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A Regularization-direct Method to Numerically Solve First Kind Fredholm Integral Equation

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Abstract. Most first kind integral equations are ill-posed, and obtaining their numerical solution often requires solving a linear system of algebraic equations of large condition number, which may be difficult or impossible. This article proposes a regularization-direct method to numerically solve first kind Fredholm integral equations. The vector forms of block-pulse functions and related properties are applied to formulate the direct method and reduce the integral equation to a linear system of algebraic equations. We include a regularization scheme to overcome the ill-posedness of integral equation and obtain a stable numerical solution. Some test problems are solved using the proposed regularization-direct method to illustrate its efficiency for solving first kind Fredholm integral equations.

1. Introduction

Integral equations beside the other forms of functional equations such as integro-differential and differential equations, are widely used for mathematical modeling many problems in physical science and engineering. Such models have often no analytical solution, hence obtaining approximate solutions requires a suitable numerical method.

On the other hand, most physical mediums used in engineering problems are linear, i.e., they can be expressed by linear constitutive relationships. Mathematical

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modeling in such mediums leads to linear functional equations. In particular, the first kind Fredholm integral equation model is an appropriate tool to model many related structures.

First kind Fredholm integral equations are inherently ill-posed, hence solutions are generally unstable. This makes numerical solutions very difficult, since a small error can lead to unbounded errors. Indeed, solutions for ill-posed problems may not exist, or may not be unique [4, 6–8, 11, 12]. This intrinsic ill-posedness is often the main challenge to analyze these problems. Therefore, it is important to provide an appropriate method to obtain stable solutions for first kind Fredholm integral equations.

This article proposes a regularization-direct method for numerical solution of first kind Fredholm integral equations using the vector forms of block-pulse functions (BPFs) and related properties. We express the first kind integral equation as a second kind Fredholm integral equation and include a regularization parameter. BPFs vector forms are then applied to the latter integral equation and the solution reduces to solving a linear system of algebraic equations due to BPF properties. The main advantages of the proposed regularization-direct method are as follows.

- Obtain a convergent numerical solution for first kind Fredholm integral equation.
- Avoids projection methods such as collocation, Galerkin, etc.
- Accuracy is satisfactory.
- The algorithm is simple and clear to use and can be easily implemented.

The remainder of this paper is organized as follows. Section 2 reviews some previous regularization methods for solving first kind integral equations. Section 3 provides a brief review on BPFs and their vector forms. Section 4 proposes the regularization-direct method for numerical solution of first kind Fredholm integral equations and Section 5 provides error analysis and evaluates convergence associated with the proposed method. Section 6 solves some test problems using the proposed method to illustrate its computational efficiency with and without regularization. Section 7 summarizes and concludes the paper.

2. Review of Regularization Methods for Solving First Kind Integral Equations

Regularization methods construct stable approximate solution for ill-posed problems, and several such methods have been previously proposed to solve ill-posed first kind Fredholm integral equations. This section reviews some related regularization methods.

Consider a general form first kind Fredholm integral equation

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad a \leqslant s \leqslant b, \tag{2.1}$$

where k is the kernel, f is a known function, and x is the unknown function to be determined.

Let us consider $\mathcal{K}x = f$, rather than (2.1), where operator $\mathcal{K} : X \longrightarrow Y$ is $\int_a^b k(s,t)x(t)\,\mathrm{d}t$. We can assume that the linear operator $\mathcal{K}$ is injective without loss of generality, since uniqueness for a linear operator can always be achieved by a suitable modification of the solution space X [11]. (A discussion on injectivity for a class of integral operators may be found in [10].)

We want to find an approximation of the unbounded inverse operator $\mathcal{K}^{-1} : Y \longrightarrow X$ by a bounded linear operator $\mathcal{R} : Y \longrightarrow X$ with regularization. This can be accomplished using the following definition.

Definition 2.1. Let X and Y be normed spaces and let $\mathcal{K} : X \longrightarrow Y$ be an injective bounded linear operator. Then a family of bounded linear operators $\mathcal{R}_\alpha : Y \longrightarrow X$, $\alpha > 0$, with the property of pointwise convergence $\lim\limits_{\alpha\to 0} \mathcal{R}_\alpha \mathcal{K}x = x$, $x \in X$, is called a *regularization scheme* for the operator $\mathcal{K}$. The parameter α is called the regularization parameter [11].

