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BOUNDS OF HANKEL DETERMINANTS FOR ANALYTIC FUNCTION

Bülent Nafi Örnek

Abstract. In this paper, we give estimates of the Hankel determinant $H_2(1)$ in a novel class $\mathcal{N}(\varepsilon)$ of analytical functions in the unit disc. In addition, the relation between the Fekete-Szegö function $H_2(1)$ and the module of the angular derivative of the analytical function $p(z)$ at a boundary point b of the unit disk will be given. In this association, the coefficients in the Hankel determinant b_2, b_3 and b_4 will be taken into consideration. Moreover, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

1. Introduction

Let p be an analytic function in the unit disc $E = \{z : |z| < 1\}$, $p(0) = 0$ and $p : E \to E$ with $p(z) = b_n z^n + ...$ In accordance with the classical Schwarz lemma, for any point z in the unit disc E, we have $|p(z)| \leq |z|^n$ for all $z \in E$ and $|b_n| \leq 1$. In addition, if the equality $|p(z)| = |z|^n$ holds for any $z \neq 0$, or $|b_n| = 1$, then p is a rotation; that is $p(z) = z^n e^{i\theta}$, θ real ([5], p.329). In electrical and electronics engineering, it is possible to encounter with applications of Schwarz lemma. As an example, the driving point impedance functions obtained as a result of

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boundary analysis of Schwarz lemma can be possibly used for circuit synthesis. Also, transfer functions in control theory and multi-notched filters in signals and systems can be considered as topics under the title of Schwarz lemma's applications [13, 14].

In order to prove our main results, we recall the following lemma [6].

Lemma 1.1 (Jack's lemma). *Let $p(z)$ be a non-constant anaytic function in E with $p(0) = 0$. If*

$$|p(z_0)| = \max\{|p(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 p'(z_0)}{p(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $p(z) = z + b_2 z^2 + b_3 z^3 + ...$ that are analytic in E. Also, let $\mathcal{N}(\varepsilon)$ be the subclass of $\mathcal{A}$ consisting of all functions $p(z)$ satisfying

$$\left|\frac{p(z)}{z}\left(\frac{2p(z)}{zp'(z)} + \frac{p(z)p''(z)}{(p'(z))^2} - 2\right)\right| < \frac{1-\varepsilon}{2\varepsilon^2},\ \frac{1}{2} \leq \varepsilon < 1,\ z \in E. \tag{1.1}$$

The certain analytic functions which is in the class of $\mathcal{N}(\varepsilon)$ on the unit disc E are considered in this paper. The subject of the present paper is to discuss some properties of the function $\phi(z)$ which belongs to the class of $\mathcal{N}(\varepsilon)$ by applying Jack's Lemma.

In this study, we give estimates of the Hankel determinant $H_2(1)$ in a novel class $\mathcal{N}(\varepsilon)$ of analytic functions in the unit disc. Moreover, the relationship between the coefficients of the hankel determinant and the angular derivative of the function p, which provides the class $\mathcal{N}(\varepsilon)$, will be examined.

Let $p \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [19] as

$$H_q(n) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix},\ b_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} = b_3 - b_2^2.$$

Here, the Hankel determinant $H_2(1) = b_3 - b_2^2$ is well-known as Fekete-Szegö functional [18]. In [19], the authors have obtained the upper bounds of the Hankel determinant $|b_2 b_4 - b_3^2|$. Also, in [16], the author have obtained the upper bounds the Hankel determinant $A_n^{(k)}$. Moreover, in [17], the authors have given bounds for the Second Hankel determinant for class $\mathcal{M}_\alpha$.

Let $p \in \mathcal{N}(\varepsilon)$ and consider the following function

$$\phi(z) = \frac{B(z) - 1}{B(z) + 1 - 2\varepsilon}, \tag{1.2}$$

where $\frac{z^2 p'(z)}{(p(z))^2} = B(z)$.

