Non Existence of PR-semi-slant Warped Product Submanifolds in a Para-Kähler Manifold

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Abstract. In this paper, we prove that there are no non-trivial PR-semi-slant warped product submanifolds with proper slant coefficients in para-Kähler manifolds $\mathcal{M}$. We also present a numerical example that illustrates the existence of a PR-warped product submanifold in $\mathcal{M}$.

1. Introduction

Warped product manifolds are arguably the most natural and significant generalization of Riemannian product manifolds. Bishop and O’Neill in [3] introduced the concept of warped product manifolds to construct manifolds of nonpositive curvature. Later O’Neill in [20] extended the theory of Riemannian warped product manifolds for pseudo-Riemannian warped product manifolds. More precisely, let $(B,g_B)$ and $(F,g_F)$ be two pseudo-Riemannian manifolds and let $f : B \to (0, \infty)$ be a positive differentiable function. Consider the product manifold $B \times F$ with projections

$$
\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F.
$$

The warped product $B \times_f F$ is the manifold $B \times F$ endowed with the pseudo-Riemannian structure such that

$$
g(X,Y) = g_B(\pi_*(X),\pi_*(Y)) + (f \circ \pi)^2 g_F(\sigma_*(X),\sigma_*(Y)),
$$

(or equivalently, $g = g_B + (f)^2 g_F$) for any $X,Y \in \Gamma(TM)$, where ‘$*$’ stands for derivation map. The function $f$ is called the warping function of the warped product.
The theory of warped products attained attention when Chen, in [7], introduced a new class of CR-submanifolds [5] called CR-warped product Kaehlerian manifolds and investigated some existence and non-existence results. Hasegawa-Mihai [15] and Munteanu [19] continued this line of research for Sasakian manifold, which can be viewed as an odd-dimensional counterpart of a K"ahler manifold. Since then, several differential geometer’s have studied the existence (or non-existence) of warped product submanifolds in Riemannian metric manifolds [9, 10, 25].

In addition to Riemannian metric manifolds, manifolds with neutral metric structures (i.e., para-Hermitian manifolds) have been explored in various geometric and physical problems, and the theory successfully applied to supersymmetric field theory, string theory and black holes [13, 16, 17, 21, 22]. The study of neutral metric manifolds was less prominent until Davidov et al. in [12] presented some analogies and differences between the structures admitting the neutral metric and the Riemannian metric. Thereafter, Chen and others initiated the study of the geometry of pseudo-Riemannian warped product submanifolds in para complex (contact) manifolds [1, 11, 24]. Motivated by the above-mentioned works, in this paper we investigate the existence (or non-existence) of \(PR\)-semi-slant warped product submanifolds in para-K"ahler manifolds by assuming proper and improper slant factor aspects.

The organization of the article is as follows. In Section 2 we recall some fundamental concepts about para-K"ahler manifolds and their submanifolds, warped product manifolds, along with a few relevant results required for the present study. Section 3 deals with the definitions of slant, \(PR\)-semi-slant submanifolds, and derivations of the necessary and sufficient conditions for integrability and totally geodesic foliation of the distributions associated with the definition of \(PR\)-semi-slant submanifolds. In Section 4, we derive non-existence results for non-trivial \(PR\)-semi-slant warped product submanifolds of the forms \(M_T \times_f M_\lambda\) and \(M_\lambda \times_f M_T\), where \(M_T\) is an invariant submanifold and \(M_\lambda\) is a proper slant submanifold in \(\mathbb{M}\). Finally in Section 5, we present an example that illustrates the existence of a \(PR\)-semi-slant warped product submanifold with an improper slant coefficient (called \(PR\)-warped product) in \(\mathbb{M}\).

2. Preliminaries

Let \(\mathbb{M}\) be an even-dimensional smooth manifold. A smooth manifold \(\mathbb{M}\) is said to have an almost product structure if

\[
P^2 = I, \tag{2.1}
\]

where \(P\) is a tensor field of type (1, 1) and \(I\) is the identity transformation on \(\mathbb{M}\). For this, the pair \((\mathbb{M}, P)\) is called an almost product manifold. An almost para-Hermitian manifold \((\mathbb{M}, \mathcal{P}, \mathcal{G})\) [18] is a smooth manifold associated with an almost product structure \(\mathcal{P}\) and a pseudo-Riemannian metric \(\mathcal{G}\) satisfying

\[
\mathcal{G}(\mathcal{P}X, \mathcal{P}Y) + \mathcal{G}(X, Y) = 0, \tag{2.2}
\]
for any $X,Y$ tangent to $\overline{M}$. Clearly, from equations (2.1) and (2.2), we conclude that the signature of $\overline{g}$ is necessarily $(m,m)$, and
\[
(2.3) \quad \overline{g}(P\! X, Y) + \overline{g}(X, P\! Y) = 0,
\]
for any $X,Y \in \Gamma(T\overline{M})$; where $\Gamma(T\overline{M})$ is the Lie algebra of vector fields in $\overline{M}$. The fundamental 2-form $\omega$ of $M$ is defined by
\[
(2.4) \quad \omega(X,Y) = \overline{g}(X, P\! Y), \quad \forall X,Y \in \Gamma(T\overline{M}).
\]

**Definition 2.1.** An almost para-Hermitian manifold $M$ is called a para-Kähler manifold [8] if $P$ is parallel with respect to $\nabla$, i.e.,
\[
(2.5) \quad (\nabla_X P\! Y) = 0, \quad \forall X,Y \in \Gamma(T\overline{M})
\]
where $\nabla$ is the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$.