It is concluded from Definition 2.1 that $\lim\limits_{\alpha\to 0} \mathcal{R}_\alpha f = \mathcal{K}^{-1} f$, for all $f \in \mathcal{K}x$.

Several approaches to solve these equations have been proposed using regularization techniques.

Phillips [16] developed a method to solve Fredholm integral equations using the quadrature rule with a regularization technique. Twomey [22] subsequently improved the method.

Lavrentiev [13] proposed a regularization method to solve first kind integral equations with square-integrable, symmetric, and positive-definite kernels.

Tikhonov [19] used the same restrictions as Lavrentiev to consider the following second kind integral equation rather than (2.1),

$$\int_a^b k^*(s,t)x(t)\,\mathrm{d}t + \gamma x(s) = f^*(s), \tag{2.2}$$

where

$$k^*(s,t) = k^*(t,s) = \int_a^b k(z,t)k(z,s)\,\mathrm{d}z, \tag{2.3}$$
$$f^*(s) = \int_a^b k(z,s)f(z)\,\mathrm{d}z,$$

and γ is the regularization parameter.

Moreover, Tikhonov introduced some other regularization methods to solve first kind integral equations in [20] and [21].

Essah and Delves [5] developed a cross-validation scheme to automatically set two introduced regularization parameters (constraints) to solve first kind integral equations by using a Chebyshev series method.

Caldwell [1] considered the simplest form of regularization

$$\int_a^b k(s,t)x_\varepsilon(t)\,\mathrm{d}t + \varepsilon x_\varepsilon(s) = f(s), \tag{2.4}$$

where ε is a small, positive parameter. He pointed out that this could establish

$$\lim_{\varepsilon\to 0} \| x_\varepsilon(s) - x(s) \| = 0 \tag{2.5}$$

in the L_2-norm, but this convergence is not uniform.

Wazwaz [23] proposed a regularization-homotopy method to solve first kind Fredholm integral equations by combining regularization with homotopy perturbation to handle the ill-posedness.

Neggal et al. [15] considered a variant of projected Tikhonov regularization to solve first kind Fredholm integral equations. They provided a theoretical analysis in Hilbert space and established some convergence rates under certain regularity assumptions on the exact solution and kernel.

Rashed [18] introduced another method to treating first and second kind Fredholm and Volterra integral equations, and tested the method for various numerical examples.

A regularization method already proposed by Wazwaz was developed by Ziyaee and Tari [24] to linear and nonlinear two-dimensional first kind Fredholm integral equations. The regularization method was used for linear equations directly, but nonlinear first kind equations were transformed to first kind linear equations by a change of variable, and then the regularization method was applied.

3. Review of Block-pulse Functions

An m-set of BPFs is defined over interval $[0,H)$ as $[2,3,9,14,17]$

$$\varphi_i(t) = \begin{cases} 1, & \frac{iH}{m} \leqslant t < \frac{(i+1)H}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $i = 0, 1, ..., m-1$, and m is a positive integer. Consider $h = H/m$, and let φ_i be the ith BPF.

The most important BPFs properties are disjointness, orthogonality, and completeness.

We can express the first m terms of BPFs as an m-vector

$$\Phi(t) = [\varphi_0(t), \varphi_1(t), \ldots, \varphi_{m-1}(t)]^T, \qquad t \in [0,H), \tag{3.2}$$

where superscript T indicates transposition. Above representation and disjointness property follows

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \tag{3.3}$$

where V is an m-vector and $\tilde{V} = \mathrm{diag}(V)$. Moreover, it can be clearly concluded that for any $m \times m$ matrix B

$$\Phi^T(t) B \Phi(t) = \hat{B}^T \Phi(t), \tag{3.4}$$

where $\hat{B}$ is an m-vector with elements equal to the diagonal entries of B. Also

$$\int_0^H \Phi(t)\,\mathrm{d}t = [h, h, \ldots, h]^T, \tag{3.5}$$

and

$$\int_0^H \Phi(t)\Phi^T(t)\,\mathrm{d}t = hI, \tag{3.6}$$

where I is the $m \times m$ identity matrix.

