Korean J. Math. **28** (2020), No. 4, pp. 717–737
http://dx.doi.org/10.11568/kjm.2020.28.4.717

ON WEIGHTED GENERALIZATION OF OPIAL TYPE INEQUALITIES IN TWO VARIABLES

Hüseyin Budak*, Mehmet Zeki Sarikaya, and Artion Kashuri

Abstract. In this paper, we establish some weighted generalization of Opial type inequalities in two independent variables for two functions. We also obtain weighted Opial type inequalities by using p-norms. Special cases of our results reduce to the inequalities in earlier study.

1. Introduction

In the year 1960, Opial established the following interesting integral inequality [11]:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\,dt \le \frac{h}{4}\int_0^h (x'(t))^2\,dt. \tag{1}$$

The constant $h/4$ is the best possible

Over the years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and

Received February 27, 2020. Revised October 27, 2020. Accepted November 5, 2020.
2010 Mathematics Subject Classification: 26D15, 26D10, 26B15.
Key words and phrases: Opial inequality, Hölder's inequality.
∗ Corresponding author.

discrete analogues of Opial inequality and its generalizations, for some of them please see [5], [6], [9], [12]- [15], [25]- [27].

In [28], Yang proved the following Opial type inequalities in two variables:

THEOREM 1.2. *If $f(t,s)$, $f_1(t,s)$ and $f_{12}(t,s)$ are continuous functions on $[a,b]\times[c,d]$ and if $f(a,s)=f(b,s)=f_1(t,c)=f_1(t,d)=0$ for $a\le t\le b$, $c\le s\le d$, then*

$$\int_a^b\int_c^d |f(t,s)|\,|f_{12}(t,s)|\,dsdt \le \frac{(b-a)(d-c)}{8}\int_a^b\int_c^d |f_{12}(t,s)|^2\,dsdt \tag{2}$$

where

$$f_1(t,s)=\frac{\partial}{\partial t}f(t,s) \text{ and } f_{12}(t,s)=\frac{\partial^2}{\partial t\partial s}f(t,s).$$

On the other hand, B. G. Pachpatte published some papers which focus on the generalizations of the inequality (2). For some of these generalizations, please see [16]- [21]. Moreover using two functions and their partial derivatives, W. S. Cheung established some generalizations of the inequality (2) in [7]. For the other Opial type inequalities in higher dimension, please see [1], [4], [8], [22]- [24].

Recently, Budak and Sarikaya have proved the following generalized Opial type inequalities in [2].

THEOREM 1.3. *Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12}, f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$ we have*

$$\begin{aligned} &\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \\ \le\ & \frac{1}{4}\left(\int_a^b\int_c^d \left[(b-a)Q(s,y)+(d-c)P(t,x)\right]|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}} \\ &\times\left(\int_a^b\int_c^d \left[(b-a)|y-s|+(d-c)|x-t|\right]|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}} \end{aligned} \tag{3}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt\right.$$
$$\left.+\int_a^b\int_c^d [(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^2\,dsdt\right]$$

where

$$P(t,x)=\begin{cases} t-a, & a\leq t\leq x \\ b-t, & x\leq t\leq b\end{cases} \quad \text{and} \quad Q(s,y)=\begin{cases} s-c, & c\leq s\leq y \\ d-s, & y\leq s\leq d.\end{cases}$$

The aim of this paper is to establish some weighted generalization of Opial type inequalities in two independent variables.

2. Weighted Generalization of Opial type inequalities

In this section, we obtain some weighted Opial type inequalities for two functions. Then we also give a new weighted Opial type inequality involving p-norms.

Theorem 2.1. *Let $w:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable and nonnegative function. Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12},f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$, we have*

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{4}$$
$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^y w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^x\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^2\,dsdt\right.$$

$$+\int_a^b\int_c^d\left[\left|\int_a^b\int_s^y w(u,v)dvdu\right|\right.$$

$$\left.\left.+\left|\int_t^x\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^2\,dsdt\right]$$

where $P(t,x)$ and $Q(s,y)$ are defined by as in Theorem 1.3.