It is an analytic function in E and $\phi(0) = 0$. Now, let us show that $|\phi(z)| < 1$ in E. From (1.2), we have

$$B(z) = \frac{1 + (1 - 2\varepsilon)\,\phi(z)}{1 - \phi(z)}$$

If the logarithm differentiation of both sides is taken in the last equation, we obtain

$$\ln(B(z)) = \ln\left(\frac{1 + (1 - 2\varepsilon)\,\phi(z)}{1 - \phi(z)}\right)$$

and

$$2 + \frac{zp''(z)}{p'(z)} - 2\frac{zp'(z)}{p(z)} = \frac{2\,(1 - \varepsilon)\,z\phi'(z)}{(1 - \phi(z))\,(1 + (1 - 2\varepsilon)\,\phi(z))}.$$

If we multiply both sides of the last equation by $\frac{1}{B(z)}$, we take

$$\frac{p(z)}{z}\left(\frac{2p(z)}{zp'(z)} + \frac{p(z)p''(z)}{(p'(z))^2} - 2\right) = \frac{2\,(1 - \varepsilon)\,z\phi'(z)}{(1 + (1 - 2\varepsilon)\,\phi(z))^2}.$$

We assume that there exists a $z_0 \in E$ such that

$$\max_{|z| \le |z_0|} |\phi(z)| = |\phi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\phi(z_0) = e^{i\theta} \text{ and } \frac{z_0 \phi'(z_0)}{\phi(z_0)} = k.$$

Thus, we have that

$$\begin{aligned}\left|\frac{p(z_0)}{z_0}\left(\frac{2p(z_0)}{z_0p'(z_0)}+\frac{p(z_0)p''(z_0)}{(p'(z_0))^2}-2\right)\right| &= \left|\frac{2(1-\varepsilon)z_0\phi'(z_0)}{(1+(1-2\varepsilon)\phi(z_0))^2}\right| \\ &= \left|\frac{2(1-\varepsilon)k\phi(z_0)}{(1+(1-2\varepsilon)\phi(z_0))^2}\right| \\ &= \left|\frac{2(1-\varepsilon)ke^{i\theta}}{(1+(1-2\varepsilon)e^{i\theta})^2}\right| \\ &= \frac{2(1-\varepsilon)k}{|1+(1-2\varepsilon)e^{i\theta}|^2}\end{aligned}$$

Since $\left|1+(1-2\varepsilon)e^{i\theta}\right|^2 \leq (1+|1-2\varepsilon|)^2$ and $\frac{1}{2}\leq\varepsilon<1$, we take

$$\left|\frac{p(z_0)}{z_0}\left(\frac{2p(z_0)}{z_0p'(z_0)}+\frac{p(z_0)p''(z_0)}{(p'(z_0))^2}-2\right)\right| \geq \frac{1-\varepsilon}{2\varepsilon^2}.$$

This contradicts the $p\in\mathcal{N}(\varepsilon)$. This means that there is no point $z_0\in E$ such that $\max_{|z|\leq|z_0|}|\phi(z)|=|\phi(z_0)|=1$. Hence, we take $|\phi(z)|<1$ in E. From the Schwarz lemma, we obtain

$$\phi(z)=\frac{B(z)-1}{B(z)+1-2\varepsilon}=\frac{(b_3-b_2^2)z^2+(2b_4-4b_2b_3+2b_2^3)z^3+...}{2(1-\varepsilon)+(b_3-b_2^2)z^2+(2b_4-4b_2b_3+2b_2^3)z^3+...},$$

$$\frac{\phi(z)}{z^2}=\frac{(b_3-b_2^2)+(2b_4-4b_2b_3+2b_2^3)z+...}{2(1-\varepsilon)+(b_3-b_2^2)z^2+(2b_4-4b_2b_3+2b_2^3)z^3+...}$$

and

$$\frac{|b_3-b_2^2|}{2(1-\varepsilon)}=\frac{|H_2(1)|}{2(1-\varepsilon)}\leq 1.$$

We thus obtain the following lemma.

Lemma 1.2. *If $p\in\mathcal{N}(\varepsilon)$, then we have the inequality*

$$|H_2(1)|\leq 2(1-\varepsilon). \tag{1.3}$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [11]:

Lemma 1.3. *If* $p(z) = b_n z^n + b_{n+1} z^{n+1} + \ldots$ *extends continuously to some boundary point* b *with* $|b| = 1$, *and if* $|p(b)| = 1$ *and* $p'(b)$ *exists, then*

$$|p'(b)| \geq n + \frac{1 - |b_n|}{1 + |b_n|} \tag{1.4}$$

and

$$|p'(b)| \geq n. \tag{1.5}$$

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4,7,9–13]. Mercer [8] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtained a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [15]).

Lemma 1.4 (Julia-Wolff lemma). *Let* p *be an analytic function in* E, $p(0) = 0$ *and* $p(E) \subset E$. *If, in addition, the function* f *has an angular limit* $p(b)$ *at* $b \in \partial E$, $|p(b)| = 1$, *then the angular derivative* $p'(b)$ *exists and* $1 \leq |p'(b)| \leq \infty$.

