KYUNGPOOK Math. J. 60(2020), 839-851
https://doi.org/10.5666/KMJ.2020.60.4.839
pISSN 1225-6951 eISSN 0454-8124

Morphic Elements in Regular Near-rings

Alex Samuel Bamunoba, Philly Ivan Kimuli* and David Ssevviiri
Department of Mathematics, Makerere University, P. O. BOX 7062, Kampala Uganda
e-mail : bamunoba@cns.mak.ac.ug, kimuli@cns.mak.ac.ug *and* ssevviiri@cns.mak.ac.ug

Abstract. We define morphic near-ring elements and study their behavior in regular near-rings. We show that the class of left morphic regular near-rings is properly contained between the classes of left strongly regular and unit-regular near-rings.

1. Introduction

Let R be a ring, M be an R-module and $A := \mathrm{End}_R(M)$ be the ring of R-module endomorphisms of M. An element α of A is called *regular* if there exists a $\beta \in A$ such that $\alpha\beta\alpha = \alpha$. A regular element α is called *unit-regular* if β can be chosen to be an R-automorphism of M, i.e., $\beta \in \mathrm{Aut}_R(M)$. If every element in A is unit-regular, then A is a unit-regular ring. In two successive papers, [6] and [7], G. Ehrlich studied the class of unit-regular rings. She showed that $\alpha \in A$ is unit-regular if and only if α is regular and has the property that $M/\mathrm{Im}(\alpha) \cong \ker(\alpha)$. Note that, the property $M/\mathrm{Im}(\alpha) \cong \ker(\alpha)$ is the *Dual of the First Isomorphism Theorem*, $M/\ker(\alpha) \cong \mathrm{Im}(\alpha)$, for $\alpha \in A$. Ehrlich's result sparked further studies concerning this dual.

Let $a \in R$. Following the case for $M = {}_RR$ in which $\alpha(r) = ra$ for all $r \in R$, we denote the left and right annihilators of a by $(0 :_l a)$ and $(0 :_r a)$ respectively. It is known that for any $a \in R$, $R/(0 :_l a) \cong Ra$. By dualisiling this isomorphism, W. K. Nicholson and C. E. Sánchez in [18] introduced the notion of morphic elements. An element a in R is said to be *left morphic* if $R/Ra \cong (0 :_l a)$. A ring is called left morphic if each of its elements is left morphic, i.e., if for every $a \in R$, the Dual

---

* Corresponding Author.
Received December 24, 2019; revised June 3, 2020; accepted June 29, 2020.
2020 Mathematics Subject Classification: 16Y30, 16E50, 12K05.
Key words and phrases: morphic regular, strongly regular, unit-regular.
This work was carried out at Makerere University with financial support from the Makerere-Sida Bilateral Program Phase IV, Project 316 "Capacity building in Mathematics and its application".

of the First Isomorphism Theorem for an endomorphism $r \mapsto ra$ of R holds. Right morphic elements and right morphic rings are defined analogously. We say that R is morphic if it is both left and right morphic. With appropriate modifications, morphic modules, morphic groups and morphic group-rings were defined. These studies exist in [5, 11, 12, 14, 20] among others. In this work we extend this study to near-rings.

Let N be a near-ring. By defining left morphic near-ring elements, several properties are obtained. For instance, we establish that where as a right near-ring is in general not Abelian as a group under addition and also lacks left-distributivity of multiplication over addition; existence of a left morphic idempotent element in a zero-symmetric right near-ring allows the two mentioned conditions above to hold under some special situations. In particular, if e is a left morphic idempotent element of a zero-symmetric near-ring N, then for all $x \in N$,

$$x(1-e) = x - xe = -xe + x.$$

Secondly, we establish in Theorem 4.8 that an idempotent e of a zero-symmetric near-ring N is left morphic if and only if e and its supplement, $1-e$, are orthogonal and their left annihilator sets have only the zero element in common. Moreover, every left morphic idempotent element of a near-ring is morphic. Surprisingly, every idempotent need not be left morphic. It turns out that there are many results already known in left morphic rings that do not carry over to left morphic near-rings. For instance, for rings, Theorem 1.1 was proved.

Theorem 1.1.([7, Ehrlich]) *A ring R is unit-regular if and only if it is both regular and left morphic.*

The group-theoretic analogue of Theorem 1.1 was proved by Y. Li and W. K. Nicholson in [13]. Here we show that the analogue of Theorem 1.1 does not hold for near-rings. Whereas a left morphic regular zero-symmetric near-ring is unit-regular, see Theorem 4.14, we show in Examples 4.15 and 4.16 that the converse does not hold in general. Also, whereas zero-symmetric left strongly regular near-rings are left morphic, see Proposition 4.6, the converse is not true in general as shown in Example 4.12. We establish and classify left morphic regular near-rings. The relationship between left strongly regular, left morphic regular and unit-regular near-rings is studied. It is explicitly illustrated that the class of left morphic regular near-rings properly lies between the classes of left strongly regular and unit-regular near-rings.