Proof. In order to prove Theorem 2.1, we consider the following four cases:

Case I: Let $g(a,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$.

Since $g(a,s)=g_1(t,c)=0$, we can write

$$g(t,s)=\int_a^t\int_c^s g_{12}(u,v)dvdu.$$

Then, by Cauchy-Schwarz inequality, we have

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{5}$$

$$=\int_a^b\int_c^d (t-a)^{\frac{1}{2}}(s-c)^{\frac{1}{2}}\,|f_{12}(t,s)|$$

$$\times(t-a)^{-\frac{1}{2}}(s-c)^{-\frac{1}{2}}\,|g(t,s)|\,w(t,s)dsdt$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (s-c)^{\frac{1}{2}} |f_{12}(t,s)| (t-a)^{-\frac{1}{2}} (s-c)^{-\frac{1}{2}}$$
$$\times \left| \int_a^t \int_c^s g_{12}(u,v) dv du \right| w(t,s) ds dt$$
$$\leq \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_a^b \int_c^d (t-a)^{-1} (s-c)^{-1} \left| \int_a^t \int_c^s g_{12}(u,v) du dv \right|^2 w(t,s) ds dt \right)^{\frac{1}{2}}.$$

By applying again the Cauchy-Schwarz inequality, we get

$$(6) \qquad (t-a)^{-1} (s-c)^{-1} \left| \int_a^t \int_c^s g_{12}(u,v) du dv \right|^2$$
$$\leq (t-a)^{-1} (s-c)^{-1} \left(\int_a^t \int_c^s du dv \right) \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 du dv \right)$$
$$= \int_a^t \int_c^s |g_{12}(u,v)|^2 du dv.$$

Substituting the inequality (6) in (5), we obtain

$$(7) \qquad \int_a^b \int_c^d |f_{12}(t,s) g(t,s)| w(t,s) ds dt$$
$$\leq \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 w(t,s) ds dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_a^b \int_c^d \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 du dv \right) w(t,s) ds dt \right)^{\frac{1}{2}}.$$

By the integration by parts, one can show that

$$\begin{aligned}
(8)\qquad & \int_a^b\int_c^d\left(\int_a^t\int_c^s |g_{12}(u,v)|^2\,dvdu\right)w(t,s)dsdt \\
&= \int_a^b\left[\left(\int_a^t\int_c^s |g_{12}(u,v)|^2\,dvdu\right)\left(\int_c^s w(t,y)dy\right)\Bigg|_c^d\right. \\
&\quad \left.-\int_c^d\left(\int_a^t |g_{12}(u,s)|^2\,du\right)\left(\int_c^s w(t,y)dy\right)ds\right]dt \\
&= \int_a^b\left(\int_a^t\int_c^d |g_{12}(u,v)|^2\,dvdu\right)\left(\int_c^d w(t,y)dy\right)dt \\
&\quad -\int_a^b\int_c^d\left(\int_a^t |g_{12}(u,s)|^2\,du\right)\left(\int_c^s w(t,y)dy\right)dsdt \\
&= \left(\int_a^t\int_c^d |g_{12}(u,v)|^2\,dvdu\right)\left(\int_a^t\int_c^d w(x,y)dydx\right)\Bigg|_a^b \\
&\quad -\int_a^b\int_c^d |g_{12}(t,v)|^2\left(\int_a^t\int_c^d w(x,y)dydx\right)dvdt \\
&\quad -\int_c^d\left[\left(\int_a^t |g_{12}(u,s)|^2\,du\right)\left(\int_a^t\int_c^s w(x,y)dydx\right)\Bigg|_a^b\right. \\
&\quad \left.-\int_a^b |g_{12}(t,s)|^2\left(\int_a^t\int_c^s w(x,y)dydx\right)dt\right]ds \\
&= \int_a^b\int_c^d |g_{12}(u,v)|^2\left(\int_a^b\int_c^d w(x,y)dydx\right)dvdu \\
&\quad -\int_a^b\int_c^d |g_{12}(u,s)|^2\left(\int_a^b\int_c^s w(x,y)dydx\right)dsdu \\
&\quad -\int_a^b\int_c^d |g_{12}(t,v)|^2\left(\int_a^t\int_c^d w(x,y)dydx\right)dvdt
\end{aligned}$$

