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The Infinite Hyper Order of Solutions of Differential Equation Related to Brück Conjecture

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Abstract. The Brück conjecture is still open for an entire function f with hyper order of no less than $1/2$, which is not an integer. In this paper, it is proved that the hyper order of solutions of a linear complex differential equation that is related to the Brück Conjecture is infinite. The results show that the conjecture holds in a special case when the hyper order of f is $1/2$.

1. Introduction

It is assumed that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory in the complex plane $\mathbb{C}$ [11, 20]. The order and hyper order of an entire function f are defined as follows:

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log^+\log^+ M(r,f)}{\log r},$$
$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log^+\log^+ T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log^+\log^+\log^+ M(r,f)}{\log r},$$

respectively, where $M(r,f)$ denotes the maximum modulus of f on the circle $|z|=r$.

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If f and g are two meromorphic functions in the complex plane, f and g share a constant a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Rubel and Yang [16] proved for a nonconstant entire function that if f and its derivative f' share two finite distinct values CM, then $f \equiv f'$. Later on, Brück [1] constructed entire functions with integer or infinite hyper order to show that f and f' share 1 CM do not satisfy $f \equiv f'$. Therefore, Brück proposed the following conjecture [1].

Brück Conjecture. *Let f be a nonconstant entire function such that its hyper order is finite and not a positive integer. If f and f' share one finite value a CM, then $f'-a = c(f-a)$, where c is a nonzero constant.*

In general, the conjecture is false for a meromorphic function f (a counterpart is presented in [10]). Brück [1] showed that the conjecture is valid for the case $a = 0$. Afterward, Gundersen and Yang [10] proved that the conjecture is true for the case in which f is of finite order. Furthermore, Chen and Shon [5] showed that the conjecture is also true when f is of hyper order strictly less than $1/2$. Thus, to solve this conjecture, the remaining case in which the hyper order of f is between $[1/2, +\infty)\backslash\mathbb{N}$ should be considered; however, many attempts have failed.

There are many results that are closely related to the Brück conjecture; they can be mainly classified into two types. One type comprises generalizing the shared value a to a nonconstant function, such as polynomial, entire small function of f, or entire functions with lower orders than that of f, (e. g. see [2, 3, 4, 12, 13, 17]); the other type comprises improving the first derivative of f to an arbitrary k-th derivative (e. g. see [2, 6, 7, 13, 18]).

To study this conjecture, $F(z) = \frac{f(z)}{a} - 1$ is chosen. Thus $F(z)$ is an entire function, and $\rho(F) = \rho(f), \rho_2(F) = \rho_2(f)$. Because f and f' share one finite value a CM, according to the Hadamard theorem,

$$\frac{f'-a}{f-a} = e^{h(z)}, \tag{1.1}$$

where $h(z)$ is an entire function. Hence, F satisfies the following linear differential equation:

$$F' - e^{h(z)}F = 1. \tag{1.2}$$

Yang [19] converted the conjecture into a question: Is it true that if $h(z)$ is a nonconstant entire function, the hyper order of F satisfying Equation (1.2) is a positive integer or infinity?

When $h(z)$ is a polynomial or transcendental entire function with order less than $1/2$, the case is true, see[2, 5]. However, the answer for the remaining case with $\rho(h) \geq 1/2$ is unknown.

The following theorem provides a partial answer to the previously presented question for the case $\rho(h) = 1/2$.

Theorem 1.1 *There exists an entire function $h(z)$ with order 1/2. Thus $-e^{h(z)}$ is of hyper order 1/2. Let it be the coefficient of a differential equation*

$$F'(z) - e^{h(z)}F(z) = 1. \tag{1.3}$$

Consequently, the hyper order of the solutions of Equation (1.3) *is infinite.*

2. Preliminary Lemmas

The Koenigs function is considered first. For $\beta \in (0, 1/e)$, the function $E_\beta(z) := e^{\beta z}$ has a repelling fixed point ξ on $\mathbb{R}$ with multiplier $\lambda = \beta\xi > 1$. Around this repelling fixed point, there exists a unique local holomorphic solution Φ of the Schröder's functional equation according to Koenigs:

$$\Phi(E_\beta(z)) = \lambda\Phi(z).$$

It is normalized by $\Phi(\xi) = 0$ and $\Phi'(\xi) = 1$ (e.g., [14]). Moreover, Φ increases on the real axis and approaches infinite, i.e. $\lim_{x\to\infty} \Phi(x) = \infty$; however, it grows more slowly than any iterate of the logarithm. In other words, the following expression can be obtained for all $m \in \mathbb{N}$:

$$\lim_{x\to\infty} \frac{\Phi(x)}{\log^m x} = 0,$$

where $\log^m$ denotes the $m-$th iterate of the logarithm [15]. For the purpose of this study, $\varepsilon : (\xi, \infty) \to (0, +\infty)$ is defined:

$$\varepsilon(x) = \frac{1}{\log \Phi(x)}. \tag{2.1}$$

This function approaches zero more slowly than any of the function $1/\log^m x$, where $m \in \mathbb{N}$. In the next step, the following expression is constructed:

$$\rho(r) = \frac{1}{2} + \varepsilon(r) \tag{2.2}$$

for some $r \geq r_0$. Thus, $\rho(r)$ is a proximate order (see [8, Lemma 3.5]). The definition of proximate order is as follow:

Definition 2.1.(Proximate order) A function $\rho(r)$ defined on $[r_0, \infty)$, where $r_0 > 0$, is called a *proximate order* if it satisfies the following conditions:

(1) $\rho(r) \geq 0$;

(2) $\lim_{r\to\infty} \rho(r) = \rho$;

(3) $\rho(r)$ is continuously differentiable on $[r_0, \infty)$;

(4) $\lim_{r\to\infty} r\rho'(r) \log r = 0$.

In [8], the author constructed an entire function by using the Weierstrass canonical product:

$$h(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right), \tag{2.3}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is a positive sequence tending to infinity such that $1 \leq a_0 \leq a_1 \leq \cdots$ and $n(r) = r^{\rho(r)} + O(\varepsilon(r)^3 r^{\rho(r)})$. Here $n(r)$ is the number of the element of the sequence $\{a_k\}$, which are contained in the disc $\{z : |z| < r\}$; $\rho(r)$ is defined in (2.2). According to the result of Borel (see [9, Theorem 3.4]), the order of the Weierstrass canonical product $h(z)$ is equal to the order of $n(r)$; thus the order of $h(z)$ is $1/2$.

Lemma 2.2.([8, Lemma 3.6]) *Based on the previously defined $h(z), \varepsilon(r)$ and $\rho(r)$, the following expression can be constructed:*

$$\log\left|h(re^{i\theta})\right| = \frac{\pi\cos((\theta-\pi)\rho(r))}{\sin(\pi\rho(r))} r^{\rho(r)} + O(\varepsilon(r)^2 r^{\rho(r)}) \tag{2.4}$$

for $\varepsilon(r) \leq \theta \leq 2\pi - \varepsilon(r)$ as $r \to \infty$.

The following expressions are true for some $r_0 > 0$:

$$\gamma^+ = \{re^{i\varepsilon(r)} : r \geq r_0\},$$
$$\gamma^- = \{re^{-i\varepsilon(r)} : r \geq r_0\},$$

and

$$G(\gamma) = \mathbb{C} \setminus \{re^{i\theta} \in \mathbb{C} : \varepsilon(r) \leq \theta \leq 2\pi - \varepsilon(r), r \geq r_0\}.$$

The following fact is a consequence of Lemma 2.2.

Lemma 2.3.([8, Lemma 3.7, 3.8]) *The function $h(z)$ is bounded on γ^+ and γ^- and in $G(\gamma)$.*

Lemma 2.4.([6]) *Let $f(z)$ be an entire function of infinite order; the hyper order is $\rho_2(f)$; note the central index of f by $\nu(r)$. Consequently, the following expression holds true:*

$$\limsup_{r\to\infty} \frac{\log\log\nu(r)}{\log r} = \rho_2(f). \tag{2.5}$$

Lemma 2.5.([6]) *Let $f(z)$ be an entire function of infinite order $\rho(f) = \infty$ and $\rho_2(f) = \alpha < +\infty$; moreover, the set $E \subset (1,\infty)$ has a finite logarithmic measure. Hence, there exists a sequence $\{z_k = r_k e^{i\theta_k}\}$ such that*

$$|f(z_k)| = M(r_k, f), \quad \lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi), \quad \theta_k \in [0, 2\pi), \quad r_k \notin E, \quad r_k \to \infty.$$

