KYUNGPOOK Math. J. 60(2020), 805-819
https://doi.org/10.5666/KMJ.2020.60.4.805
pISSN 1225-6951 eISSN 0454-8124

η-Einstein Solitons on (ε)-Kenmotsu Manifolds

Mohd Danish Siddiqi*
Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia
e-mail : msiddiqi@jazanu.edu.sa

Sudhakar Kumar Chaubey
Section of Mathematics, Department of Information Technology, Shinas College of Technology, Shinas, P. O. Box 77, Postal Code 324, Oman
e-mail : sudhakar.chaubey@shct.edu.om

Abstract. The objective of this study is to investigate η-Einstein solitons on (ε)-Kenmotsu manifolds when the Weyl-conformal curvature tensor satisfies some geometric properties such as being flat, semi-symmetric and Einstein semi-symmetric. Here, we discuss the properties of η-Einstein solitons on φ-symmetric (ε)-Kenmotsu manifolds.

1. Introduction

Hamilton proposed the use of an evolution expression, called Ricci flow, to prove Thurston's geometrization conjecture in three dimension. In 1982, he [11] popularized concept of Ricci solitons on Riemannian manifold and proved that the solitons moves under the Ricci flow simply by diffeomorphisms of the initial metric. This indicates that Ricci solitons are stationary points of the Ricci flow, which is given by

$$\frac{\partial g}{\partial t} = -2Ric(g), \tag{1.1}$$

where g is the Riemannian metric, Ric is the Ricci tensor and t is time.

Definition 1.1. A *Ricci soliton* (g, V, λ) on a Riemannian manifold M of dimension n is defined by

$$\mathcal{L}_V g + 2Ric + 2\lambda = 0, \tag{1.2}$$

* Corresponding Author.
Received October 9, 2019; revised June 26, 2020; accepted June 29, 2020.
2020 Mathematics Subject Classification: 53A31, 53C18, 53E20, 53E40.
Key words and phrases: (ε)-Kenmotsu manifold, η-Einstein solitons, Weyl-conformal curvature, Einstein manifold.

where $\mathcal{L}_V$ denotes the Lie derivative along the vector field V on M and λ is a real scalar.

A Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$, or $\lambda > 0$, respectively. In particular, if $\lambda < 0$, then the soliton is called shrinking and it generates an ancient self-similar solution to the Ricci flow with finite extinction time [7]. If the vector field V is the gradient of a potential function $-\psi$, where ψ is some smooth function $\psi : M \to \mathbb{R}$, then g is called a gradient Ricci soliton and equation (1.2) assumes the form

$$Hess\psi + Ric + \lambda g = 0. \tag{1.3}$$

Here $\mathbb{R}$ represents a the set of real numbers and $Hess$ is the Hessian of the potential function ψ. When ψ is constant, then the gradient Ricci soliton is simply an *Einstein manifold*. Thus Ricci solitons are essentially a generalization of Einstein metrics. An Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons are an important part of Hamiltonian Ricci flow as they correspond to self-similar solutions, and they often arise as singularity models. They are also linked to smooth metric measure spaces, since equation (1.3) is equivalent to $Ric\,\psi = 0$, where Ric is the ∞-Bakry-Emery Ricci tensor. In physics, a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\,\psi = \lambda$g is called a quasi-Einstein manifold. Therefore it is crucial to investigate the geometry and topology of Ricci and others solitons and their classifications.

Over the last couple of decades, many studies have analyzed self-similar solutions of geometric flows. In 2016, Catino and Mazzieri developed the conception of Einstein solitons [7], which generate self-similar solutions to the Einstein flow, which is given by

$$\frac{\partial g}{\partial t} = -2\left(Ric - \frac{\sigma}{2} g\right), \tag{1.4}$$

where σ is the scalar curvature of the Riemannian metric g. Interest in examining this equation from different perspectives originated from modern physical problems. In what follows, after describing the manifold of constant scalar curvature by the continuation of η-Einstein solitons. If an η-Einstein soliton exists, it indicate that the manifold is quasi-Einstein. It is known that the concept quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations in general theory of relativity (GR) and also in modern particle physics (astrophysics, plasma physics, nuclear physics etc).

On the other hand in 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying certain conditions, the manifolds are refered to as Kenmotsu manifolds [13]. Since then, Kenmotsu manifolds have been studied by many geometers. References [8, 10, 12, 15] and the references therein are instances of such studies. In 1993, Bejancu and Duggal [2] introduced the concept of (ε)-Sasakian manifold, and subsequently, Xufeng and Xiaoli [19] showed that these manifolds are real hypersurface of indefinite Kaehlerian manifolds. Although the theory of

(ε)-Kenmotsu manifolds was popularized by De and Sarkar [10],and they have established the perseverance of almost contact structure with indefinite metrics.