2. Preliminaries on Right Near-rings

A *right near-ring* $(N, +, \star)$ is a non-empty set N with two binary operations '$+$' and '$\star$' such that $(N, +)$ is a group, (not necessarily Abelian) with identity 0, $(N, \star)$ is a semigroup and '$\star$' satisfies the right distributive law with respect to '$+$', i.e., $(x + y) \star z = x \star z + y \star z$ for all $x, y, z \in N$. One often writes xy for $x \star y$. A natural example of a right near-ring is the following: Let G be a group, written additively but not necessarily Abelian, and $M(G)$ be the set of all mappings from G to G. Given f, g in $M(G)$, the mapping $f+g$ from G to G is defined as $(f+g)(x) = f(x)+g(x)$. The product $f \star g$, of two elements, is their composition $f(g(x))$, for all x in G. Then $(M(G), +, \star)$ is a right near-ring. A *near-field* is a near-ring in which there is a multiplicative identity and every non-zero element has a multiplicative inverse. As an immediate consequence of the right distributive law, we have $0x = 0$ for every x in N but it is not necessarily true that $x0 = 0$ for every x in N. However, if $x0 = 0$ for every $x \in N$, then N is zero-symmetric.

Let $a \in N$. For a right near-ring N, the mapping $r \mapsto ra$ from N to N determines an endomorphism of $(N, +)$ whose kernel coincides with $(0 :_l a)$ and its image with Na.

Definition 2.1. Let N be a right near-ring. A left N-*module* M is a group, $(M, +)$, written additively but not necessarily Abelian together with a map $N \times M \to M, (r, m) \mapsto rm$, such that for all $m \in M$ and for all $r_1, r_2 \in N$,

$$(r_1 + r_2)m = r_1m + r_2m \text{ and } (r_1r_2)m = r_1(r_2m).$$

Observe that, a right near-ring N has a natural structure of a left N-module.

Definition 2.2. Let N be a right near-ring and M be a left N-module. A subset L of M is called an N-*ideal* of M if $(L, +)$ is a normal subgroup of $(M, +)$ and

$$r(l+m) - rm \in L$$

for all $m \in M$, $l \in L$ and $r \in N$. The N-ideals of ${}_NN$ are called *left ideals*. In fact, if L is a left ideal of N and $LN \subseteq L$, then L is an *ideal* of N.

For any near-ring N and $a \in N$, the set $(0 :_l a)$ is an N-ideal. If M is a left N-module, one can define a factor left N-module M/L provided L is an N-ideal of M. Near-ring homomorphisms and N-homomorphisms are defined in the usual manner as for rings.

Let $a \in N$. We call a *regular* if there exists $x \in N$ such that $a = axa$. If x can be chosen to be a unit, then a is called *unit-regular*. The element a is said to be *left (right) strongly regular* if there exists $x \in N$ such that $a = xa^2$ $(a = a^2x)$ and *strongly regular* if it is left and right strongly regular. Regular, unit-regular and left (right) strongly regular near-rings are defined in the usual manner. We say that N is *reduced* if it has no non-zero nilpotent elements, i.e., $a \in N$, $a^n = 0$ implies $a = 0$. N is said to have IFP (*Insertion-of-Factors-Property*) if for $a, b \in N$, $ab = 0$ implies $anb = 0$ for every $n \in N$. For a detailed account of basic concepts concerning near-rings and near-ring modules, we refer the reader to the books [8], [16] and [21].

Convention: A near-ring N is called *unital* if there exists an element $1 \neq 0$ in N such that, $1r = r1 = r$ for every element r in N. We call N *zero-symmetric* if $r0 = 0r = 0$ for every r in N. Unless otherwise stated, all near-rings are unital zero-symmetric right near-rings and all N-modules are unitary. We denote by $U(N)$

and $U(R)$ the collection of all units of N and R respectively (i.e., invertible elements with respect to multiplication '$\star$').

3. General Results and Examples of Left Morphic Near-rings

For a near-ring N and $a \in N$, consider the multiplication map $n \mapsto na : N \to N$. By the First Isomorphism Theorem, $N/(0 :_l a) \cong Na$ considered as N-modules. If Na is an N-ideal, then the dual to this isomorphism is $N/Na \cong (0 :_l a)$.

Definition 3.1. Let N be a near-ring and $a \in N$. An element a is called *left morphic* if Na is an N-ideal and

$$N/Na \cong (0 :_l a)$$

as near-ring N-modules. A near-ring N is called *left morphic* if each of its elements is left morphic (right morphic near-rings are defined similarly).