If $\alpha > 0$, for any given $\varepsilon (0 < \varepsilon < \alpha)$, there is a sufficiently large r_k such that

$$\exp\{r_k^{\alpha-\varepsilon}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon}\};$$

If $\alpha = 0$, for any large $N > 0$, there is a sufficiently large r_k such that

$$\nu(r_k) > r_k^N.$$

3. Proof of Theorem 1.1

Evidently, the solutions of Equation (1.3) are of infinite order. The hyper order is assumed to be $\rho_2(F) = \alpha < +\infty$, and the assertion is obtained through the reduction to a contradiction. As in Lemma 2.5, a sequence $\{z_k = r_k e^{i\theta_k}\}$ can be chosen such that

$$|F(z_k)| = M(r_k, F), \quad \lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi), \quad \theta_k \in [0, 2\pi), \quad r_k \notin E, \quad r_k \to \infty.$$

According to Equation (1.3), the following expression holds true:

$$e^{h(z)} = \frac{F'}{F} - \frac{1}{F}. \tag{3.1}$$

By applying Wiman-Valiron Theorem to Equation (3.1), the following expressions can be obtained:

$$e^{h(z)} = \frac{\nu(r)}{z}(1 + o(1)) + o(1) \tag{3.2}$$

and

$$\frac{\nu(r)}{|z|}(1 + o(1)) + o(1) = |e^{h(z)}| = e^{\Re h(z)} \le e^{|h(z)|}. \tag{3.3}$$

If $\theta_0 = 0$, there exists a subsequence $\{z_{k_j} = r_{k_j} e^{i\theta_{k_j}}\}$ of $\{z_k = r_k e^{i\theta_k}\}$ such that $z_{k_j} \in G(\gamma)$. Because F is of infinite order, $\nu(r_{j_k}) > |z_{j_k}|^N$ for any large $N > 0$. In addition, according to Lemma 2.3, $|h(z_{j_k})|$ is bounded. Thus, (3.3) presents a contradiction.

If $\theta_0 \in (0, 2\pi)$, there exists a subsequence $\{z_{k_l} = r_{k_l} e^{i\theta_{k_l}}\}$ of $\{z_k = r_k e^{i\theta_k}\}$ such that $z_{k_l} \in \mathbb{C} \setminus G(\gamma)$. Based on Lemma 2.4, the following expressions can be obtained:

$$\log\left|h(r_{k_l} e^{i\theta_{k_l}})\right| = \frac{\pi \cos((\theta_{k_l} - \pi)\rho(r_{k_l}))}{\sin(\pi\rho(r_{k_l}))} r_{k_l}^{\rho(r_{k_l})} + O(\varepsilon(r_{k_l})^2 r_{k_l}^{\rho(r_{k_l})}) \tag{3.4}$$

and

$$\lim_{l\to+\infty} \frac{\pi \cos((\theta_{k_l} - \pi)\rho(r_{k_l}))}{\sin(\pi\rho(r_{k_l}))} > 0. \tag{3.5}$$

Taking the principal branch of the logarithm of Equation (3.2) on both sides leads to the following term:

$$h(z) = \log\left(\frac{\nu(r)}{z}(1 + o(1)) + o(1)\right). \tag{3.6}$$

Thus, because $\nu(r) > |z|^N$ for any large $N > 0$,

$$|h(z)| \leq |\log|\frac{\nu(r)}{z}(1 + o(1)) + o(1)|| + 2\pi \leq \log\nu(r) + O(1). \tag{3.7}$$

Lemma 2.4 and $\rho_2(F) = \alpha$ lead to the following equation:

$$\frac{\log\log\nu(r)}{\log r} \leq \alpha + 1 \tag{3.8}$$

for sufficiently large r. This results in the following expression:

$$|h(z)| \leq r^{\alpha+1} + O(1) \tag{3.9}$$

for sufficiently large r. The combination of (3.4), (3.5) and (3.9) leads to a contradiction for $l \to +\infty$. Thus, the proof is completed.

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