Sharma [16] studied the axioms of the Ricci solitons on a contact Riemannian manifold, and later, Nagaraja et al. [15] and researchers such as Bagewadi et al. [1] extensively discussed Ricci solitons on Kenmotsu manifolds. In 2009, Cho and Kimura [9] introduced the notion of η-Ricci solitons on real hypersurfaces of non-flat complex space-forms. In addition, η-Ricci solitons on manifolds with different structures [3, 5] have been studied by Blaga extensively. Furthermore, in 2018, Blaga studied the notion of η-Einstein solitons [4]. Recently, Siddiqi likewise considered some characteristics of η-Einstein solitons in [17, 18] which is closely related to this subject. Motivated by the previous research, in the present study, we investigated the geometric nature of η-Einstein soliton on an (ε)-Kenmotsu manifold.

2. Preliminaries

An $n(=2m+1)$-dimensional smooth manifold $(\mathcal{M}, g)$ is said to be an (ε)-almost contact metric manifold [6], if it admits a $(1,1)$-tensor field ϕ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

$$\varphi^2 E = -E + \eta(E)\xi, \ \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi,\xi) = \varepsilon, \ \ \eta(E) = \varepsilon g(E,\xi), \ \ g(\varphi E, \phi F) = g(E,F) - \varepsilon\eta(E)\eta(F) \tag{2.2}$$

for all vector fields E, F on $\chi(\mathcal{M})$, where ε is 1 or -1 according as ξ is a spacelike or timelike vector field and $rank\ \varphi = (n-1)$. Here $\chi(M)$ denotes the a set of all smooth vector fields of $\mathcal{M}$. If

$$d\eta(E,F) = g(E,\varphi F) \tag{2.3}$$

for every $E, F \in \chi(\mathcal{M})$, then we say that $\mathcal{M}(\varphi,\xi,\eta,g,\varepsilon)$ is an (ε)-contact metric manifold, where d is an exterior derivative. We also have

$$\varphi\xi = 0, \ \eta(\varphi E) = 0. \tag{2.4}$$

If an (ε)-contact metric manifold satisfies

$$(\nabla_E\varphi)(Y) = -g(E,\varphi F) - \varepsilon\eta(F)\varphi E, \tag{2.5}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an (ε)-Kenmotsu manifold [10]. An (ε)-almost contact metric manifold is an (ε)-Kenmotsu manifold if and only if

$$\nabla_E\xi = \varepsilon[E - \eta(E)\xi], \ \forall \in \chi(\mathcal{M}). \tag{2.6}$$

Moreover, the curvature tensor R, the Ricci tensor Ric and the Ricci operator Q on an (ε)-Kenmotsu manifold $\mathcal{M}$ with respect to the Levi-Civita connection satisfy the following relations:

$$(\nabla_X\eta)(F) = g(E,F) - \varepsilon\eta(E)\eta(F), \qquad (\nabla_\xi\eta)(F) = 0, \tag{2.7}$$

$$R(E,F)\xi = \eta(E)F - \eta(F)E, \tag{2.8}$$

$$R(\xi,E)F = \eta(F)E - \varepsilon g(E,F)\xi, \tag{2.9}$$

$$R(\xi,E)\xi = -R(E,\xi)\xi = E - \eta(E)\xi, \tag{2.10}$$

$$\eta(R(E,F)G) = \varepsilon[g(E,G)\eta(F) - g(F,G)\eta(E)], \tag{2.11}$$

$$Ric(E,\xi) = -(n-1)\eta(E), \tag{2.12}$$

$$Q\xi = -\varepsilon(n-1)\xi, \tag{2.13}$$

$$Ric(\varphi E,\varphi F) = Ric(E,F) + \varepsilon(n-1)\eta(E)\eta(F), \tag{2.14}$$

where $g(QE,F) = Ric(E,F)$ [10]. We note that if $\varepsilon = 1$ then an (ε)-Kenmotsu manifold becomes the well-known Kenmotsu manifold [13].

An (ε)-Kenmotsu manifold $\mathcal{M}$ is said to be an *η-Einstein manifold* if its Ricci tensor Ric is of the form

$$Ric(E,F) = Ag(E,F) + B\eta(E)\eta(F), \tag{2.15}$$

where A and B are scalar functions.