Let us consider that $M$ be an isometrically immersed submanifold of a para-Kähler manifold $\overline{M}$ in the sense of O'Neill [20]. Let $g$ be the induced metric on $M$ such that $g = \overline{g}|_M$ [14]. We denote the set of vector fields normal to $M$ by $\Gamma(TM^\perp)$ and the sections of tangent bundle $TM$ of $M$ by $\Gamma(TM)$. Then the Gauss-Weingarten formulas are given by, respectively,
\[
(2.6) \quad \nabla_X Y = \nabla_X Y + h(X,Y),
\]
\[
(2.7) \quad \nabla_X \zeta = -A_\zeta X + \nabla_X^\perp \zeta,
\]
for any $X,Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where $\nabla$ is the induced connection, $\nabla^\perp$ is the normal connection on the normal bundle $\Gamma(TM^\perp)$, $h$ is the second fundamental form, and the shape operator $A_\zeta$ associated with the normal section $\zeta$ is given by
\[
(2.8) \quad g(A_\zeta X, Y) = \overline{g}(h(X,Y), \zeta).
\]
If we write, for all $\tau \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$ that
\[
(2.9) \quad P\tau = t\tau + n\tau,
\]
\[
(2.10) \quad P\zeta = t'\zeta + n'\zeta,
\]
where $t\tau$ (resp., $n\tau$) is tangential (resp., normal) part of $P\tau$ and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $P\zeta$, then the submanifold $M$ is said to be invariant if $n$ is identically zero and anti-invariant if $t$ is identically zero. Now, from equations (2.3) and (2.9), we obtain for any $X,Y \in \Gamma(TM)$ that
\[
(2.11) \quad g(X, tY) = -g(tX, Y).
\]
Here, we state some important results on warped product manifolds;

**Proposition 2.2.** For $X,Y \in \Gamma(TB)$ and $Z,W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that
(i) \( \nabla_X Y \in \Gamma(TB) \),
(ii) \( \nabla_X Z = \nabla_Z X = \left( \frac{X}{f} \right) Z \),
(iii) \( \nabla_Z W = \frac{-g(Z, W)}{f} \nabla f \),

where \( \nabla \) denotes the Levi-Civita connection on \( M \) and \( \nabla f \) is the gradient of \( f \) defined by \( g(\nabla f, X) = Xf \).

**Remark 2.3.** It is also important to note that for a warped product \( M = B \times_f F \),
(1) \( M \) is said to be trivial if \( f \) is constant,
(2) \( B \) is totally geodesic and \( F \) is totally umbilical in \( M \),
(3) To reduce the complexity, we represent a vector field \( X \) on \( B \) with its lift \( \overline{X} \) and a vector field \( Z \) on \( F \) with its lift \( \overline{Z} \).

Now, we present a lemma for later use:

**Lemma 2.4.** Let \( M = B \times_f F \) be a non-trivial warped product submanifold of a para-Kähler manifold \( M \), then

\[
(2.12) \quad g(A_n Z X, W) = g(A_n W X, Z) = -tX(\ln f)g(Z, W),
\]

for any \( X, Y \in \Gamma(TB) \) and \( Z, W \in \Gamma(TF) \).

**Proof.** The proof of this lemma follows easily by using equations (2.5), (2.9), (2.11) and Gauss-Weingarten formulas. \( \square \)

3. \( \mathcal{PR} \)-semi-slant Submanifolds

In this section, motivated by [1, 24], we first state the definition of a slant submanifold in a para-Kähler manifold \( \overline{M} \), and then continue the study by introducing a generalized class of a \( \mathcal{PR} \)-submanifold [11] called \( \mathcal{PR} \)-semi-slant submanifold in \( \overline{M} \).