Corollary 1.5. *The analytic function* p *has a finite angular derivative* $p'(b)$ *if and only if* p' *has the finite angular limit* $p'(b)$ *at* $b \in \partial E$.

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma and the Hankel determinant for $\mathcal{N}(\varepsilon)$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. In addition, the relation between the Fekete-Szeg function $H_2(1)$ and the module of the angular derivative of the analytical function $p(z)$ at a boundary point b of the unit disk will be given. In this association, the coefficients in the Hankel determinant b_2, b_3 and b_4 will be taken into consideration.

THEOREM 2.1. *Let $p \in \mathcal{N}(\varepsilon)$. Assume that, for some $b \in \partial E$, p has an angular limit $p(b)$ at b, $p(b) = \frac{b}{\varepsilon}$ and $p'(b) = \frac{1}{\varepsilon}$. Then we have the inequality*

$$|p''(b)| \geq \frac{|H_2(1)|}{2\varepsilon^2}. \tag{2.1}$$

Proof. Consider the function

$$\phi(z) = \frac{B(z) - 1}{B(z) + 1 - 2\varepsilon}.$$

In addition, since $p(b) = \frac{b}{\varepsilon}$ and $p'(b) = \frac{1}{\varepsilon}$, we have

$$B(b) = \frac{b^2 p'(b)}{(p(b))^2} = \frac{b^2 \frac{1}{\varepsilon}}{\left(\frac{b}{\varepsilon}\right)^2} = \varepsilon$$

and

$$\phi(b) = \frac{B(b) - 1}{B(b) + 1 - 2\varepsilon} = \frac{\varepsilon - 1}{\varepsilon + 1 - 2\varepsilon} = -1$$

$$|\phi(b)| = 1.$$

So, from (1.5) for $n = 2$, we obtain

$$2 \leq |\phi'(b)| = \frac{2(1-\varepsilon)|B'(b)|}{|B(b) + 1 - 2\varepsilon|^2}.$$

Since

$$B'(z) = \frac{(2zp'(z) + p''(z)z^2)\, p^2(z) - 2p(z)\,(p'(z))^2 z^2}{(p(z))^4}$$

and

$$\begin{aligned} B'(b) &= \frac{(2bp'(b) + p''(b)b^2)\, p^2(b) - 2p(b)\,(p'(b))^2 b^2}{(p(b))^4} \\ &= \varepsilon^2 p''(b), \end{aligned}$$

we take

$$2 \leq |\phi'(b)| = \frac{2(1-\varepsilon)\,\varepsilon^2\,|p''(b)|}{|\varepsilon + 1 - 2\varepsilon|^2} = \frac{2\varepsilon^2\,|p''(b)|}{1-\varepsilon}$$

and

$$|p''(b)| \geq \frac{1-\varepsilon}{\varepsilon^2}.$$

Also, since $|H_2(1)| \leq 2(1-\varepsilon)$, we obtain

$$|p''(b)| \geq \frac{|H_2(1)|}{2\varepsilon^2}.$$

□

THEOREM 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|p''(b)| \geq \frac{|H_2(1)|}{4\varepsilon^2}\left(1+\frac{4(1-\varepsilon)}{2(1-\varepsilon)+|H_2(1)|}\right). \tag{2.2}$$

Proof. Let $\phi(z)$ function be the same as (1.2). So, from (1.4) for $n=2$, we obtain

$$2+\frac{1-|a_2|}{1+|a_2|} \leq |\phi'(b)| = \frac{2\varepsilon^2|p''(b)|}{1-\varepsilon},$$

where $|a_2| = \frac{|\phi''(0)|}{2!} = \frac{|H_2(1)|}{2(1-\varepsilon)}$.

Therefore, we take

$$2+\frac{1-\frac{|H_2(1)|}{2(1-\varepsilon)}}{1+\frac{|H_2(1)|}{2(1-\varepsilon)}} \leq \frac{2\varepsilon^2|p''(b)|}{1-\varepsilon},$$

$$2+\frac{2(1-\varepsilon)-|H_2(1)|}{2(1-\varepsilon)+|H_2(1)|} \leq \frac{2\varepsilon^2|p''(b)|}{1-\varepsilon},$$

and

$$\left(1+\frac{4(1-\varepsilon)}{2(1-\varepsilon)+|H_2(1)|}\right)\frac{1-\varepsilon}{2\varepsilon^2} \leq |p''(b)|.$$

