MULTIPLIERS OF DIRICHLET-TYPE SUBSPACES OF BLOCH SPACE

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Abstract. Let $M(X, Y)$ denote the space of multipliers from $X$ to $Y$, where $X$ and $Y$ are analytic function spaces. As we known, for Dirichlet-type spaces $D_p^{\alpha}$, $M(D_p^{p - 1}, D_q^{q - 1}) = \{0\}$, if $p \neq q$, $0 < p, q < \infty$. If $0 < p, q < \infty$, $p \neq q$, $0 < s < 1$ such that $p + s, q + s > 1$, then $M(D_p^{p - 2 + s}, D_q^{q - 2 + s}) = \{0\}$. However, $X \cap D_p^{p - 1} \subseteq X \cap D_q^{q - 1}$ and $X \cap D_p^{p - 2 + s} \subseteq X \cap D_q^{q - 2 + s}$ whenever $X$ is a subspace of the Bloch space $B$ and $0 < p \leq q < \infty$. This says that the set of multipliers $M(X \cap D_p^{p - 2 + s}, X \cap D_q^{q - 2 + s})$ is nontrivial. In this paper, we study the multipliers $M(X \cap D_p^{p - 2 + s}, X \cap D_q^{q - 2 + s})$ for distinct classical subspaces $X$ of the Bloch space $B$, where $X = B$, $BMOA$ or $H^\infty$.

1. Introduction

Let $D$ denote the unit disk of the complex plane $\mathbb{C}$ and $\partial D$ be the boundary of $D$, the unit circle. Denote by $H(D)$ the space of all analytic functions in $D$. The Bloch space $B$, consists of those $f \in H(D)$ for which

$$
\|f\|_B = |f(0)| + \sup_{z \in \partial D} (1 - |z|^2)|f'(z)| < \infty.
$$

Let $f \in H(D)$. For $0 < p < \infty$, $0 < r < 1$, set

$$
M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta
$$

and

$$
M_\infty(r, f) = \sup_{|z|=r} |f(z)|.
$$

The Hardy space $H^p(0 < p \leq \infty)$ is defined as the space of $f \in H(D)$ such that

$$
\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty.
$$

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For the theory about the Hardy space $H^p$, we refer the readers to [6]. The $BMOA$ space is the set of those $f \in H^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial \mathbb{D}$ [10]. It is well known that $BMOA$ is contained in the Bloch space $B$ continuously.

The weighted Dirichlet-type space $D^p_\alpha(0 < p < \infty, \alpha > -1)$ is the class of all $f \in H(\mathbb{D})$ such that

$$
\|f\|_{D^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) < \infty,
$$

here $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and $dA(z) = \frac{1}{2} dxdy$ is the normalized Lebesgue area measure. It is well known that when $p < \alpha + 1$, $D^p_\alpha = A^p_{\alpha - p}$, the Bergman space [7]. If $p > \alpha + 2$, then $D^p_\alpha \subseteq H^\infty$. Therefore, when $\alpha + 1 \leq p \leq \alpha + 2$, $D^p_\alpha$ is a proper Dirichlet-type space. The spaces $D^p_{\alpha - 1}$ are closely related with Hardy spaces. In fact, $D^1_1 = H^2$. Notice that when $0 < p \leq 2$, $D^p_{\alpha - 1} \subseteq H^p$ [7]. When $2 \leq p < \infty$, $H^p \subseteq D^p_{\alpha - 1}$ [14].

For $g \in H(\mathbb{D})$, the multiplication operator $M_g$ is defined by

$$
M_gf(z) = g(z)f(z), \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
$$

Let $X, Y$ be the norm spaces of analytic functions in $\mathbb{D}$. We denote by $M(X, Y)$ the space of multipliers from $X$ to $Y$, in other words,

$$
M(X, Y) = \{g \in H(\mathbb{D}) : fg \in Y, \; \forall f \in X\}.
$$

For convenience, we write $M(X) := M(X, X)$. Denote the norm of the multiplication operator $M_g$ by $\|M_g\|$. From [2,3], we see that

$$
M(B) = H^\infty \cap B_{\log}.
$$

Here $B_{\log}$ is the logarithmic Bloch space, consists of those $f \in H(\mathbb{D})$ for which

$$
\|f\|_{B_{\log}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| \left(\log \frac{2}{1 - |z|^2}\right) < \infty.
$$