In Lemma 3.2 below, we give other equivalent statements for a left morphic near-ring element.

Lemma 3.2. *Let N be a near-ring and $a \in N$. The following are equivalent:*

(1) *a is left morphic.*

(2) *There exists $b \in N$ such that $Na = (0 :_l b)$ and $Nb = (0 :_l a)$.*

(3) *There exists $b \in N$ such that $Na = (0 :_l b)$ and $Nb \cong (0 :_l a)$.*

Proof. (1) $\Rightarrow$ (2). By (1), if $\sigma : N/Na \to (0 :_l a)$ defined by $\sigma(1 + Na) = b$ is the N-isomorphism, then $Nb = \text{Im}(\sigma) = (0 :_l a)$ because σ is surjective and $(0 :_l b) = \ker(\sigma) = \{n + Na : nb = 0\} = Na$ because σ is injective.
(2) $\Rightarrow$ (3). This follows from the equality $Nb = (0 :_l a)$ of N-ideals.
(3) $\Rightarrow$ (1). Suppose that there exists $b \in N$ such that $Na = (0 :_l b)$ and $Nb \cong (0 :_l a)$, then Na is an N-ideal and $N/Na = N/(0 :_l b) \cong Nb \cong (0 :_l a)$, proving (1). □

Lemma 3.3. *Let N be a near-ring, $a \in N$ and $u \in U(N)$, then*

(1) *$Nu = N$,*

(2) *$(0 :_l a) = (0 :_l au^{-1})$,*

(3) *$(0 :_l a)u^{-1} = (0 :_l ua)$,*

(4) *$(0 :_l a) \cong (0 :_l a)u$.*

Proof. To prove (1), it is enough to show that $N \subseteq Nu$ since for any $u \in U(N)$, $1 = u^{-1}u \in Nu$. For every $r \in N, r = r \cdot 1 = r(u^{-1}u) = (ru^{-1})u \in Nu$, hence $N = Nu$.

For (2), let $x \in (0 :_l a)$. Then $xa = 0$. Then $0 = 0u^{-1} = (xa)u^{-1} = x(au^{-1})$; hence $x \in (0 :_l au^{-1})$ and $(0 :_l a) \subseteq (0 :_l au^{-1})$. For $x \in (0 :_l au^{-1})$, we get $x(au^{-1}) = (xa)u^{-1} = 0$. This implies that $(xa)u^{-1}u = xa = 0$. Therefore, $x \in (0 :_l a)$; which proves $(0 :_l au^{-1}) = (0 :_l a)$. The proof of (3) is similar to that of (2).

For the proof of (4), the map $\vartheta : (0 :_l a) \to (0 :_l a)u,\ x \mapsto xu$ gives the required N-isomorphism. □

Proposition 3.4. *Let N be a near-ring and $a \in N$. If a is left morphic, then the same is true of au and ua, for every $u \in U(N)$.*

Proof. Let $a \in N$ be a left morphic element. Then by Lemma 3.2 $Na = (0 :_l b)$ and $Nb = (0 :_l a)$ for some $b \in N$. Suppose $u \in N$ is a unit. Applying Lemmas 3.2 and 3.3, $N(ua) = Na = (0 :_l b) = (0 :_l bu^{-1})$ and $N(bu^{-1}) = (Nb)u^{-1} = (0 :_l a)u^{-1} = (0 :_l ua)$. So ua is left morphic. Similarly, $N(au) = (Na)u = (0 :_l b)u = (0 :_l u^{-1}b)$, and $N(u^{-1}b) = (Nu^{-1})b = Nb = (0 :_l a) = (0 :_l au)$. So, au is also left morphic. □

Proposition 3.5. *Let N be a near-ring and $a \in N$. If $a \in N$ is left morphic, then the following are equivalent:*

(1) $(0 :_l a) = \{0\}$,

(2) $Na = N$,

(3) $a \in U(N)$.

Proof. (1) ⇔ (2) is clear from N/Na isomorphic to $(0 :_l a)$. The proof of (3) ⇒ (1) is immediate. We now prove (2) ⇒ (3); from (2) we have that $1 = ba$ for some b in N and so $a = aba$. Then $(1 - ab)$ is in the left annihilator of a that is zero. Thus we obtain $1 = ab = ba$; which implies that $a \in U(N)$. □

Using condition (2) in Lemma 3.2 we get: A finite direct product $\prod_{i=1}^{n} N_i$ of near-rings is left morphic if and only if each near-ring N_i is left morphic. The proof is immediate after taking into account that the left annihilator of an element $(a_1, a_2, \ldots, a_n)$ in the product is just the product of the sets $(0 :_l a_i), i = 1, 2, \ldots, n$.

We will give examples of left morphic near-ring elements and left morphic near-rings starting with the ring theoretic ones.