Example 2.1. We consider a three dimensional manifold $\mathcal{M} = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$. We choose the vector fields

$$v_1 = z\frac{\partial}{\partial x}, \quad v_2 = z\frac{\partial}{\partial y}, \quad v_3 = -z\frac{\partial}{\partial z},$$

which are linearly independent at each point of $\mathcal{M}$. Let g be the Riemannian metric defined by

$$g(v_1,v_2) = g(v_2,v_3) = g(v_3,v_1) = 0, \qquad g(v_1,v_1) = g(v_2,v_2) = g(v_3,v_3) = \varepsilon,$$

where $\varepsilon = \pm 1$. Let η be a 1-form defined by $\eta(G) = \varepsilon g(G,v_3)$ for any vector field G on $\mathcal{M}$. Let ϕ be a $(1,1)$-tensor field defined by $\phi(v_1) = -v_2$, $\phi(v_2) = v_1$, $\phi(v_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 G = -G + \eta(G)v_3, \qquad \eta(v_3) = 1 \quad \text{and} \quad g(\phi G,\phi H) = g(G,H) - \varepsilon\eta(G)\eta(H)$$

for any vector fields G, H on $\mathcal{M}$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[v_1,v_2] = 0, \qquad [v_1,v_3] = v_1, \qquad [v_2,v_3] = v_2.$$

The use of Koszul's formula

$$2g(\nabla_E F,G) = Eg(F,G) + Fg(G,E) - Gg(E,F) \\ +g([E,F],G) - g([F,G],E) + g([G,E],F)$$

gives

$$\begin{aligned} &\nabla_{v_1} v_3 = \varepsilon v_1, && \nabla_{v_2} v_3 = \varepsilon v_2, && \nabla_{v_3} v_3 = 0, \\ &\nabla_{v_1} v_2 = 0, && \nabla_{v_2} v_2 = -\varepsilon v_3, && \nabla_{v_3} v_2 = 0, \\ &\nabla_{v_1} v_1 = -\varepsilon v_3, && \nabla_{v_2} v_1 = 0, && \nabla_{v_3} v_1 = 0. \end{aligned}$$

Using the above relations, for any vector field E on $\mathcal{M}$, we have

$$\nabla_E \xi = \varepsilon[E - \eta(E)\xi]$$

for $\xi = v_3$. Hence the manifold $\mathcal{M}$ under consideration is an (ε)-Kenmotsu manifold of dimension three.

3. η-Einstein Solitons on $(\mathcal{M}, \phi, \xi, \eta, g, \varepsilon)$

Let $(\mathcal{M}, \phi, \xi, \eta, g, \varepsilon)$ be an n-dimensional (ε)-almost contact metric manifold. Consider the equation

$$\mathcal{L}_\xi g + 2Ric + (2\lambda - \sigma)g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, and λ, and μ are real constants. For $\mu \neq 0$, the data $(g, \xi, \lambda - \frac{\sigma}{2}, \mu)$ will be called η-Einstein soliton on $\mathcal{M}$ if it satisfies equation (3.1). We remark that if the scalar curvature σ of the manifold $\mathcal{M}$ is constant, then the η-Einstein soliton $(g, \xi, \lambda - \frac{\sigma}{2}, \mu)$ reduces to an η-Ricci soliton. Moreover, if $\mu = 0$, then the η-Einstein soliton $(g, \xi, \lambda - \frac{\sigma}{2}, \mu)$ becomes a Ricci soliton $(g, \xi, \lambda - \frac{\sigma}{2})$. Therefore, the central idea of η-Einstein soliton [11] and η-Ricci soliton are different on manifolds of non constant scalar curvature.

Replacing the value of $\mathcal{L}_\xi g$ in (3.1), we obtain

$$2Ric(E, F) = -g(\nabla_E \xi, F) - g(E, \nabla_F \xi) - (2\lambda - \sigma)g(E, F) - 2\mu\eta(E)\eta(F) \tag{3.2}$$

for any $X, Y \in \chi(M)$. To prove the existence of η-Einstein soliton on an (ε)-Kenmotsu manifold, we provide the following non-trivial example:

Example 3.1. We consider a three dimensional manifold $\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. We choosing the vector fields

$$v_1 = e^{-z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad v_2 = e^{-z}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right), \quad v_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of $\mathcal{M}$ Let g be the Riemannian metric define by

$$g(v_1, v_3) = g(v_2, v_3) = g(v_2, v_2) = 0, \quad g(v_1, v_1) = g(v_2, v_2) = g(v_3, v_3) = \varepsilon,$$

where $\varepsilon = \pm 1$. The metric can be expressed un the following form

$$g = \varepsilon\left\{\frac{1}{2e^{-z}}(dx \otimes dx + dy \otimes dy) + dz \otimes dz\right\}.$$

