$$

(2.5)
$$= \frac{\Gamma(\upsilon)\cos\left(\frac{\upsilon\pi}{2}\right)}{y^\upsilon}\sum_{k=0}^\infty\left[\frac{\Theta(k)}{k!}{}_2F_3\left(\begin{array}{l}\frac{\upsilon}{2},\ \frac{\upsilon+1}{2}\ ;\\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4};\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right]$$

$$-\frac{(\lambda b)\Gamma(\upsilon+\frac{1}{2})\cos\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{y^{\upsilon+\frac{1}{2}}}\sum_{k=0}^{\infty}\left[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}\times\right.$$

$$\left.\times\,{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+1}{4},\frac{2\upsilon+3}{4};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right]$$

$$-\frac{(\lambda b)^2\Gamma(\upsilon+1)\sin\left(\frac{\upsilon\pi}{2}\right)}{2y^{\upsilon+1}}\sum_{k=0}^{\infty}\left[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^2\times\right.$$

$$\left.\times\,{}_2F_3\left(\begin{array}{l}\frac{\upsilon+1}{2},\frac{\upsilon+2}{2};\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right]$$

$$+\frac{(\lambda b)^3\Gamma(\upsilon+\frac{3}{2})\sin\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{6y^{\upsilon+\frac{3}{2}}}\sum_{k=0}^{\infty}\left[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^3\times\right.$$

$$\left.\times\,{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+3}{4},\frac{2\upsilon+5}{4};\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right]$$

Our result (2.3) or (2.4) or (2.5) is convergent in view of the convergence condition of ${}_pF_q(\cdot)$, when $p \leq q$, and $\forall\ |z| < \infty$.

Proof. The result (2.2) is obtained by the application of the integral (1.28) [with substitutions $\mu = \upsilon - 1, a = \lambda b + ck, \xi = \frac{1}{2}$] in the R.H.S. of eq.(2.1). The results (2.3), (2.4) and (2.5) are obtained by using the infinite series decomposition formulas (1.30),(1.31), Pochhammer's identity (1.32) and other algebraic properties of Pochhammer's symbols. □

3. Infinite Fourier Cosine Transforms of $x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_r\Psi_s[\cdot]$ and $x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_rF_s(\cdot)$

If we put $\Theta(k) = \dfrac{\Gamma(\alpha_1 + kA_1)...\Gamma(\alpha_r + kA_r)}{\Gamma(\beta_1 + kB_1)...\Gamma(\beta_s + kB_s)}$, for $k = 0, 1, 2, 3, ...$, in the equations (2.1) and (2.3), then after simplification we get the following:

(3.1)

$$\mathbf{J}_C(\upsilon, b, c, \lambda, y) = \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_r\Psi_s\left[\begin{array}{l}(\alpha_1, A_1), ..., (\alpha_r, A_r);\\ (\beta_1, B_1), ..., (\beta_s, B_s);\end{array}\ e^{-c\sqrt{x}}\right]\cos(xy)dx,$$

$$(3.2)\quad = y^{-\upsilon}\sum_{k=0}^{\infty}\left[\frac{\Gamma(\alpha_1+kA_1)...\Gamma(\alpha_r+kA_r)}{\Gamma(\beta_1+kB_1)...\Gamma(\beta_s+kB_s)k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+ck)^j\ \Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\ j!}\times\right.$$
$$\left.\times\cos\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right],$$

where $Re(\upsilon), c, y,$ are positive, λ and b are both positive or both negative, $\alpha_i, \beta_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R}\setminus\{0\}$ for $i = 1, 2, ..., r$ and $j = 1, 2, ..., s$, and ${}_r\Psi_s[\cdot]$ is the Fox-Wright psi function of one variable subject to suitable convergence conditions derived from the convergence conditions for (1.19),(1.20) and (1.21) given in **Case(1)** or **Case(2)** or **Case(3).**

When N is a positive integer then $\Delta(N;\lambda)$ denotes the array of N parameters given by $\frac{\lambda}{N}, \frac{\lambda+1}{N}, ..., \frac{\lambda+N-1}{N}$. When N and j are independent variables then the notation $\Delta(N; j+1)$ denotes the set of N parameters given by $\frac{j+1}{N}, \frac{j+2}{N}, ..., \frac{j+N}{N}$. When j is dependent variable that is $j = 0, 1, 2, 3, ..., N-1$, then the asterisk in $\Delta^*(N; j+1)$ represents the fact that the (denominator) parameters $\frac{N}{N}$ is always omitted (due to the need of factorial in denominator in the power series form of hypergeometric function) so that the set $\Delta^*(N; j+1)$ obviously contains only $(N-1)$ parameters [9, Chap.3, p.214].