The expansion of a function f over $[0, H)$ with respect to φ_i, $i = 0, 1, \ldots, m-1$, may be compactly written as [9, 14]

$$f(t) \simeq \sum_{i=0}^{m-1} f_i \varphi_i(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{3.7}$$

where $F = [f_0, f_1, \ldots, f_{m-1}]^T$ and f_is are defined by

$$f_i = \frac{1}{h} \int_0^H f(t) \varphi_i(t)\,\mathrm{d}t. \tag{3.8}$$

Let us assume k is a function of two variables in $\mathscr{L}^2\big([0, H) \times [0, H)\big)$, where $\mathscr{L}^2$ is the space of square integrable functions. It can be similarly expanded with respect to BPFs as

$$k(s, t) \simeq \Phi^T(s) K \Psi(t), \tag{3.9}$$

where Φ and Ψ are m_1- and m_2-dimensional BPF vectors respectively, and K is the $m_1 \times m_2$ block-pulse coefficient matrix with k_{ij}, $i = 0, 1, \ldots, m_1 - 1$, $j = 0, 1, \ldots, m_2 - 1$, as follows:

$$k_{ij} = m_1 m_2 \int_0^H \int_0^H k(s, t) \varphi_i(s) \psi_j(t)\,\mathrm{d}s\,\mathrm{d}t. \tag{3.10}$$

For convenience, we put $m_1 = m_2 = m$.

This article assumes that BPFs are defined over an arbitrary interval $[a, b)$. All the above properties and relations can be generalized over this interval.

4. Regularization-direct Method to Numerically Solve First Kind Fredholm Integral Equation

We propose an effective regularization-direct method to numerically solve first kind Fredholm integral equation using the results obtained in the previous section for BPFs.

Consider first kind Fredholm integral equation of the form

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad a \leqslant s < b, \tag{4.1}$$

where k and f are known and x is the unknown function to be determined. Moreover, $k \in \mathscr{L}^2\big([a,b) \times [a,b)\big)$ and $f \in \mathscr{L}^2\big([a,b)\big)$.

Equation (4.1) can be expressed as a second kind Fredholm integral equation

$$\alpha\, x_\alpha(s) + \int_a^b k(s,t)x_\alpha(t)\,\mathrm{d}t = f(s), \qquad a \leqslant s < b, \tag{4.2}$$

where we assume the regularization parameter $\alpha \to 0$. Solutions for (4.1) and (4.2) will be identical if $\alpha \to 0$. Therefore, we focus on the solution for (4.2).

Approximating k, f, and x_α with respect to BPFs, using (3.7) and (3.9), gives

$$\begin{aligned} k(s,t) &\simeq \Phi^T(s)K\Phi(t), \\ f(s) &\simeq F^T\Phi(s) = \Phi^T(s)F, \\ x_\alpha(s) &\simeq X_\alpha^T\Phi(s) = \Phi^T(s)X_\alpha, \end{aligned} \tag{4.3}$$

where m-vectors F and X_α, and $m \times m$ matrix K are BPFs coefficients of f, x_α, and k, respectively, and X_α is the unknown vector to be obtained.

Substituting (4.3) into (4.2),

$$\alpha\, \Phi^T(s)X_\alpha + \int_a^b \Phi^T(s)K\Phi(t)\Phi^T(t)X_\alpha\,\mathrm{d}t \simeq \Phi^T(s)F. \tag{4.4}$$

Therefore,

$$\alpha\, \Phi^T(s)X_\alpha + \Phi^T(s)K\left(\int_a^b \Phi(t)\Phi^T(t)\,\mathrm{d}t\right)X_\alpha \simeq \Phi^T(s)F, \tag{4.5}$$

and hence from (3.6)

$$\Phi^T(s)\,\alpha\, X_\alpha + \Phi^T(s)hKX_\alpha \simeq \Phi^T(s)F, \tag{4.6}$$

or

$$\alpha\, X_\alpha + hKX_\alpha \simeq F. \tag{4.7}$$

Replacing $\simeq$ with $=$,

$$(\alpha I + hK)X_\alpha = F, \tag{4.8}$$

which is a linear system of algebraic equations in terms of unknown vector $X_\alpha = [x_{\alpha_0}, x_{\alpha_1}, \ldots, x_{\alpha_{m-1}}]^T$. Solving this system and computing X_α provides solution $x_\alpha(s) \simeq X_\alpha^T \Phi(s)$ for (4.2) which will be in terms of α. When $\alpha \to 0$ the approximate solution $x(s)$ for first kind Fredholm integral equation (4.1) is

$$\begin{aligned} x(s) &= \lim_{\alpha\to 0} x_\alpha(s) \\ &\simeq \left(\lim_{\alpha\to 0} X_\alpha^T\right)\Phi(s). \end{aligned} \tag{4.9}$$