Moreover, since $|H_2(1)| \leq 2(1-\varepsilon)$, we obtain

$$|p''(b)| \geq \frac{|H_2(1)|}{4\varepsilon^2}\left(1+\frac{4(1-\varepsilon)}{2(1-\varepsilon)+|H_2(1)|}\right).$$

□

The inequality (2.2) can be strengthened as below by taking into account b_4 which is the coefficient in the expansion of the function $p(z) = z+b_2z^2+b_3z^3+...$

THEOREM 2.3. *Let $p \in \mathcal{N}(\varepsilon)$. Assume that, for some $b \in \partial E$, p has an angular limit $p(b)$ at b, $p(b)=\frac{b}{\varepsilon}$ and $p'(b)=\frac{1}{\varepsilon}$. Then we have the inequality*

$$|p''(b)| \geq \frac{|H_2(1)|}{2\varepsilon^2}\left(1+\frac{(2(1-\varepsilon)-|H_2(1)|)^2}{2\left(4(1-\varepsilon)^2-|H_2(1)|^2+4(1-\varepsilon)\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|\right)}\right). \tag{2.3}$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1 and $f(z) = z^2$. By the maximum principle, for each $z \in E$, we have the inequality $|\phi(z)| \leq |f(z)|$. So,

$$\begin{aligned} g(z) &= \frac{\phi(z)}{f(z)} = \frac{B(z)-1}{(B(z)+1-2\varepsilon)\,z^2} \\ &= \frac{(b_3-b_2^2)+(2b_4-4b_2b_3+2b_2^3)\,z+...}{2\,(1-\varepsilon)+(b_3-b_2^2)\,z^2+(2b_4-4b_2b_3+2b_2^3)\,z^3+...} \end{aligned}$$

is analytic function in E and $|g(z)| \leq 1$ for $z \in E$. In particular, we have

$$|g(0)| = \frac{|b_3-b_2^2|}{2\,(1-\varepsilon)} = \frac{|H_2(1)|}{2\,(1-\varepsilon)} \tag{2.4}$$

and

$$|g'(0)| = \frac{|2b_4-4b_2b_3+2b_2^3|}{2\,(1-\varepsilon)} = \frac{|b_4-b_2\,(b_2^2+2H_2(1))|}{(1-\varepsilon)}.$$

Furthermore, the geometric meaning of the derivative and the inequality $|\phi(z)| \leq |f(z)|$ imply the inequality

$$\frac{b\phi'(b)}{\phi(b)} = |\phi'(b)| \geq |f'(b)| = \frac{bf'(b)}{f(b)}.$$

The composite function

$$w(z) = \frac{g(z)-g(0)}{1-\overline{g(0)}g(z)}$$

is analytic in E, $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $|w(b)| = 1$ for $b \in \partial E$. For $n = 1$, from (1.4), we obtain

$$\begin{aligned} \frac{2}{1+|w'(0)|} &\leq |w'(b)| = \frac{1-|g(0)|^2}{\left|1-\overline{g(0)}g(b)\right|^2}\,|g'(b)| \\ &\leq \frac{1+\frac{|H_2(1)|}{2(1-\varepsilon)}}{1-\frac{|H_2(1)|}{2(1-\varepsilon)}}\,\{|\phi'(b)|-|f'(b)|\} \\ &= \frac{2\,(1-\varepsilon)+|H_2(1)|}{2\,(1-\varepsilon)-|H_2(1)|}\left(\frac{2\varepsilon^2\,|p''(b)|}{1-\varepsilon}-2\right). \end{aligned}$$

Since

$$w'(z) = \frac{1-|g(0)|^2}{\left(1-\overline{g(0)}g(z)\right)^2} g'(z)$$

and

$$|w'(0)| = \frac{|g'(0)|}{1-|g(0)|^2} = \frac{\frac{\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|}{(1-\varepsilon)}}{1-\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)^2} = 4(1-\varepsilon)\frac{|b_4-b_2(b_2^2+2H_2(1))|}{4(1-\varepsilon)^2-|H_2(1)|^2},$$

we obtain

$$\frac{2}{1+4(1-\varepsilon)\frac{\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|}{4(1-\varepsilon)^2-|H_2(1)|^2}} \leq \frac{2(1-\varepsilon)+|H_2(1)|}{2(1-\varepsilon)-|H_2(1)|}\left(\frac{2\varepsilon^2|p''(b)|}{1-\varepsilon}-2\right),$$