Let η be the 1-form defined by $\eta(G) = \varepsilon g(G, v_3)$ for any vector field G on $\mathcal{M}$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(v_1) = v_2, \quad \phi(v_2) = -v_1, \quad and \quad \phi(v_3) = 0$. Then, by the linearity property of ϕ and g, we have

$$\phi^2 G = -G + \eta(G)v_3, \qquad \eta(v_3) = \varepsilon \quad \text{and} \quad g(\phi G, \phi H) = g(G, H) - \varepsilon\eta(G)\eta(H)$$

for any vector fields G, H on $\mathcal{M}$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[v_1, v_2] = 0, \qquad [v_1, v_3] = v_1, \qquad [v_2, v_3] = v_2.$$

Using of Koszul's formula, we have

$$\begin{array}{lll} \nabla_{v_1} v_3 = \varepsilon v_1, & \nabla_{v_2} v_3 = \varepsilon v_2, & \nabla_{e_3} e_3 = 0, \\ \nabla_{v_1} v_2 = 0, & \nabla_{v_2} v_2 = -\varepsilon v_3, & \nabla_{v_3} v_2 = 0, \\ \nabla_{v_1} v_1 = -\varepsilon v_3, & \nabla_{v_2} v_1 = 0, & \nabla_{v_3} v_1 = 0. \end{array}$$

Using the above relations, for any vector field E on $\mathcal{M}$, we have

$$\nabla_E \xi = \varepsilon(E - \eta(E)\xi)$$

for $\xi = v_3$. Hence the manifold $\mathcal{M}$ under consideration is an ε-Kenmotsu manifold of dimension three. The non-vanishing components of the curvature tensor and the Ricci tensor can be computed as:

$$R(v_1, v_2)v_2 = -v_1, \qquad R(v_1, v_3)v_3 = -v_1, \qquad R(v_2, v_1)v_1 = -v_2,$$

$$R(v_2, v_3)v_3 = -v_2, \qquad R(v_3, v_1)v_1 = -v_3, \qquad R(v_3, v_2)v_2 = -v_3,$$

and

$$Ric(v_1, v_1) = Ric(v_2, v_2) = Ric(v_3, v_3,) = -2\varepsilon.$$

Thus the scalar curvature *scal* is given by

$$\sigma = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3,) = -6\varepsilon,$$

which is constant. From the above discussions, it is apparent that equation (3.1) is satisfied for $\lambda = 2 - 4\varepsilon$ and $\mu = 1$. Thus the data $(g, \xi, \lambda - \frac{\sigma}{2}, \mu)$ admits an η-Einstein soliton on $(\mathcal{M}^3, \phi, \xi, \eta, g, \varepsilon)$, which is shrink and expand, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively.

4. Second Order Parallel Symmetric Tensor and η-Einstein Solitons on (ε)-Kenmotsu Manifold

A well known geometrical tool used for studying Einstein solitons is a symmetric $(0,2)$-tensor field, which is parallel with respect to the Levi-Civita connection. Now, let us fix h as a symmetric parallel tensor field of $(0,2)$-type, that is, $\nabla h = 0$. Applying the well known Ricci identity [14]

$$\nabla^2 h(E,F;G,H) - \nabla^2 h(E,F;G,W) = 0,$$

we obtain the relation

$$h(R(E,F)G,H) + h(G,R(E,F)H) = 0. \tag{4.1}$$

Replacing $G = H = \xi$ in equation (4.1) and by using equation (**??**) and symmetry properties of h, we obtain $h(R(E,F)\xi,\xi) = 0$ for any $E,F \in \chi(\mathcal{M})$. Thus we have

$$\eta(E)h(\xi,F) - \eta(F)h(\xi,E) = 0. \tag{4.2}$$

Substituting $E = \xi$ in (4.2) and by the virtue of (2.1), we obtain

$$h(F,\xi) - \eta(F)h(\xi,\xi) = 0,$$

which is equivalent to

$$h(F,\xi) - \varepsilon g(F,\xi)h(\xi,\xi) = 0. \tag{4.3}$$

Differentiating this equation covariantly with respect to the Levi-Civita connection ∇ along the vector field $X \in \chi(\mathcal{M})$, we obtain

$$h(\nabla_E F,\xi) + h(F,\nabla_E \xi) = \varepsilon h(\xi,\xi)[g(\nabla_E F,\xi) + g(F,\nabla_E \xi)] + 2\varepsilon g(F,\xi)h(\nabla_E \xi,\xi). \tag{4.4}$$

Using equations (2.1), (2.2), (2.6), (2.7) and (4.3) in this equation, we find that

$$h(E,F) = \varepsilon h(\xi,\xi)g(E,F) \tag{4.5}$$

for any $E,F \in \chi(\mathcal{M})$. The covariant derivative of equation (4.5) with respect to the Levi-Civita connection ∇ along any arbitrary vector field $G \in \chi(\mathcal{M})$ indicates, with the help of the equations (2.1), (2.2) and (2.6), that $h(\xi,\xi)$ is constant. Equation (4.5) reveals that h is a constant multiple of the metric g. Thus we arrive at the following conclusion.