Example 3.6. The earliest known class of left morphic rings to be determined were the unit-regular rings in [7]. Later, in [18], [19] and [20], more examples were investigated to include all local rings where the Jacobson radical is a nilpotent principle ideal, all factor rings of commutative principal ideal domains like the rings $\mathbb{Z}_n$ of integers modulo n. Let R be a unit-regular ring, $\sigma : R \to R$ be any ring homomorphism that fixes idempotents element-wise and $R[x;\sigma]$ be the skew polynomials with commutation relation $xr = \sigma(r)x$. The rings $R[x;\sigma]/(x^n)$ and $R[x]/(x^n)$ are left morphic for all $n \geq 1$ by [11] and [12].

Example 3.7. Let N be a near-ring. The units of N are left and right morphic. To see this, let $u \in N$ be a unit. Then $Nu = N$ by Lemma 3.3 and $N/Nu = N/N \cong$

$\{0\} = (0 :_l u)$. This proves that u is left morphic. Similarly, $N/uN \cong (0 :_r u)$ and thus u is morphic. Consequently, all near-fields are morphic. If all idempotents e of N are central, then $1 - e$ is idempotent and $N(1 - e) \subseteq (0 :_l e)$. Let $xe = 0$. Then $x-(1-e)x = x+ex-x = 0$. We, therefore, have $x = (1-e)x = x(1-e) \in N(1-e)$. Thus $N(1 - e) = (0 :_l e)$. Similarly, $(0 :_l (1 - e)) = Ne$ and Lemma 3.2 implies e is left morphic. By a similar argument, e can be shown to be right morphic.

Example 3.8. A near-ring N is said to be *subcommutative* if $Na = aN$ for all $a \in N$. We call N a *generalised near field* if it is regular and subcommutative. It was proved in [17, Theorem 1.5] that, if N is a unital generalised near field, then N has a decomposition $N = (0 :_l a) \oplus Na$ for each a in N, where Na is an N-ideal. Since $N/Na \cong (0 :_l a)$ for all $a \in N$, then N is left morphic and thus all generalised near fields are left morphic.

Example 3.9. Let N be a finite *weakly divisible* near-ring, that is, for all $a, b \in N$, there exists $x \in N$ such that $xa = b$ or $xb = a$. Let $L = N \setminus U(N)$. In [3] and [4], it was shown that there exists $n, k \in \{0, 1, \ldots\}$ and $r \in L$ such that $L^k = Nr^k$, $L^n = \{0\}$ and $r^n = 0$ for some $0 \leq k < n$. Further, for each $a \in L$, $a = ur^k$ for some $u \in U(N)$. Moreover, every L^k is an N-ideal and $(0 :_l r^k) = Nr^{n-k}$. Thus we claim that N is left morphic. To see this, suppose that $a \in N$. If $a \in U(N)$, then by Example 3.6 there is nothing to prove. Otherwise, $a = ur^k$. Thus

$$Na = Nur^k = Nr^k = (0 :_l r^{n-k})$$

and by Lemma 3.3,

$$Nr^{n-k} = (0 :_l r^k) = (0 : ur^k)u = (0 :_l a)u \cong (0 :_l a).$$

Applying Lemma 3.2 completes the proof that a is left morphic and thus N is a left morphic near-ring.

With Example 3.6 in mind and following [3, Proposition 3], we hasten to note that the only morphic weakly divisible near-rings of the form $N := (\mathbb{Z}_n, +, \star)$ are those for which n is a power of a prime number.

4. Morphic Regular Elements

The aim of this section is to investigate morphic elements in regular near-rings. The following Lemmas [4.1 - 4.4] will be useful in the sequel.

Lemma 4.1.([21, Theorem 9.158]) *Let N be a non-zero regular near-ring. The following are equivalent:*

(1) *N is reduced,*

(2) *all idempotents of N are central,*

(3) *N is a sub-direct product of near-fields.*

Lemma 4.2. *A near-ring N is left strongly regular if and only if N is regular and reduced if and only if N is regular with central idempotents.*

Proof. By [1, Corollary 4.3], N is left strongly regular if and only if N is regular and reduced. Lemma 4.1 completes the proof. □

Lemma 4.3.([22, Theorem 3]) *Let N be a left strongly regular near-ring and $a \in N$. If $a = xa^2$ for some $x \in N$, then $a = axa$ and $ax = xa$.*

Lemma 4.4.([15, Proposition 5]) *Every left strongly regular near-ring is unit-regular.*

Proposition 4.5. *Let N be a left strongly regular near-ring and $a \in N$. Then a^2 is a regular element of N.*