In [15], we have that

$$
M(BMOA) = BMOA_{\log} \cap H^\infty,
$$

where $BMOA_{\log}$ is the space of those functions $f \in H^1$ such that the positive Borel measure $(1 - |z|^2)|f'(z)|^2dA(z)$ is a 2-logarithmic Carleson measure. In other words, $f \in BMOA_{\log}$ if and only if $f \in H^1$ such that

$$
\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|}\right)^2 \int_{\mathbb{B}} |f'(z)|^2(1 - |\varphi_a(z)|^2)dA(z) < \infty,
$$

where $\varphi_a$ is the disk automorphism which interchange the origin and $a$, that is

$$
\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D}.
$$

The multipliers of Dirichlet-type space $D^p_\alpha$ have been studied in [8,9,11,12]. In [8], the authors proved that for $1 < p \leq q < \infty$, a function $g \in H(\mathbb{D})$ belongs to $M(D^p_{\alpha - 2}, D^q_{\alpha - 2})$ if and only if $g \in H^q$ and the positive Borel measure $\mu$...
defined by \( d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2}dA(z) \) is a \( q \)-Carleson measure for \( \mathcal{D}_q^{\alpha} \).

If \( 1 < q < p < \infty \), then \( M(\mathcal{D}_p^{\alpha}, \mathcal{D}_q^{\alpha}) = \{0\} \).

It is standard that if \( 0 < p, q < \infty \) and \( p \neq q \), then we have
\[
M(\mathcal{D}_p^{\alpha-1}, \mathcal{D}_q^{\alpha-1}) = \{0\}.
\]

Let \( X \) be a non-zero subspace of the Bloch space \( \mathcal{B} \). The space \( X \cap \mathcal{D}_p^{\alpha} \) is equipped with the norm
\[
\|f\|_{X \cap \mathcal{D}_p^{\alpha}} = \|f\|_X + \|f\|_{\mathcal{D}_p^{\alpha}}.
\]

Lemma 1 in [5] says that if \( 0 < p \leq q < \infty \), then \( X \cap \mathcal{D}_p^{\alpha} \subseteq X \cap \mathcal{D}_q^{\alpha} \). It follows that the set of multipliers \( M(X \cap \mathcal{D}_p^{\alpha}, X \cap \mathcal{D}_q^{\alpha}) \) is nontrivial.

By Corollary 1 in [12] and Theorem 2 in [9], for all \( p \neq q \) and \( 0 < s < 1 \),
\[
M(\mathcal{D}_p^{\alpha-1+s} \cap \mathcal{D}_q^{\alpha-1+s}) = \{0\}.
\]

But when \( 0 < p \leq q < \infty \), if \( f \in X \cap \mathcal{D}_p^{\alpha} \), then
\[
\int_D |f'(z)|^p(1 - |z|^2)^{p-2+s}dA(z) \leq \|f\|_{\mathcal{B}}^p \int_D |f'(z)|^p(1 - |z|^2)^{p-2+s}dA(z)
\leq \|f\|_{\mathcal{B}}^p \|f\|_{\mathcal{D}_p^{\alpha}}^p
\leq C\|f\|_{X \cap \mathcal{D}_p^{\alpha}}^p \|f\|_{\mathcal{D}_p^{\alpha}}^p
\leq C\|f\|_{X \cap \mathcal{D}_p^{\alpha}}^p \|f\|_{\mathcal{D}_p^{\alpha}}^p.
\]

Hence \( f \in X \cap \mathcal{D}_q^{\alpha} \) and \( \|f\|_{X \cap \mathcal{D}_q^{\alpha}} \leq C\|f\|_{X \cap \mathcal{D}_p^{\alpha}} \). In other words, \( X \cap \mathcal{D}_p^{\alpha} \subseteq X \cap \mathcal{D}_q^{\alpha} \). So the set of multipliers \( M(X \cap \mathcal{D}_p^{\alpha}, X \cap \mathcal{D}_q^{\alpha}) \) is also nontrivial.