Theorem 4.1. *If an n-dimensional (ε)-Kenmotsu manifold $(\mathcal{M},\phi,\xi,\eta,g,\varepsilon)$ admits a second order parallel symmetric tensor h, then it is a constant multiple of the metric g.*

Using equation (2.6) and $\mathcal{L}_\xi g = 2\varepsilon(g - \varepsilon\eta \otimes \eta)$ in equation (3.2), we obtain

$$Ric(E,F) = -(\lambda + \varepsilon - \frac{\sigma}{2})g(E,F) - (\mu - 1)\eta(E)\eta(F). \tag{4.6}$$

In particular, for $E = \xi$ we obtain

$$Ric(E,\xi) = -(\lambda\varepsilon + \mu - \varepsilon\frac{\sigma}{2})\eta(E). \tag{4.7}$$

In this case, the Ricci operator Q defined by $g(QE,F) = S(E,F)$ given by

$$QE = -(\lambda + \varepsilon - \frac{\sigma}{2})E - \varepsilon(\mu - 1)\eta(E)\xi. \tag{4.8}$$

We remark that on an (ε)-Kenmotsu manifold, the existence of an η-Einstein soliton implies that the characteristic vector field ξ is an eigen vector of Ricci operator corresponding to the eigenvalue $-(\lambda + \mu\varepsilon - \frac{\sigma}{2})$. Now, we apply the previous results on η-Einstein soliton.

Theorem 4.2. *Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be an (ε)-Kenmotsu manifold and let us assume that the symmetric $(0,2)$-tensor field $h = \mathcal{L}_\xi g + 2Ric + 2\mu\eta \otimes \eta$ is parallel associated to g. Then $(g, \xi, -\frac{1}{2}[\varepsilon\, h(\xi,\xi) - \sigma], \mu)$ yields an η-Einstein soliton.*

Proof. Now, we can calculate

$$h(\xi,\xi) = \mathcal{L}_\xi g(\xi,\xi) + 2Ric(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -\varepsilon(2\lambda - \sigma), \tag{4.9}$$

From this equation, we have $\lambda = -\frac{1}{2}[\varepsilon\, h(\xi,\xi) - \sigma]$. Equations (2.1), (2.1) and (4.7) have been used for obtaining equation (4.9). From equation (4.5) we can conclude that

$$h(E,F) = -(2\lambda - \sigma)g(E,F)$$

for any $E, F \in \chi(\mathcal{M})$. Therefore

$$\mathcal{L}_\xi g + 2Ric + 2\mu\eta \otimes \eta = -(2\lambda - \sigma)g. \tag{4.10}$$

This gives the statement of Theorem 4.2. □

For $\mu = 0$, from equations (4.9) and (4.10), we can obtain

$$\mathcal{L}_\xi g + 2Ric - \varepsilon\, h(\xi,\xi)g = 0. \tag{4.11}$$

Hence we can state the following corollary.

Corollary 4.3. *On an (ε)-Kenmotsu manifold $(\mathcal{M}, \phi, \xi, \eta, g, \varepsilon)$ with the property that the symmetric $(0,2)$-tensor field $h = \mathcal{L}_\xi g + 2Ric$ is parallel to the connection ∇ associated with g, the relation (4.9), defines a Ricci soliton for $\mu = 0$.*

Conversely, we shall discuss the consequences of the existence of η-Einstein solitons on an (ε)-Kenmotsu manifold. From (4.10) we obtain the following conclusion:

Theorem 4.4. *If equation (4.10) define an η-Einstein soliton on an (ε)-Kenmotsu manifold $(\mathcal{M}, \phi, \xi, \eta, g, \varepsilon)$, then $(\mathcal{M}, g)$ is quasi-Einstein.*

As mentioned, a manifold is called *quasi-Einstein* if the Ricci tensor S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η satisfying $\eta(E) = \varepsilon g(E,\xi)$, where ξ is a unit vector field. The manifold is called *Einstein* if Ric is collinear with g.