Proof. Let $a \in N$. Then $a = xa^2$ for some $x \in N$. By Lemmas 4.2 and 4.3, $a = axa$, and $ax = xa$ is a central idempotent. Hence $a^2 = aa = (axa)(axa) = (aax)(xaa) = a^2(xx)a^2 = a^2x^2a^2$. □

Proposition 4.6. *Every left strongly regular near-ring N is left morphic.*

Proof. Let $a \in N$. By Lemma 4.4, $a = aua$ with $u \in U(N)$. Using Lemma 4.2 and Lemma 4.3, $au = ua$ is a central idempotent. Let v be the inverse of u. Then $a = a(uv) = (au)v$ is a product of a central idempotent and a unit which are both left morphic (according to Example 3.6). Proposition 3.4 completes the proof. □

Corollary 4.7. *Every strongly regular near-ring N is morphic.*

Proof. Strongly regular near-rings are both left and right strongly regular. By Proposition 4.6, they are both left and right morphic. □

H. E. Heatherly and J. R. Courville in [10] initiated the study of near-rings with a special condition on idempotents. In Theorem 4.8, we extend their ideas and show that the notion of left morphic idempotent elements for near-rings had been already studied in [10] although was not called so at that time.

Theorem 4.8. *Let N be a near-ring and e be an idempotent element of N. The following are equivalent:*

(1) *e is left morphic,*

(2) $Ne = (0 :_l (1 - e))$,

(3) $x(1 - e) = -xe + x$ *for all* $x \in N$,

(4) $(0 :_l e) \cap (0 :_l (1 - e)) = \{0\}$ *and* $e(1 - e) = 0$,

(5) $x(1 - e) = x - xe$ *for all* $x \in N$,

(6) $N(1 - e) = (0 :_l e)$,

(7) $(1 - e)$ *is left morphic.*

Proof. Since the implications (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are by [10, Proposition 2.1], we prove (2) $\Rightarrow$ (1) $\Rightarrow$ (7) $\Rightarrow$ (2) $\Leftrightarrow$ (6).

(2) $\Rightarrow$ (1). Suppose that (2) holds; it suffices to show that $N(1-e) = (0 :_l e)$ by Lemma 3.2. In view of [10, Corollary 2.2], $(1-e)$ is an idempotent element of N and $1-(1-e) = e$. By (2), $N(1-e) = (0 :_l (1-(1-e))) = (0 :_l e)$. Therefore, applying Lemma 3.2, $Ne = (0 :_l (1-e))$ and $N(1-e) = (0 :_l e)$ implies that e is left morphic.

(1) $\Rightarrow$ (7). If e is left morphic, then by Definition 3.1, Ne is an N-ideal of N and so $e(1-e)-e \in Ne$. That is, $e(1-e)-e = ne$ for some $n \in N$. Right multiplication by e gives $-e = ne$ and hence $e(1-e) = 0$. This proves $Ne \subseteq (0 :_l (1-e))$. Let $x \in (0 :_l (1-e))$. Then $x(1-e) = 0$ and $-x = x(1-e) - x \in Ne$ because, by (1), Ne is an N-ideal of N. Since Ne is a normal subgroup of N, $-x \in Ne$ implies $x \in Ne$ which gives $(0 :_l (1-e)) \subseteq Ne$. Hence $Ne = (0 :_l (1-e))$ and so e verifies (2). Further, $e(1-e) = 0$ implies $(1-e)$ is idempotent. By [10, Corollary 2.2], $1-(1-e) = e$ and $(1-e)$ must verify (2) as well. Hence $N(1-e) = (0 :_l (1-(1-e))) = (0 :_l e)$ and $(0 :_l (1-e)) = Ne$ which, by Lemma 3.2, proves that $(1-e)$ is left morphic.

(7) $\Rightarrow$ (2). By (7), $(1-e)$ is left morphic with $(0 :_l (1-e)) = Ne$ and so (2) follows.

(2) $\Rightarrow$ (6). Following [10, Corollary 2.2], $(1-e)$ is an idempotent element of N and $1-(1-e) = e$. Thus by (2), $N(1-e) = (0 :_l (1-(1-e))) = (0 :_l e)$.

(6) $\Rightarrow$ (2). Assume that (6) holds and let e' denote the idempotent $(1-e)$. Then $N(1-e') = (0 :_l e')$ implies that $N(1-(1-e)) = (0 :_l (1-e))$. Therefore, by [10, Corollary 2.2], $Ne = (0 :_l (1-e))$. □

Corollary 4.9. *Every left morphic idempotent element of a near-ring is morphic.*

Proof. We show that each left morphic idempotent element e of a near-ring N is also right-morphic, i.e., if an idempotent $e \in N$ is left morphic, then $eN = (0 :_r (1-e))$ and $(1-e)N = (0 :_r e)$. With the formulations of Theorem 4.8 in mind, it is enough to prove only the first equality. Since $(1-e)eN = (e-e^2)N = 0N = \{0\}$, $eN \subseteq (0 :_r (1-e))$. Suppose that $x \in (0 :_r (1-e))$. Then $(1-e)x = 0$ which gives $x = ex \in eN$. Hence $(0 :_r (1-e)) = eN$. □

Example 4.10. Consider the commutative ring N defined on the Klein four group $(N, +)$ with $N = \{0, a, b, c\}$, whose addition and multiplication is given by the two tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\star$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

We can verify that N is left morphic since $Na = (0 :_l c)$, $Nc = (0 :_l a)$, $(0 :_l b) = \{0\}$ and $(0 :_l 0) = Nb$.