$$+\int_a^b\int_c^d |g_{12}(t,s)|^2\left(\int_a^t\int_c^s w(x,y)dydx\right)dsdt$$

$$=\int_a^b\int_c^d |g_{12}(t,s)|^2\left(\int_t^b\int_s^d w(u,v)dvdu\right)dsdt.$$

By using the equality (8) in (7), we obtain the following inequality

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{9}$$

$$\leq \left(\int_a^b\int_c^d (t-a)(s-c)|f_{12}(t,s)|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left(\int_t^b\int_s^d w(u,v)dvdu\right)|g_{12}(t,s)|^2 dsdt\right)^{\frac{1}{2}}.$$

Case II: Let $g(a,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$.

We get

$$g(t,s)=-\int_a^t\int_s^d g_{12}(u,v)dvdu$$

for $(t,s)\in[a,b]\times[c,d]$. Then it follows that

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt$$

$$=\int_a^b\int_c^d (t-a)^{\frac{1}{2}}(d-s)^{\frac{1}{2}}|f_{12}(t,s)|(t-a)^{-\frac{1}{2}}(d-s)^{-\frac{1}{2}}$$

$$\times\left|\int_a^t\int_s^d g_{12}(u,v)dudv\right|w(t,s)dsdt.$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \\
\leq\ & \left(\int_a^b \int_c^d (t-a)(d-s)\, |f_{12}(t,s)|^2\, w(t,s)dsdt \right)^{\frac{1}{2}} \\
&\times \left(\int_a^b \int_c^d (t-a)^{-1} (d-s)^{-1} \left| \int_a^t \int_s^d g_{12}(u,v)dvdu \right|^2 w(t,s)dsdt \right)^{\frac{1}{2}} \\
\leq\ & \left(\int_a^b \int_c^d (t-a)(d-s)\, |f_{12}(t,s)|^2\, w(t,s)dsdt \right)^{\frac{1}{2}} \\
&\times \left(\int_a^b \int_c^d \left(\int_a^t \int_s^d |g_{12}(u,v)|^2\, dvdu \right) w(t,s)dsdt \right)^{\frac{1}{2}}.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
&\int_a^b \int_c^d \left(\int_a^t \int_s^d |g_{12}(u,v)|^2\, dudv \right) w(t,s)dsdt \\
&= \int_a^b \int_c^d \left(\int_t^b \int_c^s w(u,v)dvdu \right) |g_{12}(t,s)|^2\, dsdt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(10) \qquad &\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \\
\leq\ & \left(\int_a^b \int_c^d (t-a)(d-s)\, |f_{12}(t,s)|^2\, w(t,s)dsdt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\times \left(\int_a^b \int_c^d \left(\int_t^b \int_c^s w(u,v)dvdu \right) |g_{12}(t,s)|^2 \, dsdt \right)^{\frac{1}{2}}.$$

Case III: Let $g(b,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$. Then we have

$$g(t,s) = -\int_t^b \int_c^s g_{12}(u,v)dvdu.$$

Case IV: Let $g(b,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$. we can write

$$g(t,s) = \int_t^b \int_s^d g_{12}(u,v)dvdu.$$

By following similar to those in proof of (9) and (10), but with suitable modifications, we establish the following inequalities in Case III and Case IV:

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)| \, w(t,s)dsdt \tag{11}$$

$$\leq \left(\int_a^b \int_c^d (b-t)(s-c) |f_{12}(t,s)|^2 \, w(t,s)dsdt \right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_c^d \left(\int_a^t \int_s^d w(u,v)dvdu \right) |g_{12}(t,s)|^2 \, dsdt \right)^{\frac{1}{2}},$$

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)| \, w(t,s)dsdt \tag{12}$$

$$\leq \left(\int_a^b \int_c^d (b-t)(d-s) |f_{12}(t,s)|^2 \, w(t,s)dsdt \right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left(\int_a^t\int_c^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}},$$

respectively.