5. Weyl Semi-symmetric (ε)-Kenmotsu Manifold

This section describes a study of Weyl semi-symmetric (ε)-Kenmotsu manifolds admitting η-Einstein solitons. For an n-dimensional (ε)-almost contact metric manifold, the Weyl conformal curvature tensor C is given by:

$$C(E,F)G = R(E,F)G - \frac{1}{(n-2)}[Ric(F,G)E - Ric(E,G)F + g(F,G)QE$$
$$-g(F,G)QF] + \frac{\sigma}{(n-1)(n-2)}[g(F,G)E - g(E,G)F]. \tag{5.1}$$

From this equation (5.1), we can easily find that

$$C(\xi,E)F = -\frac{1}{(n-1)(n-2)}[\sigma\,\varepsilon + (n-1)][\eta(F)E - g(E,F)\xi]$$
$$-\frac{1}{n-2}\{Ric(E,F)\xi - \varepsilon\eta(F)QE\}, \tag{5.2}$$

and

$$C(\xi,E)\xi = -\frac{1}{(n-1)(n-2)}[\sigma\,\varepsilon + (n-1)][E - \varepsilon\eta(E)\xi]$$
$$+\frac{1}{n-2}\{QE + \varepsilon(n-1)\eta(F)\xi\}, \tag{5.3}$$

where equations (2.1), (2.2), (2.9) and (2.12) have been used. Before proceeding to prove our results, we provide the following definition.

Definition 5.1. An n-dimensional (ε)-Kenmotsu manifold $\mathcal{M}$ is said to be a *Weyl semi-symmetric* if $R(E,F)\cdot C = 0$, $\forall$ E, F $\in \chi(\mathcal{M})$.

Let us assume that the manifold $\mathcal{M}$ is Weyl semi-symmetric (ε)-Kenmotsu manifold. Then from Definition (5.1), we have

$$R(E,F)C(U,V)W - C(R(E,F)U,V)W$$
$$- C(U,R(E,F)V)W - C(U,V)R(E,F)W = 0. \tag{5.4}$$

Substituting $E = U = \xi$ in this equation and using (2.9), we obtain

$$\eta(C(\xi,V)W)F - \varepsilon g(F,C(\xi,V)W)\xi - C(F,V)W$$
$$+\eta(F)C(\xi,V)W - \eta(V)C(\xi,F)W + \varepsilon g(F,V)C(\xi,\xi)W$$
$$-\eta(W)C(\xi,V)F + \varepsilon g(F,W)C(\xi,V)\xi = 0. \tag{5.5}$$

Using equations (5.1), (5.2) and equation (5.3) in (5.5), we have

$$R(F,V)W = \varepsilon[g(F,W)V - g(V,W)F]. \tag{5.6}$$

Taking the inner product of (5.6) with Z, we have

$$g(R(F,V)W,G) = \varepsilon[g(F,W)g(V,G) - g(V,W)g(F,G)]. \tag{5.7}$$

Taking $V = W = e_i$, where $\{e_i,\ i = 1,2,...,n\}$ be a set of orthonormal vector fields of the tangent space of $\mathcal{M}$, and summing over i, $i = 1,2,...,n$, in equation (5.7), we obtain

$$Ric(F,G) = -\varepsilon(n-1)g(F,G) \implies \sigma = -\varepsilon n(n-1). \tag{5.8}$$

This shows that the manifold under consideration has a space of constant curvature and that it is therefore an Einstein manifold. Conversely we suppose that the (ε)-Kenmotsu manifold of dimension n satisfies equations (5.7) and (5.8). It is well known that

$$\begin{aligned}(R(E,F)\cdot C)(U,V)W &= R(E,F)C(U,V)W - C(R(E,F)U,V)W \\ &\quad - C(U,R(E,F)V)W - C(U,V)R(E,F)W.\end{aligned} \tag{5.9}$$

From equations (5.7)-(5.9), we obtain

$$R\cdot C = 0. \tag{5.10}$$

This shows that an n-dimensional (ε)-Kenmotsu manifold $\mathcal{M}$ is Weyl semi-symmetric. Hence we can state the following:

Theorem 5.2. *An n-dimensional (ε)-Kenmotsu manifold is Weyl semi-symmetric if and only if it is a space of constant curvature.*

Suppose that the Weyl semi-symmetric (ε)-Kenmotsu manifold M admits an η-Einstein soliton $(g,\xi,\lambda-\frac{\sigma}{2},\mu)$. Then from equations (3.2) and (5.8), we have

$$-2\varepsilon(n-1)g(E,F) = -g(\nabla_E\xi,F) - g(E,\nabla_F\xi) - (2\lambda-\sigma)g(E,F) - 2\mu\eta(E)\eta(F).$$

Replacing E with ξ in the above equation and using equations (2.1), (2.2), (2.6) and (2.7), we obtain

$$\{\varepsilon(n-1) - \lambda + \frac{\sigma}{2} - \mu\varepsilon\}\eta(F) = 0.$$

Since $\eta(F) \neq 0$ (in general) on an ε-contact metric manifold, the above equation gives

$$\lambda = -\frac{\varepsilon}{2}[2\mu + (n-1)(n-2)].$$

On the basis of these facts, we can state the following theorem.