From [5], we see that if \( q > 1 \) and \( 0 < p \leq q < \infty \), then
\[
M(\mathcal{B} \cap \mathcal{D}_p^{\alpha}, \mathcal{B} \cap \mathcal{D}_q^{\alpha}) = M(\mathcal{B})
\]
and
\[
M(BMOA \cap \mathcal{D}_p^{\alpha}, BMOA \cap \mathcal{D}_q^{\alpha}) = M(BMOA).
\]

If \( 0 < p \leq q < \infty \), then
\[
M(\mathcal{H}_p^{\infty} \cap \mathcal{D}_p^{\alpha}, \mathcal{H}_q^{\infty} \cap \mathcal{D}_q^{\alpha}) = \mathcal{H}_p^{\infty} \cap \mathcal{D}_q^{\alpha}.
\]

Motivated by [8] and [5], it is natural to ask what is the set of multipliers \( M(X \cap \mathcal{D}_p^{\alpha}, X \cap \mathcal{D}_q^{\alpha}) \) when \( 0 < s < 1 \). In this paper, we characterize the multipliers \( M(X \cap \mathcal{D}_p^{\alpha}, X \cap \mathcal{D}_q^{\alpha}) \) when \( 0 < s < 1 \). Our main results are stated as follows.

**Theorem 1.1.** Suppose that \( g \in \mathcal{H}(\mathbb{D}) \), \( 0 < p \leq q < \infty \), \( 0 < s < 1 \) satisfying \( p + s > 1 \). Define the positive Borel measure \( \mu \) by \( d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2}dA(z) \), then

(i) \( g \in M(\mathcal{B} \cap \mathcal{D}_p^{\alpha}, \mathcal{B} \cap \mathcal{D}_q^{\alpha}) \) if and only if \( g \in M(\mathcal{B}) \) and \( \mu \) is a \( q \)-Carleson measure for \( \mathcal{B} \cap \mathcal{D}_p^{\alpha} \).
(ii) $g \in M(BMOA \cap D_{p-2+s}^p, BMOA \cap D_{q-2+s}^q)$ if and only if $g \in M(BMOA)$ and $\mu$ is a $q$-Carleson measure for $BMOA \cap D_{p-2+s}^p$.

(iii) $M(\mathcal{H}_\infty \cap D_{p-2+s}^p, \mathcal{H}_\infty \cap D_{q-2+s}^q) = \mathcal{H}_\infty \cap D_{q-2+s}^q$.

**Theorem 1.2.** Suppose $0 < q < p < \infty$, $0 < s < 1$ with $q+s > 1$. Then

(i) $M(B \cap D_{p-2+s}^p, B \cap D_{q-2+s}^q) = \{0\}$.

(ii) $M(BMOA \cap D_{p-2+s}^p, BMOA \cap D_{q-2+s}^q) = \{0\}$.

(iii) $M(\mathcal{H}_\infty \cap D_{p-2+s}^p, \mathcal{H}_\infty \cap D_{q-2+s}^q) = \{0\}$.

Throughout this paper, $C$ denotes a positive constant depending only on indexes $p, q, s, \ldots$, it is not necessary to be the same from one line to another. Let $f$ and $g$ be two positive functions. For convenience, we write $f \preceq g$, if $f \leq Cg$ holds, where $C$ is a positive constant independent of $f$ and $g$. If $f \sim g$ and $g \preceq f$, then we say $f \asymp g$.

2. Preliminary

In this section, we state some definitions and lemmas which will be used in the paper. Let $I$ be an arc of $\partial \mathbb{D}$. Denote the normalized Lebesgue measure of $I$ by $|I|$, that is, $|I| = \frac{1}{\pi} \int_I |d\xi|$. For an arc $I \subseteq \partial \mathbb{D}$, the Carleson square based on $I$ is defined by

$$S(I) := \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}.$$ 

If $I = \partial \mathbb{D}$, then we set $S(I) = \mathbb{D}$. Let $\mu$ be a positive Borel measure on $\mathbb{D}$. For $0 \leq \alpha < \infty, 0 < s < \infty$, we say that $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure if there exists a constant $C > 0$ such that for all arcs $I \subseteq \partial \mathbb{D}$,

$$\mu(S(I)) \leq C \frac{|I|^s}{(\log \frac{1}{|I|})^\alpha}.$$ 

If $\alpha = 0$, then $\mu$ is called an $s$-Carleson measure. If $\alpha = 0$ and $s = 1$, then $\mu$ is said to be a Carleson measure. Recall that an $f \in H^1$ belongs to the space $BMOA$ if and only if the positive Borel measure $|f'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure.