Since $g(a,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (9) for the rectangles $[a,b]\times[c,y]$ and $[a,x]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\begin{aligned}(13)\qquad &\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt\\ \le\ &\left(\int_a^b\int_c^y (t-a)(s-c)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}\\ &\times\left(\int_a^b\int_c^y\left(\int_t^b\int_s^y w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}(14)\qquad &\int_a^x\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt\\ \le\ &\left(\int_a^x\int_c^d (t-a)(s-c)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}\\ &\times\left(\int_a^x\int_c^d\left(\int_t^x\int_s^d w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}\end{aligned}$$

respectively.

As $g(a,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we apply the inequality (10) for the rectangles $[a,b]\times[y,d]$ and $[a,x]\times[c,d]$ for $(x,y)\in$

$[a,b]\times[c,d]$, then we get

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \tag{15}$$

$$\leq \left(\int_a^b\int_y^d (t-a)(d-s)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_y^d \left(\int_t^b\int_y^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}}$$

and

$$\int_a^x\int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \tag{16}$$

$$\leq \left(\int_a^x\int_c^d (t-a)(d-s)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^x\int_c^d \left(\int_t^x\int_c^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}}.$$

Similarly, since $g(b,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (11) for the rectangles $[a,b]\times[c,y]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \tag{17}$$

$$\leq \left(\int_a^b\int_c^y (b-t)(s-c)\,|f_{12}(t,s)|^2\, w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^y\left(\int_a^t\int_s^y w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

and

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{18}$$

$$\le \left(\int_x^b\int_c^d (b-t)\,(s-c)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d\left(\int_x^t\int_s^d w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}.$$

Finally, as $g(b,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we apply the inequality (12) for the rectangles $[a,b]\times[y,d]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{19}$$

$$\le \left(\int_x^b\int_c^d (b-t)\,(d-s)\,|f_{12}(t,s)|^2\,w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d\left(\int_x^t\int_c^s w(u,v)dvdu\right)|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

and

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{20}$$

$$\leq \left(\int_a^b \int_y^d (b-t)(d-s)|f_{12}(t,s)|^2 w(t,s)dsdt\right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_y^d \left(\int_a^t \int_y^s w(u,v)dvdu\right)|g_{12}(t,s)|^2 dsdt\right)^{\frac{1}{2}}.$$

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be reel numbers. Then we have the following Cauchy-Scwarz inequality
(21)

$$a_1b_1 + a_2b_2 + ... + a_nb_n \leq \left(a_1^2 + a_2^2 + ... + a_n^2\right)^{\frac{1}{2}} \left(b_1^2 + b_2^2 + ... + b_n^2\right)^{\frac{1}{2}}.$$

If we add the inequalities (13)-(20), then by using the Cauchy-Schwarz inequality (21), we obtain

$$4\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt$$

$$\leq \left[\int_a^b \int_c^y (b-a)(s-c)|f_{12}(t,s)|^2 w(t,s)dsdt\right.$$

$$+\int_a^x \int_c^d (t-a)(d-c)|f_{12}(t,s)|^2 w(t,s)dsdt$$

$$+\int_a^b \int_y^d (b-a)(d-s)|f_{12}(t,s)|^2 w(t,s)dsdt$$

$$\left.+\int_x^b \int_c^d (b-t)(d-c)|f_{12}(t,s)|^2 w(t,s)dsdt\right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b \int_c^y \left(\int_a^b \int_s^y w(u,v)dvdu\right)|g_{12}(t,s)|^2 dsdt\right.$$