Near-rings in which every element is idempotent are called *Boolean.* In light of Example 4.10 and a property exhibited by a left morphic idempotent element of a near-ring in Theorem 4.8, one is led to the following:

Proposition 4.11. *Let N be a Boolean near-ring. Then N is a commutative morphic ring.*

Proof. By [2, Theorem 3], $(N, +)$ is commutative. N is (left and right) strongly regular and so is morphic by Corollary 4.7. By Lemma 4.2, all the elements of N are central. Now

$$x(y+z) = (y+z)x = yx + zx = xy + xz$$

and we have the left distributive law. □

A near-ring N is called a *left duo near-ring* if the left ideals of N are ideals of N. Every left strongly regular near-ring is a left duo near-ring.

Example 4.12. Let D be a division ring and $R := M_n(D)$ be the $n \times n$ matrix ring over $D, n \geq 2$. It is well known that R is a left and right morphic regular ring, see [18]. However, R is neither left nor right duo, hence neither left nor right strongly regular. In particular, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Example 4.12 is a left morphic regular ring which is not left strongly regular. However, if N is a reduced near-ring, then it fulfills the IFP-property. But [8, Proposition 1.6.35] implies that N has the IFP-property if and only if for all $a \in N$, $(0 :_l a)$ and $(0 :_r a)$ are ideals of N. Hence regularity conditions and left morphic left duo near-rings are linked by Proposition 4.13 below.

Proposition 4.13. *Let N be a near-ring. Then the following are equivalent:*

(1) *N is a reduced left morphic near-ring,*

(2) *N is a left strongly regular near-ring,*

(3) *N is a regular and left duo near-ring.*

Proof. (1) $\Rightarrow$ (2). Let $a \in N$. By the left morphic hypothesis, there exists some $b \in N$ such that $Na = (0 :_l b)$. Further, N fulfills the IFP property because it is reduced. Thus $(0 :_l (0 :_r a)) = (0 :_l (0 :_r Na)) = (0 :_l (0 :_r (0 :_l b))) = (0 :_l b) = Na$ so that $(0 :_l (0 :_r Na)) = Na$ for every $a \in N$. By [8, Proposition 3.4.14], N is regular and therefore left strongly regular by Lemma 4.2.

(2) $\Rightarrow$ (1). Given (2), N is left morphic by Proposition 4.6 and reduced by Lemma 4.2.

(3) $\Rightarrow$ (2). Let $a \in N$. Then each left annihilator set of a is an ideal because N is left duo. In this case N fulfills the IFP-property and so by [1, Proposition 4.2], N is left strongly regular.

(2) $\Rightarrow$ (3). This always holds since every left strongly regular near-ring is regular by Lemma 4.2 and every N-ideal of N is an ideal. □

Theorem 4.14. *Every left morphic regular near-ring is unit-regular.*

Proof. Let $a \in N$, $a = axa$ for some $x \in N$, $Na = (0 :_l b)$ and $Nb = (0 :_l a)$ for some $b \in N$. The argument is similar to the proof of [18, Proposition 5]. Observe that $a \in Na = (0 :_l b)$ and $b \in Nb = (0 :_l a)$ give

$$ab = 0 = ba. \tag{4.1}$$

The elements, xa and ax are idempotents in N, they are left morphic and so they satisfy the conditions of Theorem 4.8. In particular,

$$a(1 - ax) = a - a(ax) = -a(ax) + a$$

for all $a \in N$. From $a = axa$ we have $(a - axa) = (1 - ax)a = 0$. This implies that $(1 - ax) \in (0 :_l a) = Nb$. Therefore, $(1 - ax) = rb$ for some $r \in N$. Now, we write

$$u = xax + b$$

and verify that u is a unit. In view of Proposition 3.4, it is enough to show that $(0 :_l u) = \{0\}$. Suppose $y \in (0 :_l u)$. Then

$$yu = y(xax + b) = 0. \tag{4.2}$$

Right multiply (4.2) by $(1 - ax)$ to get

$$0 = y(xax + b)(1 - ax) = y(b(1 - ax) + xax(1 - ax)).$$

Applying Theorem 4.8 and the fact that $ab = 0 = ba$ leads to $0 = y(b - bax + (xax - xaxax)) = yb$. Hence $y \in (0 :_l b) = Na$, say $y = na$, for some $n \in N$. Then