Let $(X, \| \cdot \|_X)$ be a normed space of analytic functions. Then a positive Borel measure $\mu$ on $\mathbb{D}$ is said to be an $s$-Carleson measure for $X$, if there exists a constant $C > 0$ such that for all $f \in X$,

$$\int_{\mathbb{D}} |f(z)|^sd\mu(z) \leq C\|f\|_X^s.$$ 

The following lemma can be found in Theorem 2 of [17], which plays an important role in the proofs of theorems.

**Lemma 2.1.** Suppose that $0 \leq \alpha < \infty$ and $0 < s < \infty$. Then a positive Borel measure $\mu$ on $\mathbb{D}$ is an $\alpha$-logarithmic $s$-Carleson measure if and only if

$$\sup_{a \in \partial \mathbb{D}} \left( \log \frac{2}{1 - |a|} \right)^\alpha \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - a\bar{z}} \right)^s d\mu(z) < \infty.$$
We will make use of the lacunary power series (also called power series with Hadamard gaps) of a function \( f \in \mathcal{H}(\mathbb{D}) \), that is, \( f \) is of the form

\[
f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad z \in \mathbb{D},
\]

with \( \frac{n_{k+1}}{n_k} \geq \lambda > 1 \) for all \( k \). Several known results on lacunary power series will be used in this paper. We put them together in the following statement, see [1,4,5,13,19].

**Lemma 2.2.** Suppose that \( 0 < p < \infty, \alpha > -1 \). \( f \in \mathcal{H}(\mathbb{D}) \) which is given by a lacunary power series,

\[
f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad z \in \mathbb{D}.
\]

Then

(i) \( f \in D_p^{\alpha} \) if and only if \( \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p < \infty \), and

\[
\| f - f(0) \|_{D_p^{\alpha}} \asymp \sum_{k=0}^{\infty} n_k^{p-\alpha-1} |a_k|^p.
\]

(ii) \( f \in H^\infty \) if and only if \( \sum_{k=0}^{\infty} |a_k| < \infty \), and

\[
\| f \|_{H^\infty} \asymp \sum_{k=0}^{\infty} |a_k|.
\]

(iii) \( f \in B \) if and only if \( \sup_k |a_k| < \infty \), and

\[
\| f \|_B \asymp \sup_k |a_k|.
\]

The following estimate can be found in [13].

**Lemma 2.3.** Suppose that \( \beta > -1, s > 0 \) and \( f \in \mathcal{H}(\mathbb{D}) \) with \( f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \ z \in \mathbb{D} \). Then

\[
\sum_{k=1}^{\infty} n_k^{-(\beta+1)} |a_k|^s \asymp \int_0^1 (1 - r)^\beta |f(re^{i\theta})|^s \, dr
\]

for all \( \theta \in \mathbb{R} \).

The following lemma is useful in theory of analytic function spaces and operator theory, see [18].

**Lemma 2.4.** Suppose that \( z \in \mathbb{D}, c \) is real, \( t > -1 \), and

\[
I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - z\bar{w}|^{2t+2+c}} dA(w).
\]

(i) If \( c < 0 \), then as a function of \( z \), \( I_{c,t} \) is bounded on \( \mathbb{D} \).

(ii) If \( c = 0 \), then

\[
I_{c,t}(z) \asymp \log \frac{1}{1 - |z|^2} \quad \text{as} \quad |z| \to 1^-.
\]

(iii) If \( c > 0 \), then

\[
I_{c,t}(z) \asymp \frac{1}{(1 - |z|^2)^c} \quad \text{as} \quad |z| \to 1^-.
\]
We will use the following estimate to prove our results, which can be found in [16].

**Lemma 2.5.** For $s > -1$, $r, t > 0$ with $0 < r + t - s - 2 < r$, there exists a constant $C > 0$ such that for any $a, b \in D$,

$$
\int_D \frac{(1 - |z|^2)^s}{|1 - \bar{a}z|^r|1 - bz|^t} dA(z) \leq \frac{C}{(1 - |a|^2)^{r+t-s-2}}.
$$