$$+\int_a^x\int_c^d\left(\int_t^x\int_c^d w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt$$

$$+\int_a^b\int_y^d\left(\int_a^b\int_y^s w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt$$

$$\left.+\int_x^b\int_c^d\left(\int_x^t\int_c^d w(u,v)dvdu\right)\left|g_{12}(t,s)\right|^2 dsdt+\right]^{\frac{1}{2}}$$

$$=\left[(b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^2 dsdt\right.$$

$$\left.+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^2 dsdt\right]^{\frac{1}{2}}$$

$$\times\left[\int_a^b\int_c^d\left|\int_a^b\int_s^y w(u,v)dvdu\right|\left|g_{12}(t,s)\right|^2 dsdt\right.$$

$$\left.+\int_a^b\int_c^d\left|\int_t^x\int_c^d w(u,v)dvdu\right|\left|g_{12}(t,s)\right|^2 dsdt\right]^{\frac{1}{2}}.$$

This proves the first inequality in (4).

The proof of the second inequality in (4) is obvious from the fact that $\sqrt{pq}\leq\frac{1}{2}(p+q)$, for $p,q>0$. □

Remark 2.2. If we choose $w(t,s)=1$ for all $(t,s)\in[a,b]\times[c,d]$ in Theorem 2.1, then the inequalities (4) reduce to the inequalities (3).

Corollary 2.3. *If we choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Theorem 2.1, then we have the following inequality*

$$\int_a^b\int_c^d\left|f_{12}(t,s)g(t,s)\right|w(t,s)dsdt$$

$$\le \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\le \frac{1}{8}\left[\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right.$$

$$\left.+\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]\left|g_{12}(t,s)\right|^2 dsdt\right]$$

where

$$P(t)=\begin{cases} t-a, & a\le t\le \frac{a+b}{2}\\ b-t, & \frac{a+b}{2}\le t\le b\end{cases} \quad \text{and} \quad Q(s)=\begin{cases} s-c, & c\le s\le \frac{c+d}{2}\\ d-s, & \frac{c+d}{2}\le s\le d.\end{cases}$$

REMARK 2.4. If we choose $f(t,s)=g(t,s)$ for $(t,s)\in[a,b]\times[c,d]$ in Corollary 2.3, then we have the following inequality

$$\int_a^b\int_c^d \left|f_{12}(t,s)f(t,s)\right| w(t,s)dsdt$$

$$\le \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\left|f_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}}w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{8}\left[\int_a^b\int_c^d[(b-a)Q(s)+(d-c)P(t)]\,|f_{12}(t,s)|^2\,dsdt\right.$$

$$\left.+\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}}w(u,v)dvdu\right|+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^2\,dsdt\right].$$

THEOREM 2.5. *Let $w:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable and nonnegative function. Let $f(t,s)$, $f_1(t,s)$, $f_{12}(t,s)$, $g(t,s)$, $g_1(t,s)$ and $g_{12}(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12},f_{12}\in L_2([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$, we have*

$$\int_a^b\int_c^d|f_{12}(t,s)g(t,s)|\,w(t,s)dsdt \tag{22}$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{y}w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{x}\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{4p}\left[\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]$$

$$+\frac{1}{4q}\left[\int_a^b\int_c^d\left[\left|\int_a^b\int_s^y w(u,v)dvdu\right|+\left|\int_t^x\int_c^d w(u,v)dvdu\right|\right]|g_{12}(t,s)|^q\,dsdt\right]$$

where $P(t,x)$ and $Q(s,y)$ are defined by as in Theorem 1.3 and where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Hölder inequality instead of Cauchy-Schwarz inequality and by following similar to those in proof of Theorem 2.1, but with suitable modifications, one can prove the first inequality in (22). The details left to the reader. The proof of the second inequality in (22) is obvious from the Young inequality as defined by