$$0 = 0a = (yu)a = y(xax + b)a = y(0 + xa) = yxa = naxa = na = y.$$

Since $(0 :_l u) = \{0\}$, u is a unit in N. Lastly, observe that

$$aua = a(xax + b)a = a(ba + xaxa) = a(xa + 0) = axa = a.$$

Hence a is unit-regular. □

The near-ring in Example 4.10 is morphic and unit-regular near-ring. Clearly, for rings, all morphic regular rings are unit-regular and the converse holds by [7].

In Examples 4.15 and 4.16, we construct unit-regular near-ring elements and unit-regular near-rings which are not left morphic and hence the analogue of Ehrlich's theorem does not, in general, hold for near-rings.

Example 4.15. Consider the group $\mathbb{Z}_3 := \{\bar{0}, \bar{1}, \bar{2}\}$ and the set of all functions

$$M_0(\mathbb{Z}_3) := \{f \in M_0(\mathbb{Z}_3) \mid f : \mathbb{Z}_3 \to \mathbb{Z}_3 \text{ such that } f(\bar{0}) = \bar{0}\}.$$

Recall, $M_0(\mathbb{Z}_3)$ is a zero-symmetric near-ring with respect to componentwise addition $+$ and composition $\circ$ of functions.

Claim 1. *If $N := M_0(\mathbb{Z}_3)$, then the near-ring $(N, +, \circ)$ is unit-regular but not left morphic.*

Proof. We begin by investigating the elements of N as follows. Let $\bar{x} \in \mathbb{Z}_3,\ f_i \in N$ with $i = 1, \ldots, 9$, then

$$\begin{array}{lcl} f_1(\bar{1}) = \bar{0} & \text{and} & f_1(\bar{2}) = \bar{0}; \\ f_2(\bar{1}) = \bar{0} & \text{and} & f_2(\bar{2}) = \bar{1}; \\ f_3(\bar{1}) = \bar{0} & \text{and} & f_3(\bar{2}) = \bar{2}; \\ f_4(\bar{1}) = \bar{1} & \text{and} & f_4(\bar{2}) = \bar{0}; \\ f_5(\bar{1}) = \bar{1} & \text{and} & f_5(\bar{2}) = \bar{1}; \\ f_6(\bar{1}) = \bar{1} & \text{and} & f_6(\bar{2}) = \bar{2}; \\ f_7(\bar{1}) = \bar{2} & \text{and} & f_7(\bar{2}) = \bar{0}; \\ f_8(\bar{1}) = \bar{2} & \text{and} & f_8(\bar{2}) = \bar{1}; \\ f_9(\bar{1}) = \bar{2} & \text{and} & f_9(\bar{2}) = \bar{2}. \end{array}$$

Note that $f_1 = 0_N$ is the zero map and $f_6 = 1_N$ is the identity map. The only non-identity unity is f_8. These three maps are trivially unit-regular. Since f_3, f_4, f_5 and f_9 are idempotent, they are unit-regular and f_2 and f_7 are unit-regular using f_8. Hence N is a unit-regular near-ring. But N is not left morphic in view of Theorem 4.8 since $(f_5(1_N - f_5))(\bar{2}) \neq \bar{0}$. □

Example 4.16. Let D be a division ring and $R := M_n(D)$ be the $n \times n$ matrix ring over $D, n \geq 2$. R is unit-regular [9, Corollary 4.7]. For any non-zero R-module M, let $N := R \times M$ and define addition and multiplication, respectively, in the following manner:

$$\begin{aligned} \langle a_1, m_1 \rangle + \langle a_2, m_2 \rangle &= \langle a_1 + a_2, m_1 + m_2 \rangle \\ \langle a_1, m_1 \rangle \star \langle a_2, m_2 \rangle &= \langle a_1 a_2, a_1 m_2 + m_1 \rangle. \end{aligned}$$