3. **Proof of main results**

**Proof of Theorem 1.1.** (i) First suppose that $g \in M(B \cap D_{p-2+s}^p, B \cap D_{q-2+s}^q)$. For any $a \in D$, let $\varphi_a$ be defined by (3) and $f_a$ be defined by

$$
f_a(z) = \log \frac{1}{1 - \bar{a}z}, \quad z \in D.
$$

A simple computation shows that $\sup_{a \in D} \|\varphi_a\|_B < \infty$ and $\sup_{a \in D} \|\varphi_a\|_{D_{p-2+s}^p} < \infty$. This implies that $\varphi_a \in B \cap D_{p-2+s}^p$ and $\sup_{a \in D} \|\varphi_a\|_{B \cap D_{p-2+s}^p} < \infty$. We have $g\varphi_a \in B \cap D_{q-2+s}^q$ and

$$
(1 - |z|^2)|(g\varphi_a)'(z)| \leq \|g\varphi_a\|_B
\leq \|g\varphi_a\|_{B \cap D_{q-2+s}^q}
\leq \|M_g\| \|\varphi_a\|_{B \cap D_{p-2+s}^p} \leq C\|M_g\|,
$$

that is,

$$
(1 - |z|^2)g'(z)\varphi_a(z) + g(z)\varphi'_a(z) \leq C\|M_g\|.
$$

Taking $z = a$, using the fact that $\varphi_a(a) = 0$ and $|\varphi'_a(a)| = \frac{1}{1 - |a|^2}$ we get

$$
|g(a)| \leq C\|M_g\|,
$$

which implies that $g \in \mathcal{H}^\infty$.

It is obvious that $f'_a(z) = \frac{a}{1 - \bar{a}z}$ and $\sup_{a \in D} \|f_a\|_B < \infty$. By Lemma 2.4, there is a constant $C > 0$ independent of $a$ such that

$$
\int_D |f'_a(z)|^p(1 - |z|^2)^{-2p+s}dA(z) \leq \int_D \frac{(1 - |z|^2)^p}{|1 - \bar{a}z|^p}dA(z)
\leq \int_D \frac{(1 - |z|^2)^p}{|1 - az|^{2p+2p-2s}}dA(z)
\leq C.
$$

This implies that $\sup_{a \in D} \|f_a\|_{D_{p-2+s}^p} < \infty$. Hence, we have $f_a \in B \cap D_{p-2+s}^p$ and $\sup_{a \in D} \|f_a\|_{B \cap D_{q-2+s}^q} < \infty$. So $g f_a \in B \cap D_{q-2+s}^q$ and

$$
(1 - |z|^2)|(gf_a)'(z)| \leq \|gf_a\|_{B \cap D_{q-2+s}^q} \leq \|M_g\| \|f_a\|_{B \cap D_{p-2+s}^p} \leq C\|M_g\|.
$$
On the other hand, since $g \in \mathcal{H}^\infty$, 
\[
(1 - |z|^2)|g(z)f_a(z)| \leq \|g\|_{\mathcal{H}^\infty} \|f_a\|_{\mathcal{B}} \leq C\|g\|_{\mathcal{H}^\infty}.
\]
Combining (4) and (5) we deduce that 
\[
(1 - |z|^2)|g'(z)f_a(z)| \leq C(\|M_g\| + \|g\|_{\mathcal{H}^\infty}).
\]
Taking $z = a$ we obtain 
\[
(1 - |a|^2)|g'(a)| \log \frac{1}{1 - |a|^2} \leq C,
\]
which shows that $g \in \mathcal{B}_{log}$. From (1) we see that $g \in M(\mathcal{B})$.

We next show that $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a $q$-Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$. Let $f \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$. Since $g \in \mathcal{H}^\infty$, we have 
\[
\int_{\mathcal{D}} |g(z)|^q |f'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) \leq \|g\|^q_{\mathcal{H}^\infty} \|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p} \leq \|g\|^q_{\mathcal{H}^\infty} \|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}.
\]
Note that $gf \in \mathcal{B} \cap \mathcal{D}_{q-2+s}^q$,
\[
\int_{\mathcal{D}} |(gf)'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) \leq \|gf\|^q_{\mathcal{B} \cap \mathcal{D}_{q-2+s}^q} \leq \|M_g\|^q \|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}.
\]
Combining (6) and (7) implies 
\[
\int_{\mathcal{D}} |f(z)|^q |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) \leq C(\|g\|^q_{\mathcal{H}^\infty} + \|M_g\|^q) \|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}.
\]
That is, $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a $q$-Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$.