$$a_1^{1/p}a_1^{1/q}\le\frac{1}{p}a_1+\frac{1}{q}a_2,$$

for $a_1,a_2>0$, where $\frac{1}{p}+\frac{1}{q}=1$. □

Remark 2.6. If we choose $w(t,s)=1$ for all $(t,s)\in[a,b]\times[c,d]$ in Theorem 2.5, then we have the following inequality

$$\begin{aligned}
&\int_a^b\int_c^d|f_{12}(t,s)g(t,s)|\,dsdt\\
\le\ &\frac{1}{4}\left(\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b\int_c^d[(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}\\
\le\ &\frac{1}{4p}\left[\int_a^b\int_c^d[(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]\\
&+\frac{1}{4q}\left[\int_a^b\int_c^d[(b-a)\,|y-s|+(d-c)\,|x-t|]\,|g_{12}(t,s)|^q\,dsdt\right]
\end{aligned}$$

which is proved by Budak in [3].

Corollary 2.7. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 2.5, then we have the following inequality*

$$\begin{aligned}
&\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, w(t,s)dsdt \\
\leq\ & \frac{1}{4}\left(\int_a^b \int_c^d [(b-a)Q(s)+(d-c)P(t)]\, |f_{12}(t,s)|^p\, dsdt\right)^{\frac{1}{p}} \\
&\times \left(\int_a^b \int_c^d \left[\left|\int_a^b \int_s^{\frac{c+d}{2}} w(u,v)dvdu\right| \right.\right. \\
&\left.\left. + \left|\int_t^{\frac{a+b}{2}} \int_c^d w(u,v)dvdu\right|\right] |g_{12}(t,s)|^q\, dsdt\right)^{\frac{1}{q}} \\
\leq\ & \frac{1}{4p}\left[\int_a^b \int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\, |f_{12}(t,s)|^p\, dsdt\right] \\
&+\frac{1}{4q}\left[\int_a^b \int_c^d \left[\left|\int_a^b \int_s^{\frac{c+d}{2}} w(u,v)dvdu\right| \right.\right. \\
&\left.\left. + \left|\int_t^{\frac{a+b}{2}} \int_c^d w(u,v)dvdu\right|\right] |g_{12}(t,s)|^q\, dsdt\right]
\end{aligned}$$

where $P(t)$ and $Q(s)$ are defined by as in Corollary 2.3.

Remark 2.8. If we choose $f(t,s) = g(t,s)$ for $(t,s) \in [a,b]\times[c,d]$ in Corollary 2.7, then we have the following inequality

$$\int_a^b \int_c^d |f_{12}(t,s)f(t,s)|\, w(t,s)dsdt$$

$$\leq \frac{1}{4}\left(\int_a^b\int_c^d [(b-a)Q(s)+(d-c)P(t)]\,|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{4p}\left[\int_a^b\int_c^d [(b-a)Q(s,y)+(d-c)P(t,x)]\,|f_{12}(t,s)|^p\,dsdt\right]$$

$$+\frac{1}{4q}\left[\int_a^b\int_c^d\left[\left|\int_a^b\int_s^{\frac{c+d}{2}} w(u,v)dvdu\right|\right.\right.$$

$$\left.\left.+\left|\int_t^{\frac{a+b}{2}}\int_c^d w(u,v)dvdu\right|\right]|f_{12}(t,s)|^q\,dsdt\right].$$

References

[1] R.P. Agarwal and P. Y. H. Pang, *Sharp opial-type inequalities in two variables*, Appl Anal. **56** (3) (1996), 227–242.

[2] H. Budak and Sarikaya, *Refinements of Opial type inequalities in two variables*, ResearchGate Article: www.researchgate.net/publication/329091454.

[3] H. Budak, *Generalizations of Opial type inequalities in two variables using p-norms*, Transylvanian Journal of Mathematics and Mechanics, **11** (1-2) (2019), 63–75.

[4] Z. Changjian, and W. Cheung, *On improvements of Opial-type inequalities*, Georgian Mathematical Journal, **21** (4) (2014), 415–419.

[5] W.S. Cheung, *Some new Opial-type inequalities*, Mathematika **37** (1990), 136–142.

