SMOOTH POINTS OF $L_s(nl_2^\infty)$

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Abstract. For $n \geq 2$, we characterize the smooth points of the unit ball of $L_s(nl_2^\infty)$.

1. Introduction

The main result about smooth points is known as the Mazur density theorem. Recall that the Mazur density theorem [5, p. 171] says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. We denote by $B_E$ the closed unit ball of a real Banach space $E$ and also by $E^*$ the dual space of $E$. $x \in B_E$ is called a smooth point of $B_E$ if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $smB_E$ the set of smooth points of $B_E$. For $n \in \mathbb{N}$, we denote by $L(nE)$ the Banach space of all continuous n-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x\| = 1} |T(x_1, \ldots, x_n)|$. A n-linear form $T$ is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation $\sigma$ on $\{1, 2, \ldots, n\}$. We denote by $L_s(nE)$ the Banach space of all continuous symmetric n-linear forms on $E$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in L_s(nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in L_s(nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \hat{P}$. We denote by $P(nE)$ the Banach space of all continuous n-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\| = 1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [3].

Choi and Kim ([1, 2]) initiated and characterized the smooth points of the unit balls of $P(2l_2^2)$ and $P(2l_2^2)$. Greus [4] characterized the smooth 2-homogeneous polynomials on Hilbert spaces. Kim ([6, 8]) characterized the smooth points of the unit balls of $L_s(2l_2^\infty)$ and $L_s(3l_2^\infty)$. Kim [7] classified the smooth points of the unit ball of $P(2d(1, w)^2)$, where $d(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $w$. Recently, Kim [9] classified the smooth points...
of the unit ball of $\mathcal{P}(2^2, \mathbb{R}^2)$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$.

Let

$$u_j := [(1, -1), (1, -1), \ldots, (1, -1), (1, 1), \ldots, (1, 1)]$$

for $0 \leq j \leq n$, where $-1$ appears $j$-times in $u_j$. Let

$$U_n := \{u_j : 0 \leq j \leq n\}.$$ 

In this paper, for $n \geq 2$, we characterize the smooth points of the unit ball of $\mathcal{L}_s(n^2l_\infty^2)$ as follows: Let $n \geq 2$ and $T \in \mathcal{L}_s(n^2l_\infty^2)$ with $\|T\| = 1$. Then, $T \in \text{sm}B_{\mathcal{L}_s(n^2l_\infty^2)}$ if and only if $|\text{Norm}(T) \cap U_n| = 1$, where

$$\text{Norm}(T) := \{(a_1, b_1), \ldots, (a_n, b_n) \in B_{l_\infty^2} \times \cdots \times B_{l_\infty^2} : |T((a_1, b_1), \ldots, (a_n, b_n))| = \|T\|\}.$$ 

2. Results

Let

$$F_0((x_1, y_1), \ldots, (x_n, y_n)) := x_1 \cdots x_n, \quad F_n((x_1, y_1), \ldots, (x_n, y_n)) := y_1 \cdots y_n$$

and for $1 \leq j \leq n - 1,$

$$F_j((x_1, y_1), \ldots, (x_n, y_n)) := \sum_{\{k_1, \ldots, k_j, s_{j+1}, \ldots, sn\} = \{1, \ldots, n\}} y_{k_1} \cdots y_{k_j} x_{s_{j+1}} \cdots x_{sn}.$$ 

Note that $\{F_0, F_1, \ldots, F_n\}$ is a basis for $\mathcal{L}_s(n^2l_\infty^2).$ If $T \in \mathcal{L}_s(n^2l_\infty^2)$, then

$$T((x_1, y_1), \ldots, (x_n, y_n)) = \sum_{0 \leq j \leq n} a_j F_j((x_1, y_1), \ldots, (x_n, y_n))$$

for some $a_j \in \mathbb{R}$. For simplicity, we write $T = (a_0, a_1, \ldots, a_n)^t$. Note that $F_0(u_j) = 1$ for every $0 \leq j \leq n.$

**Theorem 2.1.** Let $n \geq 2$ and $T \in \mathcal{L}_s(n^2l_\infty^2).$ Then

$$\|T\| = \sup\{|T(u_j)| : 0 \leq j \leq n\}.$$ 

**Proof.** It follows from the Krein-Milman theorem, symmetry and $n$-linearity of $T$. \hfill \square

**Theorem 2.2.** Let $n \geq 2$ and $T \in \mathcal{L}_s(n^2l_\infty^2)$ with $\|T\| = 1$. Then, $T \in \text{sm}B_{\mathcal{L}_s(n^2l_\infty^2)}$ if and only if $|\text{Norm}(T) \cap U_n| = 1$.

**Proof.** ($\Rightarrow$) Otherwise. There exist $u_{k_1}, u_{k_2} \in \text{Norm}(T) \cap U_n$ for some $0 \leq k_1 \neq k_2 \leq n$. Then,

$$\text{sign}(T(u_{k_1})) \delta_{u_{k_1}} \neq \text{sign}(T(u_{k_2})) \delta_{u_{k_2}}$$

and

$$\text{sign}(T(u_{k_i})) \delta_{u_{k_i}}(T) = |T(u_{k_i})| = 1 = \|\text{sign}(T(u_{k_i})) \delta_{u_{k_i}}\| (i = 1, 2),$$
where \( \delta_{u_k} \in L_s(n1^2) \) is the evaluation functional by \( u_k \). Hence,

\[
T \notin \text{smB}_{L_s(n1^2)}.
\]

This is a contradiction.