Suppose that $g \in M(\mathcal{B})$ and $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a $q$-Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, we prove that $g \in M(\mathcal{B} \cap \mathcal{D}_{p-2+s}^p, \mathcal{B} \cap \mathcal{D}_{q-2+s}^q)$. For any $f \in \mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, we have $gf \in \mathcal{B}$. It remains to prove that $gf \in \mathcal{D}_{q-2+s}^q$. Since $d\mu(z) = |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z)$ is a $q$-Carleson measure for $\mathcal{B} \cap \mathcal{D}_{p-2+s}^p$, there is a constant $C > 0$ independent of $f$ such that 
\[
\int_{\mathcal{D}} |f(z)|^q |g'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) \leq C\|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p}.
\]
Combining (6) and (8) we see that 
\[
\int_{\mathcal{D}} |(gf)'(z)|^q(1 - |z|^2)^{q-2+s}dA(z) \leq C\|gf\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p} \leq C\|f\|^q_{\mathcal{B} \cap \mathcal{D}_{p-2+s}^p},
\]
which implies that $gf \in \mathcal{D}_{q-2+s}^q$.

The idea of proofs of (ii) and (iii) is similar to that of (i). For the completeness of the paper, we give their proofs briefly below.

(ii) Assume that $g \in M(BMOA \cap \mathcal{D}_{p-2+s}^p, BMOA \cap \mathcal{D}_{q-2+s}^q)$. For any $a \in \mathbb{D}$, let $\varphi_a$ and $f_a$ be defined as in the proof of (i). An easy computation shows
Since \( \sup_{a \in \mathbb{D}} \| f_a \|_{\mathcal{D}^q_{p-2+s}} < \infty \). Since \( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{1 - a e^{i\theta}} |d\theta| < \infty \), we have \( f_a \in \mathcal{H}^1 \).

Since \( f'_a(z) = \frac{a}{1 - az} \), by Lemma 2.5, there exists a constant \( C > 0 \) such that

\[
\int_{\mathbb{D}} |f'_a(z)|^2 (1 - |\varphi(z)|^2) dA(z) = \int_{\mathbb{D}} \frac{|a|^2}{|1 - \overline{a}z|^2} \frac{(1 - |b|^2)(1 - |z|^2)}{|1 - bz|^2} dA(z) \\
\leq (1 - |b|^2) \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \overline{a}z|^2|1 - bz|^2} dA(z) \\
\leq C.
\]

Hence, the Borel measure \( |f'_a(z)|^2 (1 - |z|^2) dA(z) \) is a Carleson measure by Lemma 2.1, so \( f_a \in BMOA \). Since \( C \) is independent of \( a \), we deduce that \( \sup_{a \in \mathbb{D}} \| f_a \|_{BMOA} < \infty \). Hence, \( f_a \in BMOA \cap \mathcal{D}^p_{p-2+s} \) and \( \sup_{a \in \mathbb{D}} \| f_a \|_{BMOA \cap \mathcal{D}^p_{p-2+s}} < \infty \). In addition, a similar argument implies \( g \in \mathcal{H}^\infty \). So \( g f_a \in BMOA \cap \mathcal{D}^q_{q-2+s} \). Hence, there exists a constant \( C > 0 \) such that for any arc \( I \),

\[
(9) \int_{S(I)} |(g f_a)'(z)|^2 (1 - |z|^2) dA(z) \leq C |I|
\]

and

\[
(10) \int_{S(I)} |f'_a(z)|^2 (1 - |z|^2) dA(z) \leq C |I|.
\]

Then by \( g \in \mathcal{H}^\infty \), (9) and (10) we obtain

\[
(11) \int_{S(I)} |g'(z)|^2 |f_a(z)|^2 (1 - |z|^2) dA(z) \leq C |I|.
\]

Take \( a = (1 - |I|) e^{i\theta} \), where \( e^{i\theta} \) is the center of \( I \), then for any \( z \in S(I) \),

\[
|1 - \overline{a}z| = 1 - |a| = |I|, \quad |f_a(z)| \asymp \log \frac{1}{|I|}.
\]

Thus (11) implies that

\[
\left( \log \frac{1}{|I|} \right)^2 \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dA(z) \leq C |I|,
\]

in other words, \( g \in BMOA_{q, \log} \). Therefore \( g \in M(BMOA) \) from (2).