[6] W.S. Cheung, *Some generalized Opial-type inequalities*, J. Math. Anal. Appl. **162** (1991), 317– 321.

[7] W.S. Cheung, *On Opial-type inequalities in two variables*, Aequationes Mathematicae **38** (1989), 236–244.
[8] W.S. Cheung, *Opial-type inequalities with m functions in n variables*, Mathematika **39** (2) (1992), 319–326.
[9] S. S. Dragomir, *Generalizations of Opialś inequalities for two functions and applications*, Preprint RGMIA Res. Rep. Coll. **21** (2018), Art. 64.
[10] C. T. Lin and G. S.Yang, *A generalized Opial's inequality in two variables*, Tamkang J. Math. **15** (1984), 115–122.
[11] Z. Opial, *Sur une inegaliti*, Ann. Polon. Math. **8** (1960), 29–32.
[12] B. G. Pachpatte, *On Opial-type integral inequalities*, J. Math. Anal. Appl. **120** (1986), 547–556.
[13] B. G. Pachpatte, *Some inequalities similar to Opial's inequality*, Demonstratio Math. **26** (1993), 643–647.
[14] B. G. Pachpatte, *A note on some new Opial type integral inequalities*, Octogon Math. Mag. **7** (1999), 80–84.
[15] B. G. Pachpatte, *On some inequalities of the Weyl type*, An. Stiint. Univ. "Al.I. Cuza" Iasi **40** (1994), 89–95.
[16] B. G. Pachpatte, *On Opial type integral inequalities*, J. Math. Analy. Appl. **120**, 547–556 (1986).
[17] B. G. Pachpatte, *On two inequalities similar to Opial's inequality in two independent variables*, Periodica Math. Hungarica **18** (1987), 137–141.
[18] B. G. Pachpatte, *On an inequality of opial type in two variables*, Indian J. Pure Appl. Math. **23** (9) (1992), 657–661.
[19] B.C. Pachpatte, *On two independent variable Opial-type integral inequalities*, J. Math. Anal. Appl. **125** (1987), 47–57.
[20] B.C. Pachpatte, *On Opial type inequalities in two independent variables*, Proc. Royal Soc. Edinburgh, **100A** (1985), 263–270.
[21] B.C. Pachpatte, *On certain two dimensional integral inequalities*, Chinese J. Math. **17** (4) (1989), 273–279.
[22] B.C. Pachpatte, *On multidimensional Opial-type inequalities*, J. Math. Anal. Appl. **126** (1) (1987), 85–89.
[23] B.C. Pachpatte, *On some new integral inequalities in ceveral independent variables*, Chinese Journal of Mathematics **14** (2) (1986), 69–79.
[24] B.C. Pachpatte, *Inequalities of Opial type in three independent variables*, Tamkang Journal of Mathematics **35** (2) (2004), 145–158.
[25] H. M. Srivastava, K.-L. Tseng, S.-J. Tseng and J.-C. Lo, *Some weighted Opial-type inequalities on time scales*, Taiwanese J. Math. **14** (2010), 107–122.
[26] J. Traple, *On a boundary value problem for systems of ordinary differential equations of second order*, Zeszyty Nauk. Univ. Jagiello. Prace Mat. **15** (1971), 159–168.
[27] C.-J. Zhao and W.-S. Cheung, *On Opial-type integral inequalities and applications*, Math. Inequal. Appl. **17** (1) (2014), 223–232.
[28] G. S. Yang. *Inequality of Opial-type in two variables*, Tamkang J. Math. **13** (1982), 255–259.

Hüseyin Budak
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce, Turkey
E-mail: hsyn.budak@gmail.com

Mehmet Zeki Sarikaya
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce, Turkey
E-mail: sarikayamz@gmail.com

Artion Kashuri
Department of Mathematics
Faculty of Technical Science
University Ismail Qemali
Vlora, Albania
E-mail: artionkashuri@gmail.com