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2+\frac{(2(1-\varepsilon)-|H_2(1)|)^2}{4(1-\varepsilon)^2-|H_2(1)|^2+4(1-\varepsilon)\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|}\right)$$

and

$$|p''(b)| \geq \frac{1-\varepsilon}{\varepsilon^2}\left(1+\frac{(2(1-\varepsilon)-|H_2(1)|)^2}{2\left(4(1-\varepsilon)^2-|H_2(1)|^2+4(1-\varepsilon)\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|\right)}\right).$$

Since $|H_2(1)| \leq 2(1-\varepsilon)$, we obtain the inequality (2.3). □

If $p(z)-z$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

THEOREM 2.4. *Let $p(z) \in \mathcal{N}(\varepsilon)$ and $b_3 > b_2^2$ $(b_2 > 0, b_3 > 0)$. Also, $p(z)-z$ has no zeros in E except $z=0$. Assume that, for some $b \in \partial E$, p has an angular limit $p(b)$ at b, $p(b) = \frac{b}{\varepsilon}$ and $p'(b) = \frac{1}{\varepsilon}$. Then we have the inequality*

(2.5)

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2-\frac{\ln^2\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|-|b_4-b_2(b_2^2+2H_2(1))|}\right)$$

and

$$\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right| \leq \left|H_2(1)\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)\right|. \tag{2.6}$$

Proof. Let $b_3 > b_2^2$ and $\phi(z)$, $g(z)$ be as in the proof of Theorem 2.3. Having in mind inequality (2.4), we denote by $\ln g(z)$ the analytic branch of the logarithm normed by the condition

$$\ln g(0) = \ln\left(\frac{|H_2(1)|}{2\,(1-\varepsilon)}\right) < 0.$$

The function

$$d(z) = \frac{\ln g(z) - \ln g(0)}{\ln g(z) + \ln g(0)}$$

is analytic in the unit disc E, $|d(z)| < 1$ for $z \in E$, $d(0) = 0$ and $|d(b)| = 1$ for $b \in \partial E$. From (1.4) for $n = 1$, we obtain

$$\begin{aligned} \frac{2}{1+|d'(0)|} &\leq |d'(b)| = \frac{|2\ln g(0)|}{|\ln g(b) + \ln g(0)|^2}\left|\frac{g'(b)}{g(b)}\right| \\ &= \frac{-2\ln g(0)}{\ln^2 g(0) + \arg^2 g(b)}\left\{|\phi'(b)| - |f'(b)|\right\}. \end{aligned}$$

Since

$$\begin{aligned} |d'(0)| &= \frac{1}{|2\ln g(0)|}\left|\frac{g'(0)}{g(0)}\right| = \frac{-1}{2\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)} \frac{\frac{\left|b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right|}{(1-\varepsilon)}}{\frac{|H_2(1)|}{2(1-\varepsilon)}} \\ &= \frac{-1}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)} \frac{\left|b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right|}{|H_2(1)|}, \end{aligned}$$

we take

$$\frac{1}{1 - \frac{\left|b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right|}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|}} \leq \frac{-\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)}{\ln^2\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right) + \arg^2 g(b)}\left(\frac{2\varepsilon^2\,|p''(b)|}{1-\varepsilon} - 2\right).$$

Replacing $\arg^2 g(b)$ by zero, we take

$$\frac{1}{1 - \frac{\left|b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right|}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|}} \leq \frac{-1}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)}\left(\frac{2\varepsilon^2\,|p''(b)|}{1-\varepsilon} - 2\right),$$

$$2 - \frac{\ln^2\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)| - \left|b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right|} \leq \frac{2\varepsilon^2\,|p''(b)|}{1-\varepsilon}$$

and

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2-\frac{\ln^2\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)|H_2(1)|-|b_4-b_2(b_2^2+2H_2(1))|}\right).$$

Similarly, the function $g(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$\begin{aligned} 1 &\geq |d'(0)| = \frac{|2\ln g(0)|}{|\ln g(0)+\ln g(0)|^2}\left|\frac{g'(0)}{g(0)}\right| = \frac{-1}{2\ln g(0)}\left|\frac{g'(0)}{g(0)}\right| \\ &= \frac{-1}{2\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)}\frac{\frac{\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right|}{(1-\varepsilon)}}{\frac{|H_2(1)|}{2(1-\varepsilon)}} \\ &= \frac{-1}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)}\frac{|b_4-b_2(b_2^2+2H_2(1))|}{|H_2(1)|} \end{aligned}$$

and

$$\left|b_4-b_2\left(b_2^2+2H_2(1)\right)\right| \leq \left|H_2(1)\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)\right|.$$