We turn to show that \( |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \) is a \( q \)-Carleson measure for \( BMOA \cap \mathcal{D}^p_{p-2+s} \). For every \( f \in BMOA \cap \mathcal{D}^p_{p-2+s} \), we have \( g f \in BMOA \cap \mathcal{D}^q_{q-2+s} \) and

\[
\int_{\mathbb{D}} |(g f)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \| g f \|_{\mathcal{D}^q_{q-2+s}}^q \\
\leq \| g \|_{BMOA \cap \mathcal{D}^p_{p-2+s}}^q \| f \|_{BMOA \cap \mathcal{D}^p_{p-2+s}}^q \\
\leq \| M_g \|_{BMOA \cap \mathcal{D}^p_{p-2+s}} \| f \|_{BMOA \cap \mathcal{D}^p_{p-2+s}}.
\]

(12)
A similar argument as in the proof of (i) shows that

\[ (13) \quad \int_D |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \|g\|_{BMOA}^q \|f\|_{BMOA \cap D_{p-2+s}^q}^q. \]

Combining (12) and (13) yields

\[ \int_D |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C (\|g\|_{BMOA}^q + \|M_q g\|_{BMOA}^q) \|f\|_{BMOA \cap D_{p-2+s}^q}^q. \]

We conclude that \( d\mu(z) = |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \) is a \( q \)-Carleson measure for \( BMOA \cap D_{p-2+s}^q \).

Conversely, for any \( f \in BMOA \cap D_{p-2+s}^q \), we have \( gf \in BMOA \). We only need to prove \( gf \in D_{q-2+s}^q \). By hypothesis, there exists a constant \( C > 0 \) independent of \( f \) such that

\[ (14) \quad \int_D |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{BMOA \cap D_{p-2+s}^q}^q. \]

By (13) and (14) we obtain

\[ \int_D |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C \|f\|_{BMOA \cap D_{p-2+s}^q}^q. \]

That is, \( gf \in D_{q-2+s}^q \).

(iii) We only need to show

\[ M(\mathcal{H}^\infty \cap D_{p-2+s}^q, \mathcal{H}^\infty \cap D_{q-2+s}^q) \supseteq \mathcal{H}^\infty \cap D_{q-2+s}^q, \]

since the converse is obvious.

Let \( g \in \mathcal{H}^\infty \cap D_{q-2+s}^q \). For any \( f \in \mathcal{H}^\infty \cap D_{p-2+s}^q \), we have \( gf \in \mathcal{H}^\infty \). It remains to prove that \( gf \in D_{q-2+s}^q \). These hypothesis imply

\[ \int_D |f(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \|f\|_{\mathcal{H}^\infty}^q \|g\|_{D_{p-2+s}^q}^q \leq \|f\|_{\mathcal{H}^\infty \cap D_{p-2+s}^q}^q \|g\|_{D_{q-2+s}^q}^q. \]

and

\[ \int_D |g(z)|^q |f'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq \|g\|_{\mathcal{H}^\infty}^q \|f\|_{\mathcal{H}^\infty \cap D_{p-2+s}^q}^q. \]

Hence

\[ \int_D |(gf)'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \leq C (\|g\|_{D_{p-2+s}^q}^q + \|g\|_{\mathcal{H}^\infty}^q) \|f\|_{\mathcal{H}^\infty \cap D_{p-2+s}^q}^q. \]

The proof is complete. \( \square \)

Proof of Theorem 1.2. (i) Suppose that \( g \in M(\mathcal{B} \cap D_{p-2+s}^q, \mathcal{B} \cap D_{q-2+s}^q) \) and \( g \neq 0 \), then \( g \in \mathcal{B} \cap D_{q-2+s}^q \). Let

\[ f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad a_k = n_k^{\frac{1}{q}}, \quad z \in \mathbb{D}, \]

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\[ \sum_{k=0}^{\infty} a_k z^{n_k}, \quad a_k = n_k^{\frac{1}{q}}, \quad z \in \mathbb{D}, \]
with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k$. Since $\sum_{k=1}^{\infty} |a_k| < \infty$, by Lemma 2.2 we have $f \in H^\infty \subseteq B$. It is not difficult to see that $\sum_{k=0}^{\infty} n_k^{1-s} |a_k|^p < \infty$, Lemma 2.2 yields $f \in D_{p-2+s}^p$. Hence $f \in B \cap D_{p-2+s}^p$ and $fg \in B \cap D_{q-2+s}^q$. We have
\[
\int_D (1 - |z|^2)^{q-2+s}|(gf)'(z)|^q dA(z) \leq \|gf\|_{D_{q-2+s}^q}^q < \infty
\]
and
\[
\int_D (1 - |z|^2)^{q-2+s}|g'(z)|^q dA(z) \leq \|g\|_{D_{q-2+s}^q}^q < \infty.
\]
These imply
\[
(15) \quad \int_D (1 - |z|^2)^{q-2+s}|g(z)f'(z)|^q dA(z) < \infty.
\]
On the other hand, $f'(z) = \sum_{k=0}^{\infty} a_k n_k z^{n_k-1}$, by Lemma 2.3 we see that
\[
\int_0^1 (1 - r)^{q-2+s} |f'(re^{i\theta})|^q dr \approx \sum_{k=0}^{\infty} n_k^{-(q+s-1)} |a_k n_k|^q = \infty.
\]
Since $g \in D_{q-2+s}^q \subseteq H^q$ (see [9], p. 1877), $g$ has a finite and nonzero radial limit almost everywhere on the boundary of $D$. Thus
\[
\int_0^1 (1 - r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty
\]
for almost all $\theta \in \mathbb{R}$ (see [9], p. 1878). This is in contradiction to (15).