5. Error Analysis

This section investigates the proposed regularization-direct method convergence.

Error for the proposed method has two parts, one related to the direct method (particularly BPF expansion) and the other due to the regularization method.

Consider $\mathcal{K}x = f$ (defined in Section 2), where $\mathcal{K} : X \longrightarrow Y$. Assume $\mathcal{R}_\alpha : Y \longrightarrow X$ is the regularization operator. Let $x_\alpha^m = X_\alpha^T \Phi(s)$ be the approximate solution for (4.1), computed by the proposed method, where m is the number of BPFs and α is the regularization parameter, and x is the exact solution of (4.1). Then

$$\begin{aligned} x_\alpha^m - x &= \mathcal{R}_\alpha f^m - x \\ &= \mathcal{R}_\alpha f^m - \mathcal{R}_\alpha f + \mathcal{R}_\alpha f - x \\ &= \mathcal{R}_\alpha f^m - \mathcal{R}_\alpha f + \mathcal{R}_\alpha \mathcal{K} x - x, \end{aligned} \tag{5.1}$$

and from the triangle inequality

$$\| x_\alpha^m - x \| \leqslant \| \mathcal{R}_\alpha f^m - \mathcal{R}_\alpha f \| + \| \mathcal{R}_\alpha \mathcal{K} x - x \| . \tag{5.2}$$

Therefore,

$$\| x_\alpha^m - x \| \leqslant \| \mathcal{R}_\alpha \| \, \| f^m - f \| + \| \mathcal{R}_\alpha \mathcal{K} - I \| \, \| x \|, \tag{5.3}$$

where I is the identity operator.

This decomposition shows that the error consists of two parts. The first term reflects error from the BPF expansion and the second term is due to approximation error between $\mathcal{R}_\alpha$ and $\mathcal{K}^{-1}$, where $\lim_{\alpha\to 0} \mathcal{R}_\alpha = \mathcal{K}^{-1}$.

From Definition 2.1, the second error term will decrease as α approaches zero, i.e., $\lim_{\alpha\to 0} \| \mathcal{R}_\alpha \mathcal{K} - I \| = 0$.

The following lemma applies to the first term.

Lemma 5.1. *The representation error (or the residual error) when a differentiable function $f(t)$ is represented in a series of m BPFs is [17]*

$$\| f^m - f \| \leqslant \frac{1}{2\sqrt{3}m} \underset{[0,H)}{SUP} \left(f'(t) \right). \tag{5.4}$$

Hence, from Lemma 5.1, the first error term in (5.3) will decrease as m increases. Therefore, we should choose an appropriate m to achieve an acceptable total error. Finally, the relation (5.3) shows that the proposed regularization-direct method has satisfactory convergence.

6. Test Problems and Numerical Results

We present several test problems solved using the proposed regularization-direct method with their related numerical results to illustrate the proposed method's computational efficiency.

Approximate results were calculated for each test problem for two sets of points as follows.

- **i.** The set of mid-points of the m subintervals $\big[a + ih, a + (i+1)h\big)$ of interval $[a, b)$, where $i = 0, 1, \ldots, m-1$, and $h = (b-a)/m$.
- **ii.** The set of ten specific points s_i in interval $[a, b)$ such that $s_i = a + ih'$, where $i = 0, 1, \ldots, 9$, and $h' = (b-a)/10$.

The results are presented as mean-absolute error. If we obtain the approximate solution at N points s_i for a given m, then the mean-absolute error for this m can be expressed as

$$E = \frac{1}{N} \sum_{i=0}^{N-1} \big| x(s_i) - \bar{x}(s_i) \big|, \tag{6.1}$$

where x and $\bar{x}$ are the exact and approximate solutions, respectively, and we set $N = m$ and $N = 10$ to calculate E at the mid-points and ten points, respectively.