□

Theorem 2.5. *Under hypotheses of Theorem 2.4, we have*

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2-\frac{1}{2}\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)\right). \tag{2.7}$$

Proof. From the proof of Theorem 2.4, using the inequality (1.5) for the function $g(z)$, for $p=1$ we obtain

$$1 \leq |d'(b)| = \frac{|2\ln g(0)|}{|\ln g(b)+\ln g(0)|^2}\left|\frac{g'(b)}{g(b)}\right| = \frac{-2}{\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)}\left(\frac{2\varepsilon^2|p''(b)|}{1-\varepsilon}-2\right)$$

and

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2-\frac{1}{2}\ln\left(\frac{|H_2(1)|}{2(1-\varepsilon)}\right)\right).$$

□

If $p(z)-z$ a have zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.6. *Let $p \in \mathcal{N}(\varepsilon)$. Assume that, for some $b \in \partial E$, p has an angular limit $p(b)$ at b, $p(b) = \frac{b}{\varepsilon}$ and $p'(b) = \frac{1}{\varepsilon}$. Let $z_1, z_2, ..., z_n$ be zeros of the function $p(z) - z$ in E that are different from zero. Then we have the inequality*

$$|p''(b)| \geq \frac{1-\varepsilon}{2\varepsilon^2} \left(2 + \sum_{i=1}^{n} \frac{1-|z_i|^2}{|b-z_i|^2} \right. \tag{2.8}$$

$$\left. + \frac{2\left(\prod_{i=1}^{n}|z_i| - |H_2(1)|\right)^2}{\left(2(1-\varepsilon)\prod_{i=1}^{n}|z_i|\right)^2 - |H_2(1)|^2 + 2(1-\varepsilon)\prod_{i=1}^{n}|z_i| \left| 2\left(b_4 - b_2\left(b_2^2 + 2H_2(1)\right)\right) + H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i} \right|} \right).$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1 and $z_1, z_2, ..., z_n$ be zeros of the function $p(z) - z$ in E that are different from zero. Let

$$t(z) = z^2 \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}.$$

$t(z)$ is an analytic function in E and $|t(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in E$, we have $|\phi(z)| \leq |t(z)|$. Consider the function

$$\begin{aligned} r(z) &= \frac{\phi(z)}{t(z)} = \left[\frac{B(z)-1}{B(z)+1-2\varepsilon}\right] \frac{1}{z^2 \prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}} \\ &= \frac{\left(b_3 - b_2^2\right) z^2 + \left(2b_4 - 4b_2 b_3 + 2b_2^3\right) z^3 + ...}{\left(2(1-\varepsilon) + \left(b_3 - b_2^2\right) z^2 + \left(2b_4 - 4b_2 b_3 + 2b_2^3\right) z^3 + ...\right) z^2 \prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}, \\ &= \frac{\left(b_3 - b_2^2\right) + \left(2b_4 - 4b_2 b_3 + 2b_2^3\right) z + ...}{\left(2(1-\varepsilon) + \left(b_3 - b_2^2\right) z^2 + \left(2b_4 - 4b_2 b_3 + 2b_2^3\right) z^3 + ...\right) \prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}. \end{aligned}$$

$r(z)$ is analytic in E and $|r(z)| < 1$ for $z \in E$. In particular, we have

$$|r(0)| = \frac{|b_3 - b_2^2|}{2(1-\varepsilon)\prod_{i=1}^{n}|z_i|} = \frac{|H_2(1)|}{2(1-\varepsilon)\prod_{i=1}^{n}|z_i|}$$

and

$$|r'(0)| = \frac{\left| 2b_4 - 4b_2 b_3 + 2b_2^3 + \left(b_3 - b_2^2\right)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i} \right|}{2(1-\varepsilon)\prod_{i=1}^{n}|z_i|}.$$