Remark 3.1. When $A_1 = ... = A_r = B_1 = ... = B_s = 1$ in (3.1), (3.2) then we get

$$(3.3)\quad \mathbf{K}_C(\upsilon, b, c, \lambda, y) = \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_rF_s\left(\begin{array}{l}\alpha_1, ..., \alpha_r;\\ \beta_1, ..., \beta_s;\end{array}\ e^{-c\sqrt{x}}\right)\cos(xy)dx,$$

$$(3.4)\quad = y^{-\upsilon}\sum_{k=0}^{\infty}\left[\frac{(\alpha_1)_k...(\alpha_r)_k}{(\beta_1)_k...(\beta_s)_k\ k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+ck)^j\ \Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\ j!}\cos\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right)\right.$$
$$\left.\times\ {}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\right],$$

where $Re(\upsilon), c, y,$ are positive, λ and b are both positive or both negative, $r \le s+1$, and $\alpha_i, \beta_j \in \mathbb{C}$ for $i = 1, 2, ..., r$ and $j = 1, 2, ..., s$.

4. Fourier Cosine Transform of $x^{\upsilon-1}\{\exp(b\sqrt{x})-1\}^{-\lambda}$

For the generalization $\mathbf{I}_C(\upsilon, b, \lambda, y)$ of Ramanujan's integral $\mathbf{R}_C(m, n)$ in terms of ordinary hypergeometric functions ${}_2F_3$, the following holds:

$$(4.1)\quad \mathbf{I}_C(\upsilon, b, \lambda, y) = \int_0^{\infty}\frac{x^{\upsilon-1}\cos(xy)}{\{\exp(b\sqrt{x})-1\}^{\lambda}}dx,$$

$$(4.2)\qquad = y^{-\upsilon}\sum_{k=0}^{\infty}\left[\frac{(\lambda)_k}{k!}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\lambda b+bk)^{\ell}\Gamma\left(\upsilon+\frac{\ell}{2}\right)}{y^{\frac{\ell}{2}}\ \ell!}\cos\left(\frac{\upsilon\pi}{2}+\frac{\ell\pi}{4}\right)\right],$$

$$(4.3)\qquad = y^{-\upsilon}\sum_{k=0}^{\infty}\left[\frac{(\lambda)_k}{k!}\sum_{j=0}^{3}\frac{(-1)^{j}(\lambda b+bk)^{j}\ \Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\ j!}\cos\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right)\right.$$
$$\left.\times\ {}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\right],$$

$$(4.4) = \frac{\Gamma(\upsilon)\cos\left(\frac{\upsilon\pi}{2}\right)}{y^{\upsilon}}\sum_{k=0}^{\infty}\left[\frac{(\lambda)_k}{k!}{}_2F_3\left(\begin{array}{l}\frac{\upsilon}{2},\frac{\upsilon+1}{2};\\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4};\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\right]$$

$$-\frac{(\lambda b)\Gamma(\upsilon+\frac{1}{2})\cos\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{y^{\upsilon+\frac{1}{2}}}\sum_{k=0}^{\infty}\left[\frac{(\lambda+1)_k}{k!}{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+1}{4},\frac{2\upsilon+3}{4};\\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4}\ ;\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\right]$$
$$-\frac{(\lambda b)^2\Gamma(\upsilon+1)\sin\left(\frac{\upsilon\pi}{2}\right)}{2y^{\upsilon+1}}\sum_{k=0}^{\infty}\left[\frac{\{(\lambda+1)_k\}^2}{(\lambda)_k\ k!}{}_2F_3\left(\begin{array}{l}\frac{\upsilon+1}{2},\frac{\upsilon+2}{2};\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2}\ ;\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\right]$$
$$+\frac{(\lambda b)^3\Gamma(\upsilon+\frac{3}{2})\sin\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{6y^{\upsilon+\frac{3}{2}}}\sum_{k=0}^{\infty}\left[\frac{\{(\lambda+1)_k\}^3}{k!\{(\lambda)_k\}^2}{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+3}{4},\frac{2\upsilon+5}{4};\\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4}\ ;\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\right],$$

where $Re(\upsilon), y, \lambda, b > 0$.

Proof. In eq.(2.1), put $\Theta(k) = (\lambda)_k$ and $c = b$, we obtain

$$(4.5)\qquad \mathbf{I}_C(\upsilon,b,\lambda,y) = \int_0^{\infty} x^{\upsilon-1}e^{-(\lambda b)\sqrt{x}}\left\{\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}e^{-(bk)\sqrt{x}}\right\}\cos(xy)dx.$$

Using the binomial expansion (1.18) in (4.5), after simplification we get (4.1). Equations (4.2), (4.3) and (4.4) are obtained from (2.2), (2.3) and (2.5) by putting $\Theta(k) = (\lambda)_k$ and $c = b$. □

5. Ramanujan's Integral $\mathbf{R}_C(m, n)$

The analytical solution of the integral $\mathbf{R}_C(m, n)$ is given by

$$(5.1)\qquad \mathbf{R}_C(m,n) = \int_0^{\infty}\frac{x^m\cos(\pi n x)}{\exp\left(2\pi\sqrt{x}\right)-1}dx,$$

$$(5.2)$$
$$= -(n\pi)^{-m-1}\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left\{\frac{-(2\pi+2\pi k)}{\sqrt{n\pi}}\right\}^{\ell}\Gamma\left(m+1+\frac{\ell}{2}\right)\sin\left(\frac{m\pi}{2}+\frac{\ell\pi}{4}\right)\right],$$

(5.3)

$$= -(n\pi)^{-m-1}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{3}\frac{1}{j!}\left\{\frac{-(2\pi+2\pi k)}{\sqrt{n\pi}}\right\}^{j}\Gamma\left(m+1+\frac{j}{2}\right)\sin\left(\frac{m\pi}{2}+\frac{j\pi}{4}\right)\right.$$
$$\left.\times\,{}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2m+j+2}{2}\right);\\ \Delta^*\left(4;1+j\right);\end{array}\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right],$$

$$= -\frac{m!\,\sin\left(\frac{m\pi}{2}\right)}{(n\pi)^{m+1}}\sum_{k=0}^{\infty}\left[{}_2F_3\left(\begin{array}{l}\frac{m+1}{2},\frac{m+2}{2};\\ \frac{1}{4},\frac{1}{2},\frac{3}{4};\end{array}\ -\frac{\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] \tag{5.4}$$
$$+\frac{\left(\frac{3}{2}\right)_m\sin\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{(\pi)^m(n)^{m+\frac{3}{2}}}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_2F_3\left(\begin{array}{l}\frac{2m+3}{4},\frac{2m+5}{4};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{array}\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]$$
$$-\frac{(2)(m+1)!\cos\left(\frac{m\pi}{2}\right)}{(\pi)^m(n)^{m+2}}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_2F_3\left(\begin{array}{l}\frac{m+2}{2},\frac{m+3}{2};\\ \frac{3}{4},\frac{5}{4},\frac{3}{2};\end{array}\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]$$
$$+\frac{\left(\frac{5}{2}\right)_m\cos\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{(\pi)^{m-1}(n)^{m+\frac{5}{2}}}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_2F_3\left(\begin{array}{l}\frac{2m+5}{4},\frac{2m+7}{4};\\ \frac{5}{4},\frac{3}{2},\frac{7}{4};\end{array}\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right],$$

where m is a non-negative integer and n is positive rational number.

Proof. The results (5.1), (5.2), (5.3) and (5.4) are obtained from (4.1), (4.2), (4.3) and (4.4) by putting $\upsilon = m+1,\ b = 2\pi,\ \lambda = 1$ and $y = n\pi$. □

6 Applications of Ramanujan's Integrals

In this section we establish the following nine new infinite summation formulas associated with hypergeometric series ${}_0F_1$, ${}_1F_2$ and ${}_2F_3$:

$$\sum_{k=0}^{\infty}\left[{}_2F_3\left(\begin{array}{l}1,\frac{3}{2};\\ \frac{1}{4},\frac{1}{2},\frac{3}{4};\end{array}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] \tag{6.1}$$
$$-\frac{3\pi}{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_1F_2\left(\begin{array}{l}\frac{7}{4};\\ \frac{1}{2},\frac{3}{4};\end{array}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]$$
$$+5\pi^2\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_1F_2\left(\begin{array}{l}\frac{9}{4};\\ \frac{5}{4},\frac{3}{2};\end{array}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] = \frac{1}{32}\left(4\pi-13\right),$$