(iii) Assume that $g \in M(BMOA \cap D_{p-2+s}^p, BMOA \cap D_{q-2+s}^q)$ and $g \neq 0$, then $g \in BMOA \cap D_{q-2+s}^q$. Let $a_k = (2^k) \frac{1}{k^s}$, $k = 1, 2, \ldots$, and
\[
f(z) = \sum_{k=0}^{\infty} a_k z^{2^k}, \quad z \in D.
\]
Then $f \in H^\infty \cap D_{p-2+s}^p$, by Lemma 2.2. Hence $f \in BMOA \cap D_{p-2+s}^p$ and $fg \in BMOA \cap D_{q-2+s}^q$. So
\[
\int_D (1 - |z|^2)^{q-2+s}|(gf)'(z)|^q dA(z) \leq \|gf\|_{D_{q-2+s}^q}^q < \infty
\]
and
\[
\int_D (1 - |z|^2)^{q-2+s}|g'(z)f(z)|^q dA(z) \leq \|f\|_{H^\infty}^q \|g\|_{D_{q-2+s}^q}^q < \infty.
\]
We get
\[
\int_D (1 - |z|^2)^{q-2+s}|g(z)f'(z)|^q dA(z) < \infty.
\]
Since \( f'(z) = \sum_{k=0}^{\infty} 2^k a_k z^{2^k - 1} \), from Lemma 2.3,
\[
\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q dr \leq \sum_{k=0}^{\infty} (2^k)^{-s} |a_k 2^k|^q = \infty.
\]
Therefore, for almost all \( \theta \in \mathbb{R} \),
\[
\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty.
\]
This is a contradiction.

(iii) Assume \( g \in \mathcal{M}(\mathcal{H}^\infty \cap \mathcal{D}_p^{p-2+s}, \mathcal{H}^\infty \cap \mathcal{D}_q^{q-2+s}) \) and \( g \neq 0 \), then \( g \in \mathcal{H}^\infty \cap \mathcal{D}_q^{q-2+s} \). Let \( f \in \mathcal{H}(\mathbb{D}) \) be defined as in the proof of (i). The same argument as in the proof of (i) shows that \( f \in \mathcal{H}^\infty \cap \mathcal{D}_p^{p-2+s} \). So \( fg \in \mathcal{H}^\infty \cap \mathcal{D}_q^{q-2+s} \), i.e.,
\[
\int_\mathbb{D} (1-|z|^2)^{q-2+s} |(gf)'(z)|^q dA(z) \leq \|gf\|_{\mathcal{D}_q^{q-2+s}}^q.
\]
In addition,
\[
\int_\mathbb{D} (1-|z|^2)^{q-2+s} |g(z)f'(z)|^q dA(z) \leq \|f\|_{\mathcal{H}^\infty}^q \|g\|_{\mathcal{D}_q^{q-2+s}}^q.
\]
We have
\[
\int_\mathbb{D} (1-|z|^2)^{q-2+s} |g(z)f'(z)|^q dA(z) < \infty.
\]
On the other hand, by Lemma 2.3 we deduce that
\[
\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q dr = \infty.
\]
This together with \( g \in \mathcal{D}_q^{q-2+s} \subseteq \mathcal{H}^\infty \) yields
\[
\int_0^1 (1-r)^{q-2+s} |f'(re^{i\theta})|^q |g(re^{i\theta})|^q dr = \infty
\]
for almost all \( \theta \in \mathbb{R} \) ([9], p. 1878). We obtain a contradiction. This finishes the proof. \( \square \)

References


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