Moreover, with the simple calculations, we get

$$\frac{b\phi'(b)}{\phi(b)} = |\phi'(b)| \geq |t'(b)| = \frac{bt'(b)}{t(b)}$$

and

$$|t'(b)| = 2 + \sum_{i=1}^{n} \frac{1-|z_i|^2}{|b-z_i|^2}.$$

The auxiliary function

$$\varkappa(z) = \frac{r(z)-r(0)}{1-\overline{r(0)}r(z)}$$

is analytic in the unit disc E, $\varkappa(0) = 0$, $|\varkappa(z)| < 1$ for $z \in E$ and $|\varkappa(b)| = 1$ for $b \in \partial E$. From (1.3) for $n = 1$, we obtain

$$\begin{aligned} \frac{2}{1+|\varkappa'(0)|} &\leq |\varkappa'(b)| = \frac{1-|r(0)|^2}{\left|1-\overline{r(0)}r(b)\right|^2}|r'(b)| \\ &\leq \frac{1+|r(0)|}{1-|r(0)|}\left|\frac{\phi'(b)}{t(b)} - \frac{\phi(b)t'(b)}{t^2(b)}\right| \\ &= \frac{1+|r(0)|}{1-|r(0)|}\left\{|\phi'(b)| - |t'(b)|\right\}. \end{aligned}$$

Since

$$\begin{aligned} |\varkappa'(0)| &= \frac{|r'(0)|}{1-|r(0)|^2} = \frac{\frac{\left|2b_4-4b_2b_3+2b_2^3+\left(b_3-b_2^2\right)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{2(1-\varepsilon)\prod_{i=1}^{n}|z_i|}}{1-\left(\frac{|H_2(1)|}{2(1-\varepsilon)\prod_{i=1}^{n}|z_i|}\right)^2} \\ &= 2(1-\varepsilon)\prod_{i=1}^{n}|z_i|\frac{\left|2b_4-4b_2b_3+2b_2^3+(b_3-b_2^2)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(2(1-\varepsilon)\prod_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2}, \end{aligned}$$

we get

$$\frac{2}{1+2(1-\varepsilon)\prod_{i=1}^{n}|z_i|\frac{\left|2b_4-4b_2b_3+2b_2^3+\left(b_3-b_2^2\right)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(2(1-\varepsilon)\prod_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2}} \leq$$

$$\frac{1+\frac{|H_2(1)|}{2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|}}{1-\frac{|H_2(1)|}{2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|}}\left\{\frac{2\varepsilon^2|p''(b)|}{1-\varepsilon}-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right\},$$

$$\frac{\left(2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|^2-|H_2(1)|\right)^2}{\left(2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|\left|2b_4-4b_2b_3+2b_2^3+\left(b_3-b_2^2\right)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq\left\{\frac{2\varepsilon^2|p''(b)|}{1-\varepsilon}-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right\},$$

and

$$|p''(b)|\geq\frac{1-\varepsilon}{2\varepsilon^2}\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|b-z_i|^2}\right.$$

$$\left.+\frac{\left(2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|^2-|H_2(1)|\right)^2}{\left(2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+2(1-\varepsilon)\prod\limits_{i=1}^{n}|z_i|\left|2b_4-4b_2b_3+2b_2^3+\left(b_3-b_2^2\right)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right). \quad\square$$

In the following theorem, we give the estimate of the boundary Schwarz lemma involving the boundary fixed point.

THEOREM 2.7. *Let $p\in\mathcal{N}(\varepsilon)$. Assume that, for $-1\in\partial E$, p has an angular limit $p(-1)$ at b, $p(-1)=\frac{-1}{\varepsilon}$ and $p'(-1)=\frac{1}{\varepsilon}$. Then we have the inequality*

(2.9)

$$p''(-1)\geq\frac{1-\varepsilon}{2\varepsilon^2}\left(2+\frac{|2(1-\varepsilon)+H_2(1)|^2}{4(1-\varepsilon)^2-|H_2(1)|^2}\frac{2}{1+\Re\left(\frac{2(1-\varepsilon)+\overline{H_2(1)}}{2(1-\varepsilon)+H_2(1)}\frac{b_4-b_2\left(b_2^2+2H_2(1)\right)}{1-\left|\frac{H_2(1)}{2(1-\varepsilon)}\right|^2}\right)}\right).$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. Therefore, from the assumptions, we have

$$\phi(z)=\frac{B(z)-1}{B(z)+1-2\varepsilon},\ B(z)=\frac{z^2p'(z)}{(p(z))^2}$$

and

$$\phi(-1)=-1,$$

where $b=-1$ is a boundary fixed point of $\phi(z)$. Also, we have

$$\begin{aligned}\phi(z) &= \frac{B(z)-1}{B(z)+1-2\varepsilon} = \frac{\left(b_3-b_2^2\right)z^2+\left(2b_4-4b_2b_3+2b_2^3\right)z^3+...}{2\left(1-\varepsilon\right)+\left(b_3-b_2^2\right)z^2+\left(2b_4-4b_2b_3+2b_2^3\right)z^3+...}\\ &= \frac{\left(b_3-b_2^2\right)}{2\left(1-\varepsilon\right)}z^2+\left(2b_4-4b_2b_3+2b_2^3\right)z^3+....\\ &= c_2z^2+c_3z^3+c_4z^4+...\end{aligned}$$