Theorem 5.3. *Let $\mathcal{M}$ be an n-dimensional Weyl semi-symmetric (ε)-Kenmotsu manifold with $n \geq 3$. Then the η-Einstein soliton $(g,\xi,\lambda-\frac{\sigma}{2},\mu)$ on $\mathcal{M}$ is said to be shrink, expand, and remain steady if $2\mu\varepsilon + \varepsilon(n-1)(n-2)$ is $>$, $<$, and $=0$,* *respectively.*

6. Einstein Semi-symmetric (ε)-Kenmotsu Manifold

This section concerns with the study of η-Einstein soliton on Einstein semi-symmetric (ε)-Kenmotsu manifold. To prove our results, we recall the following definition.

Definition 6.1. An n-dimensional (ε)-Kenmotsu manifold M is called *Einstein semi-symmetric* if $R \cdot Ein = 0$, where Ein is the Einstein tensor defined by

$$Ein(E,F) = Ric(X,Y) - \frac{\sigma}{n} g(E,F), \tag{6.1}$$

where Ric is the Ricci tensor and σ is the scalar curvature.

Let M be an n-dimensional Einstein semi-symmetric (ε)-Kenmotsu manifold. Then from the Definition (6.1), we have

$$(R(E,F) \cdot Ein)(G,H) = 0. \tag{6.2}$$

This equation can be written as

$$Ein(R(E,F)G,H) + Ein(G,R(E,F)H) = 0. \tag{6.3}$$

Now, using equation (6.1) in equation (6.3), we obatin

$$\begin{aligned} &Ric(R(E,F)G,H) + Ric(G,R(E,F)H) \\ &-\frac{\sigma}{n}[g(R(E,F)G,H) + g(G,R(E,F)H)] = 0, \end{aligned} \tag{6.4}$$

which is equivalent to the Ricci semi-symmetric (ε)-Kenmotsu manifold. Replacing E and H with ξ in equation (6.4) and then using equations (2.1)-(2.10) and (2.12), we obtain

$$Ric(E,G) = (n-1)\varepsilon g(E,G). \tag{6.5}$$

This equation shows that the manifold under consideration is an Einstein manifold. The converse is obvious. Hence we state the following.

Theorem 6.2. *An n-dimensional (ε)-Kenmotsu manifold $\mathcal{M}$ is Einstein semi-symmetric if and only if it is an Einstein manifold.*

Now, let us consider an n-dimensional (ε)-Kenmotsu manifold $\mathcal{M}$ admits an η-Einstein soliton $(g, V, \lambda - \frac{\sigma}{2}, \mu)$ on $\mathcal{M}$. If V is a conformal Killing vector field on M, then by definition we have

$$(\mathcal{L}_V g)(E,F) = \rho g(E,F) \tag{6.6}$$

for some scalar function ρ. From equation (3.1), we have

$$Ric = -\left(\lambda - \frac{\sigma}{2}\right) g - \frac{1}{2}\mathcal{L}_V g - \mu\eta\otimes\eta.$$

From equation (6.6) and the above equation, we have

$$Ric(E,F) = -\left[\left(\lambda - \frac{\sigma}{2}\right) + \frac{\rho}{2}\right] g(E,F) - \mu\eta(E)\eta(F). \tag{6.7}$$

From equations (6.1) and (6.4), we can obtain

$$\begin{aligned}(R(E,F)\cdot Ein)(G,H) = &- Ric(R(E,F)G,H) - Ric(G,(R(E,F)H) \\ &+ \frac{r}{n}[g(R(E,F)G,H) + g(G,R(E,F)H)].\end{aligned} \tag{6.8}$$

Using equations (2.11), and (6.7) and the curvature identity in equation (6.8), we obatin

$$\begin{aligned}(R(E,F)\cdot Ein)(G,H) = \varepsilon\mu\{&\eta(H)\eta(F)g(E,G) - \eta(E)\eta(H)g(F,G) \\ &+ \eta(G)\eta(H)g(E,F) - \eta(G)\eta(E)g(F,H)\}.\end{aligned} \tag{6.9}$$

On an ε-contact metric manifold, in general, $\eta(H)\eta(F)g(E,G) - \eta(E)\eta(H)g(F,G) + \eta(G)\eta(H)g(E,F) - \eta(G)\eta(E)g(F,H) \neq 0$. Therefore from equation (6.9) we have $\mu = 0$. This shows that an n-dimensional (ε)-Kenmotsu manifold $\mathcal{M}$ with the conformal Killing vector field is Einstein semi-symmetric if and only if $\mu = 0$. Thus we can state the following theorem.

Theorem 6.3. *Suppose M is an n-dimensional, $n \geq 3$, (ε)-Kenmotsu manifold admitting a conformal Killing vector field V, then the η-Einstein soliton $(g, V, \lambda - \frac{\sigma}{2}, \mu)$ on M is an Einstein soliton if and only if M is Einstein semi-symmetric. Moreover, the Einstein soliton $(g, V, \lambda - \frac{\sigma}{2})$ is shrink, expand, or steady according as $\rho > (n-1)(n-2)\varepsilon$, $\rho < (n-1)(n-2)\varepsilon$, or $\rho = (n-1)(n-2)\varepsilon$, respectively.*

7. η-Einstein Solitons on φ-Ricci Symmetric (ε)-Kenmotsu Manifold

Definition 7.1. An (ε)-Kenmotsu manifold is said to be *ϕ-Ricci symmetric* if the Ricci operator Q satisfies

$$\varphi^2(\nabla_E Q)F = 0$$

for all vector fields E, F on $\mathcal{M}$.

Let us assume that $\mathcal{M}$ is an n-dimensional φ-Ricci symmetric (ε)-Kenmotsu manifold. Then from Definition (7.1) and equation (2.1) we have

$$-(\nabla_E Q)F + \eta((\nabla_E Q)F)\xi = 0. \tag{7.1}$$

Taking the inner product of (7.1) with G and using equation (2.2), we have

$$-g((\nabla_X Q)(F), G) + \varepsilon\eta((\nabla_E Q)(F))\eta(G) = 0,$$

from which we can obtain

$$-g(\nabla_E QF, G) + Ric(\nabla_E F, G) + \varepsilon\eta((\nabla_E Q)(F))\eta(G) = 0. \tag{7.2}$$

Substituting $F = \xi$ in equation (7.2) and using equations (2.2), (2.6) and (2.13), we obtain

$$(n-1)[g(E,G) - \eta(E)\eta(G)] + \varepsilon[Ric(E,G) + (n-1)\eta(E)\eta(G)] = 0,$$

which gives

$$Ric(E,G) = -\varepsilon(n-1)g(E,G) + (\varepsilon-1)(n-1)\eta(E)\eta(G). \tag{7.3}$$

This shows that the manifold under consideration is an η-Einstein manifold for $\varepsilon \neq 1$. Hence we can state the following theorem.

Theorem 7.2. *An n-dimensional φ-Ricci symmetric (ε)-Kenmotsu manifold is an η-Einstein manifold.*

From equations (3.2) and (7.3), we have

$$\begin{aligned} &g(\nabla_E \xi, G) + g(E, \nabla_G \xi) + (2\lambda - \sigma)g(E,G) + 2\mu\eta(E)\eta(G) \\ &\qquad = 2(n-1)\{\varepsilon g(E,F) - (\varepsilon-1)\eta(E)\eta(G)\}. \end{aligned} \tag{7.4}$$

Substituting $E = \xi$ in equation (7.4), we obtain

$$\begin{aligned} &g(\nabla_E \xi, \xi) + g(E, \nabla_\xi \xi) + (2\lambda - \sigma)g(E,\xi) + 2\mu\eta(E)\eta(\xi) \\ &\qquad = 2(n-1)\{\varepsilon g(E,\xi) - (\varepsilon-1)\eta(E)\eta(\xi)\}, \end{aligned}$$

which gives

$$[2(\varepsilon-2)(n-1) + (2\lambda-\sigma)\varepsilon + 2\mu]\,\eta(E) = 0. \tag{7.5}$$

Since $\eta(E) \neq 0$ on an almost contact metric manifold, therefore equation (7.5) takes the form

$$\lambda = \frac{\sigma}{2} - (\varepsilon-2)(n-1) - \mu.$$

From equation (7.3), we have

$$\sigma = -(n-1)[(n-1)\varepsilon + 1].$$

From the last two equations, we have

$$\lambda = -\frac{1}{2}[(n-1)(n\varepsilon + \varepsilon - 3) + 2\mu].$$

This shows that the η-Einstein soliton on M under consideration shrink, expand, or remain steady if $(n-1)(n\varepsilon+\varepsilon-3)+2\mu$ is $>$, $<$, $=0$, respectively. Thus we can state the following theorem.

Theorem 7.3. *Let $(g,\xi,\lambda-\frac{\sigma}{2},\mu)$ be an η-Einstein soliton on an n-dimensional φ-symmetric (ε)-Kenmotsu manifold $\mathcal{M}$. Then $(g,\xi,\lambda-\frac{\sigma}{2},\mu)$ expand, shrink, or remains steady if $(n-1)(n\varepsilon+\varepsilon-3)+2\mu$ is $<$, $>$, $=0$, respectively.*

Acknowledgements. The authors are thankful to the referee's for their valuable comments and suggestions for the improvement of the paper.

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