\((\Leftarrow)\) Let \( T = \sum_{0 \leq j \leq n} a_j F_j = (a_0, a_1, \ldots, a_n)^t \) for some \( a_j \in \mathbb{R} \). Suppose that

\[
\text{Norm}(T) \cap \mathcal{U}_a = \{ u_{k_0} \}
\]

for some \( 0 \leq k_0 \leq n \). Then,

\[
|T(u_{k_0})| = 1 > |T(u_k)| (0 \leq k \neq k_0 \leq n).
\]

Let \( f \in L_s(n1^2)^* \) be such that \( f(T) = 1 = \|f\| \). For simplicity we denote

\[
f = (f(F_0), f(F_1), \ldots, f(F_n)).
\]

Let

\[
\alpha_j := f(F_j) \text{ for } 0 \leq j \leq n.
\]

Claim: \( \alpha_l = F_l(u_{k_0}) \alpha_0 \) for \( 1 \leq l \leq n \).

**Case 1:** \( F_l(u_{k_0}) \neq 0 \).

Let \( m \in \mathbb{N} \) be such that

\[
|T(u_k)| + \frac{1}{m} |1 - \frac{F_l(u_k)}{F_l(u_{k_0})}| < 1 (0 \leq k \neq k_0 \leq n)
\]

Let

\[
T_1 := T + \frac{1}{m} (F_0 - \frac{1}{F_l(u_{k_0})} F_l)
\]

and

\[
T_2 := T - \frac{1}{m} (F_0 - \frac{1}{F_l(u_{k_0})} F_l).
\]

By Theorem 2.1, it follows that, for \( i = 1, 2 \),

\[
\|T_i\| = \max\{|T(u_j)| + \frac{1}{m} (F_0(u_j) - \frac{F_l(u_j)}{F_l(u_{k_0})}) : 0 \leq j \leq n\}
\]

\[
= \max\{|T(u_{k_0})|, |T(u_k)| + \frac{1}{m} |1 - \frac{F_l(u_k)}{F_l(u_{k_0})}| : 0 \leq k \neq k_0 \leq n\}
\]

(because of \( F_0(u_{k_0}) = F_0(u_k) = 1\))

\[
= 1.
\]

Since

\[
1 \geq f(T_i) = f(T) \pm \frac{1}{m} f(F_0 - \frac{1}{F_l(u_{k_0})} F_l) = 1 \pm \frac{1}{m} (\alpha_0 - \frac{1}{F_l(u_{k_0})} \alpha_l) (i = 1, 2),
\]

\[
\alpha_0 = \frac{1}{F_l(u_{k_0})} \alpha_l,
\]

so

\[
\alpha_l = F_l(u_{k_0}) \alpha_0.
\]

**Case 2:** \( F_l(u_{k_0}) = 0 \).
Let $m \in \mathbb{N}$ be such that
\[ |T(u_k)| + \frac{1}{m}|F_l(u_k)| < 1 \quad (0 \leq k \neq k_0 \leq n). \]

Let
\[ R_1 := T + \frac{1}{m}F_l, \quad R_2 := T - \frac{1}{m}F_l. \]

By Theorem 2.1, it follows that, for $i = 1, 2$,
\[ \|R_i\| = \max\{|T(u_j)| + \frac{1}{m}|F_l(u_j)| : 0 \leq j \leq n\} \]
\[ = \max\{|T(u_{k_0})|, |T(u_k)| + \frac{1}{m}|F_l(u_k)| : 0 \leq k \neq k_0 \leq n\} \]
\[ = 1. \]

Since
\[ 1 \geq f(R_i) = f(T) \pm \frac{1}{m}f(F_l) = 1 \pm \frac{1}{m}\alpha_l \quad (i = 1, 2), \]
which shows that $\alpha_i = 0 = F_l(u_{k_0})\alpha_0$. We have shown the claim. It follows that
\[ 1 = f(T) \]
\[ = \sum_{0 \leq j \leq n} a_j\alpha_j \]
\[ = a_0\alpha_0 + (\sum_{1 \leq j \leq n} a_jF_j(u_{k_0}))\alpha_0 \]
\[ = (a_0F_0(u_{k_0}) + \sum_{1 \leq j \leq n} a_jF_j(u_{k_0}))\alpha_0 \quad \text{(because of } F_0(u_{k_0}) = 1) \]
\[ = T(u_{k_0})\alpha_0, \]
which shows that $\alpha_0 = \text{sign}(T(u_{k_0}))$. Therefore,
\[ \alpha_l = \text{sign}(T(u_{k_0}))F_l(u_{k_0}) \]
for $1 \leq l \leq n$. Since $f$ is unique, $T \in \text{sm}B_{L,1}(a_{10})$. \hfill \Box

References


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