Consider the function

$$\varphi(z) = -\frac{1+\overline{c_2}}{1+c_2}\frac{c_2z^2-\phi(z)}{z^2-\overline{c_2}\phi(z)}.$$

$\varphi(z)$ is analytic in E and $|\varphi(z)|<1$ for $z\in E$ and $b=-1$ is a boundary fixed point of $\varphi(z)$. That is, $\varphi(-1)=-1$. Moreover, with the simple calculations, we obtain

$$\varphi'(-1) = \frac{1-|c_2|^2}{|1+c_2|^2}\left(\phi'(-1)-2\right).$$

On the other hand, we get

$$\begin{aligned}\varphi(z) &= -\frac{1+\overline{c_2}}{1+c_2}\frac{c_2z^2-\phi(z)}{z^2-\overline{c_2}\phi(z)}\\ &= -\frac{1+\overline{c_2}}{1+c_2}\frac{c_2z^2-\left(c_2z^2+c_3z^3+c_4z^4+...\right)}{z^2-\overline{c_2}\left(c_2z^2+c_3z^3+c_4z^4+...\right)}\\ &= -\frac{1+\overline{c_2}}{1+c_2}\frac{-c_3z^3-c_4z^4-...}{z^2-|c_2|^2z^2-\overline{c_2}c_3z^3-...}\\ &= -\frac{1+\overline{c_2}}{1+c_2}\frac{-c_3z-c_4z^2-...}{1-|c_2|^2-\overline{c_2}c_3z-},\end{aligned}$$

and

$$\varphi'(0) = \frac{1+\overline{c_2}}{1+c_2}\frac{c_3}{1-|c_2|^2}.$$

In particular, from (1.4) for $n=1$, we have

$$\varphi'(-1) \geq \frac{2}{1+\Re\varphi'(0)}. \tag{2.10}$$

Let us substitute the values of $\varphi'(-1)$ and $\varphi'(0)$ into (2.10). Therefore, we take

$$\frac{1-|c_2|^2}{|1+c_2|^2}\left(\phi'(-1)-2\right) \geq \frac{2}{1+\Re\left(\frac{1+\overline{c_2}}{1+c_2}\frac{c_3}{1-|c_2|^2}\right)}$$

and

$$\phi'(-1) \geq 2+\frac{|1+c_2|^2}{1-|c_2|^2}\frac{2}{1+\Re\left(\frac{1+\overline{c_2}}{1+c_2}\frac{c_3}{1-|c_2|^2}\right)}.$$

Since

$$\phi'(-1)=\frac{2\varepsilon^2 p''(-1)}{1-\varepsilon},\ c_2=\frac{b_3-b_2^2}{2(1-\varepsilon)},\ c_3=2b_4-4b_2b_3+2b_2^3, H_2(1)=b_3-b_2^2,$$

we obtain

$$\frac{2\varepsilon^2 p''(-1)}{1-\varepsilon} \geq 2+\frac{\left|1+\frac{b_3-b_2^2}{2(1-\varepsilon)}\right|^2}{1-\left|\frac{b_3-b_2^2}{2(1-\varepsilon)}\right|^2}\frac{2}{1+\Re\left(\frac{1+\overline{\frac{b_3-b_2^2}{2(1-\varepsilon)}}}{1+\frac{\left(b_3-b_2^2\right)}{2(1-\varepsilon)}}\frac{2b_4-4b_2b_3+2b_2^3}{1-\left|\frac{b_3-b_2^2}{2(1-\varepsilon)}\right|^2}\right)}$$

and

$$p''(-1) \geq \frac{1-\varepsilon}{2\varepsilon^2}\left(2+\frac{|2(1-\varepsilon)+H_2(1)|^2}{4(1-\varepsilon)^2-|H_2(1)|^2}\frac{2}{1+\Re\left(\frac{2(1-\varepsilon)+\overline{H_2(1)}}{2(1-\varepsilon)+H_2(1)}\frac{b_4-b_2\left(b_2^2+2H_2(1)\right)}{1-\left|\frac{H_2(1)}{2(1-\varepsilon)}\right|^2}\right)}\right).$$

□

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Bülent Nafi Örnek
Department of Computer Engineering
Amasya University
Merkez-Amasya, 05100, Turkey
E-mail: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr