Quasi-reversibility of the Ring of $2 \times 2$ Matrices over an Arbitrary Field

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Abstract. A ring $R$ is quasi-reversible if $0 \neq ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$, where $I(R)$ is the set of all idempotents in $R$. In this short paper, we prove that the ring of $2 \times 2$ matrices over an arbitrary field is quasi-reversible, which is an answer to the question given by Da Woon Jung et al. in [Bull. Korean Math. Soc., 56(4) (2019) 993-1006].

1. Introduction

Let $R$ be a ring. Use $I(R)$ to denote the set of all idempotents in $R$ and $I(R)' = I(R) \setminus \{0\}$. Let $\text{Mat}_n(R)$ be $n \times n$ matrix ring over $R$. Following Da Woon Jung et. al. [1] a ring $R$ is quasi-reversible provided that if $ab \in I(R)'$ for $a, b \in R$, then $ba \in I(R)$.

Theorem 1.1. ([1, Theorem 1.8]) $\text{Mat}_2(\mathbb{Z}_2)$ is quasi-reversible.

2. Main Result

Now, we propose an answer to Question 1 stated in Da Woon Jung et. al. [1].

Question 2.1. Let $K$ be a field. Is $\text{Mat}_2(K)$ quasi-reversible?

Theorem 2.2. Let $K$ be a field. Then $\text{Mat}_2(K)$ is quasi-reversible.

Proof. Let $K$ be a field and $A, B \in \text{Mat}_2(K)$ such that $AB \in I(\text{Mat}_2(K))'$. If $A$ is invertible, then

$$BA = A^{-1}(AB)A = A^{-1}(ABAB)A = BABA.$$
Hence $BA \in I(Mat_2(K))$. Similarly, if $B$ is invertible, then $BA$ is idempotent. It remains to consider the case that $A$ and $B$ are non-invertible. To this end we prove the following claims:

**Claim 1.** If $M = (m_{ij})$ is a singular $2 \times 2$ matrix, then $M^2 = tr(M)M$ where, $tr(M) = m_{11} + m_{22}$.

*Proof of Claim 1.* Since $|M| = 0$, it follows that $m_{11}m_{22} = m_{12}m_{21}$. Consequently, we have

$$M^2 = \begin{pmatrix} m_{11}^2 + m_{12}m_{21} & m_{11}m_{12} + m_{12}m_{22} \\ m_{11}m_{21} + m_{21}m_{12} & m_{12}m_{21} + m_{22}^2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 + m_{11}m_{22} & m_{12}(m_{11} + m_{22}) \\ m_{21}(m_{11} + m_{22}) & m_{11}m_{22} + m_{22}^2 \end{pmatrix} = (m_{11} + m_{22})M = tr(M)M.$$

**Claim 2.** If $M = (m_{ij})$ is a non-zero singular and idempotent $2 \times 2$ matrix, then $tr(M) = 1$.

*Proof of Claim 2.* By Claim 1, we have $0 \neq M = M^2 = tr(M)M$ thus $(tr(M) - 1)M = 0$ and hence $tr(M) = 1$.

Now, Let $A$ and $B$ be two non-invertible matrices such that $AB \in I(Mat_2(K))'$.

Then, Claim 2 concludes $tr(AB) = 1$. Thus, by Claim 1 we deduce that

$$(BA)^2 = tr(BA)BA = tr(AB)BA = BA.$$

Therefore, $BA$ is idempotent and the proof is completed. □

**References**