$$(6.2)\quad \sum_{k=0}^{\infty}\left[{}_2F_3\left(\begin{array}{c}1,\frac{3}{2}\ ;\\ \frac{1}{4},\frac{1}{2},\frac{3}{4};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] - \frac{3\pi}{4}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_1F_2\left(\begin{array}{c}\frac{7}{4}\ ;\\ \frac{1}{2},\frac{3}{4};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] + \frac{5\pi^2}{8}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_1F_2\left(\begin{array}{c}\frac{9}{4}\ ;\\ \frac{5}{4},\frac{3}{2};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] = \frac{\pi^2}{16}\left(\frac{3}{\pi}-\frac{1}{2}-\frac{5}{\pi^2}\right),$$

$$(6.3)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_2F_3\left(\begin{array}{c}\frac{7}{4},\frac{9}{4}\ ;\\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] - \frac{16}{5}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_2F_3\left(\begin{array}{c}2,\frac{5}{2}\ ;\\ \frac{3}{4},\frac{5}{4},\frac{3}{2};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] + \frac{7\pi}{6}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_2F_3\left(\begin{array}{c}\frac{9}{4},\frac{11}{4}\ ;\\ \frac{5}{4},\frac{3}{2},\frac{7}{4};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] = \frac{\pi^2}{60}\left(\frac{5}{\pi}-\frac{5}{\pi^2}-1\right),$$

$$(6.4)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_0F_1\left(\begin{array}{c}-\!\!-;\\ \frac{1}{2};\end{array}\ -\frac{\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] - 2\sqrt{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{array}{c}1;\\ \frac{3}{4},\frac{5}{4};\end{array}\ \frac{-\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] + \pi\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{array}{c}-\!\!-;\\ \frac{3}{2}\ ;\end{array}\ \frac{-\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] = \frac{\sqrt{2}-1}{4},$$

$$(6.5)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_0F_1\left(\begin{array}{c}-\!\!-;\\ \frac{1}{2};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] - 2\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{array}{c}1;\\ \frac{3}{4},\frac{5}{4};\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] + \frac{\pi}{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{array}{c}-\!\!-;\\ \frac{3}{2}\ ;\end{array}\ \frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right] = \frac{1}{4},$$

$$(6.6)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}\ {}_0F_1\left(\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{-\pi^2}{64}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
-\sqrt{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{matrix}1;\\ \frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{-\pi^2}{64}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
+\frac{\pi}{4}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{-\pi^2}{64}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{3\sqrt{2}-2}{4},$$

$$(6.7)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}\ {}_0F_1\left(\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{-\pi^2}{144}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
-\frac{2\sqrt{3}}{3}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{matrix}1;\\ \frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{-\pi^2}{144}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
+\frac{\pi}{6}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{-\pi^2}{144}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{13\sqrt{3}-12}{12},$$

$$(6.8)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}\ {}_0F_1\left(\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
-4\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{matrix}1;\\ \frac{3}{4},\frac{5}{4};\end{matrix}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
+2\pi\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ -\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{1}{8\pi},$$

$$(6.9)\quad \sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}\ {}_0F_1\left(\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{-25\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
-2\sqrt{5}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_1F_2\left(\begin{matrix}1;\\ \frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{-25\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]
+\frac{5\pi}{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{-25\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{8\sqrt{5}-15}{100}.$$

The results (6.1) to (6.3) are obtained by putting $m = 1, n = \frac{1}{2}$; $m = 1, n = 2$ and $m = 2, n = 2$ in the equations (5.1) and (5.4) and finally comparing with equations

(1.2), (1.3) and (1.4). When $m = 0$ with $n = 1,\ 2,\ 4,\ 6,\ \frac{1}{2},\ \frac{2}{5}$ in the equations (5.1) and (5.4) and comparing with equations (1.9), (1.10), (1.11), (1.12), (1.13) and (1.14), we get the remaining results (6.4) to (6.9) respectively. In view of the hypergeometric functions (1.34), (1.35) and (1.36), we can express the above results (6.4) to (6.9) in terms of cosine, sine and Lommel functions. Our results (6.1) to (6.9) are convergent in view of the convergence condition of ${}_pF_q(\cdot)$ series, when $p \leq q$, and for all $|z| < \infty$.

7. Conclusion

Here, we have described some infinite Fourier cosine transforms of Ramanujan. Various Ramanujan integrals, which may be different from those of presented here, can also be evaluated in a similar way. The results established above seem significant. We conclude our observation by remarking that various new results and applications can be obtained from our general theorem by appropriate choice of parameters v, λ, b, c, y and bounded sequence $\{\Theta(k)\}_{k=0}^{\infty}$ in $\mathbf{I}_C^*(v, b, c, \lambda, y)$. This work is in continuation to our earlier work [6] on infinite Fourier sine transforms.

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