Claim 2. *$(N, +, \star)$ is a unit-regular near-ring that is not left morphic.*

Proof. It can be verified that N is a right near-ring with identity $\langle 1, 0 \rangle$. We proceed to show that N is unit-regular. Let $\langle u, -um \rangle \in N$, where u is a unit element in R and $m \in M$. Then for an arbitrary element $\langle a, m \rangle \in N, a \in R$, it is easy to verify that $\langle a, m \rangle \star \langle u, -um \rangle \star \langle a, m \rangle = \langle a, m \rangle$ and $\langle u, -um \rangle$ is a unit with inverse $\langle u^{-1}, m \rangle$. This gives N unit-regular. To prove that N is not left morphic it suffices to show that all elements $\langle a, m \rangle \in N$ with $0 \neq m \in M$ are not left morphic. Let $\langle a, m \rangle \in N, m \neq 0$. If $\langle a, m \rangle$ is left morphic, then

$$\langle a, m \rangle \in N \langle a, m \rangle = (0 :_l \langle b, n \rangle) \text{ and } \langle b, n \rangle \in N \langle b, n \rangle = (0 :_l \langle a, m \rangle)$$

for some $\langle b, n \rangle \in N$. This gives $\langle a, m \rangle \star \langle b, n \rangle = \langle 0, 0 \rangle = \langle b, n \rangle \star \langle a, m \rangle$ which yields $ab = 0 = ba, -n = bm$ and $m = -an$. Therefore, $m = -a(n) = -a(-bm) = (ab)m = 0m = 0$, which is a contradiction. Thus $\langle a, m \rangle$ is not left morphic. □

By Proposition 4.6 and Theorem 4.14, the picture of relationships among all these classes of near-rings is as follows:

$$\left\{\begin{array}{c}\text{Left strongly}\\ \text{regular near-rings}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{Left morphic}\\ \text{regular near-rings}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{Unit-}\\ \text{regular near-rings}\end{array}\right\}.$$

However, if N has IFP we get from [1, Proposition 4.2] that such a unit-regular near-ring is left strongly regular. Hence we have:

Proposition 4.17. *Let N be a near-ring which has IFP. The following are equivalent:*

(1) *N is left strongly regular.*

(2) *N is left morphic and regular.*

(3) *N is unit-regular.*

Acknowledgements. The authors are grateful to the referee for the helpful comments and suggestions that greatly improved the exposition.

References

[1] N. Argac and N. Groenewald, *Weakly and strongly regular near-rings*, Algebra Colloq., **12**(2005), 121–130.

[2] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Aust. Math. Soc., **2**(1970), 363–368.

[3] A. Benini and S. Pellegrini, *Weakly divisible nearrings*, Discrete Math., **208/209**(1999), 49–59.

[4] A. Benini and S. Pellegrini, *Finite weakly divisible nearrings*, Riv. Mat. Univ. Parma (8), **2**(2009), 101–116.

[5] J. Chen, Y. Li and Y. Zhou, *Morphic group rings*, J. Pure Appl. Algebra, **205**(2006), 621–639.

[6] G. Ehrlich, *Unit-regular rings*, Portugal. Math., **27**(1968), 209–212.

[7] G. Ehrlich, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc., **216**(1976), 81–90.

[8] G. Ferrero, *Nearrings : some developments linked to semigroups and groups*, Springer Publishing Co. Inc., 2012.

[9] K. R. Goodearl, *Von Neumann regular rings,* Monographs and studies in Mathematics **4**, Advanced Publishing Program, Boston, Mass London, 1979.

[10] H. E. Heatherly and J. R. Courville, *Near-rings with a special condition on idempotents*, Math. Pannon., **10**(1999), 197–209.

[11] T.-K. Lee and Y. Zhou, *Morphic rings and unit regular rings*, J. Pure Appl. Algebra, **210**(2007), 501–510.

[12] T.-K. Lee and Y. Zhou, *A theorem on unit-regular rings*, Canad. Math. Bull., **53**(2010), 321–326.

[13] Y. Li and W. K. Nicholson, *Ehrlich's theorem for groups*, Bull. Aust. Math. Soc., **81**(2010), 304–309.

[14] Y. Li, W. K. Nicholson and L. Zan, *Morphic groups*, J. Pure Appl. Algebra, **214**(2010), 1827–1834.

[15] G. Mason, *Strongly regular near-rings*, Proc. Edinburgh Math. Soc., **23**(1980), 27–35.

[16] J. D. P. Meldrum, *Near-rings and their links with groups*, Research Notes in Mathematics **134**, Pitman, London, 1985.

[17] P. Nandakumar, *On weakly π-subcommutative near-rings*, Bull. Malays. Math. Sci. Soc., **32**(2009), 131–136.

[18] W. K. Nicholson and C. E. Sánchez, *Rings with the dual of the isomorphism theorem*, J. Algebra, **271**(2004), 391–406.

[19] W. K. Nicholson and C. E. Sánchez, *Principal rings with the dual of the isomorphism theorem*, Glasgow Math. J., **46**(2004), 181–191.

[20] W. K. Nicholson and C. E. Sánchez, *Morphic modules*, Comm. Algebra, **33**(2005), 2629–2647.

[21] G. Pilz, *Near-rings, the theory and its applications*, Revised ed. North Holland, 1983.

[22] Y. V. Reddy and C. V. L. N. Murty, *On strongly regular near-rings*, Proc. Edinburgh Math. Soc., **27**(1984), 61–64.