Test Problem 6.1.
Consider the first kind Fredholm integral equation

$$\int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{6.2}$$

where $k(s,t) = \sin(s+t)$ and $f(s) = \frac{1}{4}\Big(\big(1 - \cos(2)\big)\cos(s) + \big(2 + \sin(2)\big)\sin(s)\Big)$, with exact solution $x(s) = \cos(s)$. Table 1 shows mean-absolute errors for the direct method, with and without regularization, for different m and $\alpha \to 0$.

Test Problem 6.2.
Consider the first kind Fredholm integral equation

$$\int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{6.3}$$

where $k(s,t) = t\exp(t-s)$ and $f(s) = \frac{1}{2}\exp(-s)$, with solution $x(s) = \exp(-s)$. Table 2 shows the results for $\alpha \to 0$.

Test Problem 6.3.
Consider the first kind Fredholm integral equation

$$\int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{6.4}$$

where $k(s,t) = 2st^2$ and $f(s) = \frac{1}{2}s$, with exact solution $x(s) = s$. Table 3 shows the results for $\alpha \to 0$.

Test Problem 6.4.
Consider the first kind Fredholm integral equation [18]

$$\int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{6.5}$$

where $k(s,t) = \begin{cases} t(s-1), & t \leqslant s, \\ s(t-1), & t > s, \end{cases}$ and $f(s) = \exp(s) + \big(1-\exp(1)\big)s - 1$, with exact solution $x(s) = \exp(s)$. Figure 1 shows the results obtained by the proposed method at mid-points for $m = 4$ and $\alpha \to 0$.

Test Problem 6.5.
Consider the first kind Fredholm integral equation [15]

$$\int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{6.6}$$

where $k(s,t) = \exp(s^2 t)$ and $f(s) = \big(\exp(s^2+1)-1\big)/(s^2+1)$, with exact solution $x(s) = \exp(s)$. Figure 2 shows the results at mid-points for $m = 8$ and $\alpha = 1\,e-8$.

The numerical results generally confirm that the proposed regularization-direct method provides a stable approximate solution with sufficient accuracy for first kind Fredholm integral equation; whereas only unstable solutions are found without regularization. The reduction rate for the mean absolute errors confirms the proposed method has satisfactory convergence.

Table 1: Mean-absolute error for the solution to test problem 6.1 at mid-points and ten specific points, for different m and $\alpha \to 0$.

	Mean-absolute error at mid-points		**Mean-absolute error at ten points**	
$\boldsymbol{m}$	Proposed regularization-direct method	Direct method Without regularization	Proposed regularization-direct method	Direct method Without regularization
2	7.3 e−3	7.3 e−3	6.2 e−2	6.2 e−2
4	2.9 e−3	6.8 e−1	2.7 e−2	1.6 e0
8	1.4 e−3	3.8 e0	1.3 e−2	5.4 e0
16	7.0 e−4	1.7 e+1	6.3 e−3	1.6 e+1
32	3.5 e−4	7.0 e+1	3.0 e−3	3.4 e+1
64	1.7 e−4	2.9 e+2	1.5 e−3	1.4 e+2

Table 2: Mean-absolute error for the solution to test problem 6.2 at mid-points and ten specific points, for different m and $\alpha \to 0$.

	Mean-absolute error at mid-points		**Mean-absolute error at ten points**	
$\boldsymbol{m}$	Proposed regularization-direct method	Direct method Without regularization	Proposed regularization-direct method	Direct method Without regularization
2	6.2 e−3	2.5 e−1	9.2 e−2	1.3 e0
4	3.2 e−3	1.4 e0	4.6 e−2	4.2 e0
8	1.6 e−3	6.2 e0	2.3 e−2	1.2 e+1
16	8.2 e−4	2.5 e+1	1.1 e−2	2.5 e+1
32	4.1 e−4	1.0 e+2	5.7 e−3	1.0 e+2
64	2.1 e−4	4.1 e+2	2.8 e−3	4.1 e+2

Table 3: Mean-absolute error for the solution to test problem 6.3 at mid-points and ten specific points, for different m and $\alpha \to 0$.

	Mean-absolute error at mid-points		**Mean-absolute error at ten points**	
$\boldsymbol{m}$	Proposed regularization-direct method	Direct method Without regularization	Proposed regularization-direct method	Direct method Without regularization
2	9.1 e−3	6.5 e−1	1.4 e−1	3.3 e0
4	4.3 e−3	5.0 e0	6.7 e−2	1.5 e+1
8	2.1 e−3	3.9 e+1	3.3 e−2	7.7 e+1
16	1.0 e−3	3.1 e+2	1.6 e−2	3.1 e+2
32	5.2 e−4	2.5 e+3	8.1 e−3	2.5 e+3
64	2.6 e−4	2.0 e+4	4.1 e−3	2.0 e+4

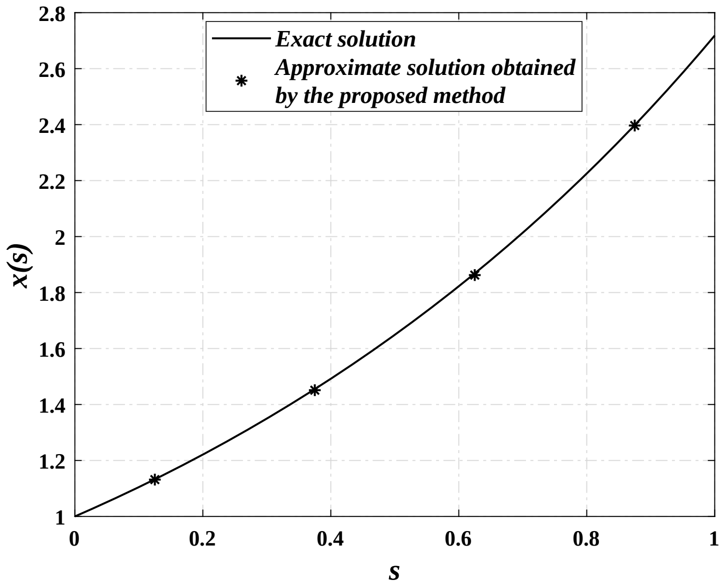

Figure 1: Numerical results for test problem 6.4 with $m = 4$ and $\alpha \to 0$, using the proposed method.

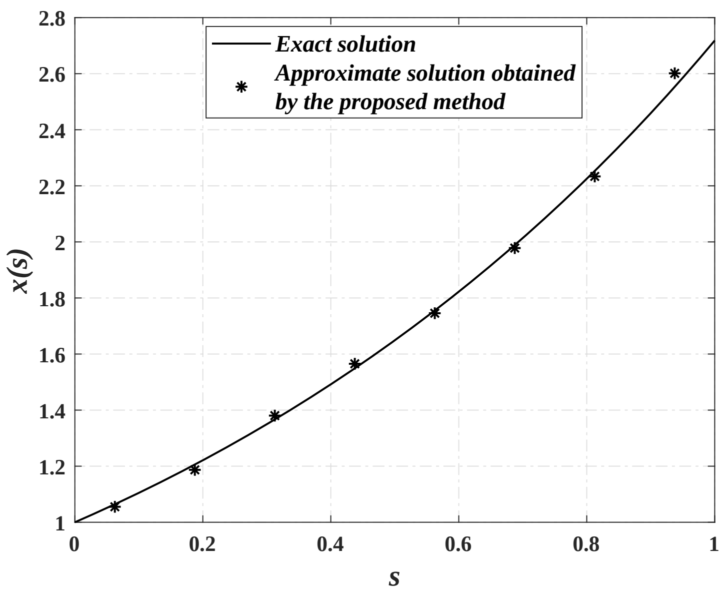

Figure 2: Numerical results for test problem 6.5 with $m = 8$ and $\alpha = 1\,e - 8$, using the proposed method.

7. Conclusion

We proposed a regularization-direct method to numerically solve first kind Fredholm integral equations using vector forms of BPFs and their related properties. The proposed method adopts a regularization scheme to overcome the integral equation's ill-posedness. Various test problems confirmed that the proposed method can provide stable and convergent numerical solutions with reasonable accuracy for first kind Fredholm integral equations.

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