Let \( M \) be a non-degenerate submanifold of a para-Kähler manifold \( \overline{M} \) such that \( t^2 X = \lambda X = \lambda \mathcal{P}_2 X \), \( g(tX, Y) = -g(X, tY) \) for any \( X, Y \in \Gamma(TM) \), where \( \lambda \) is a real coefficient. Then with the help of equation (2.11), we have

\[
(3.1) \quad \frac{g(\mathcal{P}X, tY)}{||\mathcal{P}X|| ||tY||} = -\frac{g(X, \mathcal{P}tY)}{||\mathcal{P}X|| ||tY||} = -\frac{g(X, t^2 Y)}{||\mathcal{P}X|| ||tY||} = -\lambda \frac{g(X, \mathcal{P}^2 Y)}{||\mathcal{P}X|| ||tY||} = \lambda \frac{g(\mathcal{P}X, \mathcal{P}Y)}{||\mathcal{P}X|| ||tY||}.
\]

On the other hand,

\[
(3.2) \quad \frac{g(\mathcal{P}X, tY)}{||\mathcal{P}X|| ||tY||} = \frac{g(tX, tY)}{||\mathcal{P}X|| ||tY||}.
\]
In particular, from equations (3.1) and (3.2), we obtain for $X = Y$ that $g(PX, tX) = g(PY, tY) = \frac{\sqrt{\lambda}}{|PX|}$. Here we call $\lambda$ a slant coefficient and consequently $M$ a slant submanifold. Conversely, assume that $M$ is a slant submanifold then $\lambda \frac{|PX|}{|tX|} = \frac{|tX|}{|PX|}$, where $X$ is a non-lightlike vector field. We obtain by the consequence of previous equation for any $X, Y \in \Gamma(TM)$ that $-\lambda g(X, P^2 Y) = g(PX, tY)$, which yields $g(X, t^2 Y) = \lambda g(X, P^2 Y)$. Hence, $t^2 = \lambda I$, $g(tX, Y) = -g(X, tY)$ by virtue of the fact that structure is para-Kähler and $X$ is any non-lightlike vector field.

**Remark 3.1.** The slant coefficient $\lambda$ is sometimes chosen as $\cos^2 \theta$ or $\cosh^2 \theta$ or $-\sinh^2 \theta$ for vector fields tangent to $M$, where $\theta$ is a real constant, called slant angle. The nature of the $\lambda$ equals to $\cos^2 \theta$ or $\cosh^2 \theta$ or $-\sinh^2 \theta$ depending on the consideration of vector fields (that is, angle between spacelike–spacelike or timelike–timelike or timelike–spacelike vector fields). The authors in [1, 2] distinguished different cases for $\lambda$, depending on the behaviour of vector fields.

As a consequence of the above characterization, we define slant submanifold in an almost para-Hermitian manifold;

**Definition 3.2.** Let $M$ be an isometrically immersed submanifold of an almost para-Hermitian manifold $\overline{M}$ and let $D_{\lambda}$ be the distribution on $M$. Then $D_{\lambda}$ is said to be slant distribution on $M$, accordingly $M$ slant submanifold, if there exist a real valued constant $\lambda$ such that

$$t^2 = \lambda I, \quad g(tX, Y) = -g(X, tY),$$

for any non-lightlike tangent vector fields $X, Y \in D_{\lambda}$ on $M$. Here, $\lambda$ will be called slant coefficient that is globally constant and independent of the choice of point on $M$ in $\overline{M}$.

**Remark 3.3.** Since the manifold $M$ is non-degenerate (i.e., $M$ includes either spacelike vector fields or timelike vector fields), thus our definition of slant submanifold can be considered as the generalization of definitions given in [6, 23] that covers only the spacelike vector fields which implies $\lambda = \cos^2 \theta$, where $\theta$ is a slant angle.

Now, we have the following definition.

**Definition 3.4.** Let $M$ be an isometrically immersed submanifold of an almost para-Hermitian metric manifold $\overline{M}(\mathcal{P}, \mathcal{g})$. Then the submanifold $M$ is said to be a $\mathcal{P}\mathcal{R}$-semi-slant if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathcal{D}_T, \mathcal{D}_\lambda)$ satisfying the following conditions:

(i) $TM = \mathcal{D}_T \oplus \mathcal{D}_\lambda$,

(ii) the distribution $\mathcal{D}_T$ is invariant under $\mathcal{P}$, i.e., $\mathcal{P}(\mathcal{D}_T) \subset \mathcal{D}_T$ and

(iii) the distribution $\mathcal{D}_\lambda$ is slant distribution with slant coefficient $\lambda$. 
A $\mathcal{PR}$-semi-slant submanifold is

1. $\mathcal{PR}$-submanifold if $\mathcal{D}_T \neq \{0\}$ and $\mathcal{D}_{\lambda} \neq \{0\}$ with $\lambda = 0$ [11],
2. proper if $\mathcal{D}_T \neq \{0\}$, $\mathcal{D}_{\lambda} \neq \{0\}$ and $\lambda \neq 0, 1$.

Here, we give an example for demonstrating the existence of proper $\mathcal{PR}$-semi-slant submanifold $M$ in a para-Kähler manifold $\overline{M}$.

**Example 3.5.** Let $\overline{M} = \mathbb{R}^8$ be a 8-dimensional manifold with the standard Cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$.

Define the para-Kähler pseudo-Riemannian metric structure $(\mathcal{P}, \mathcal{g})$ on $\overline{M}$ by

$$\mathcal{P}e_1 = e_5, \mathcal{P}e_2 = e_6, \mathcal{P}e_3 = e_7, \mathcal{P}e_4 = e_8, \mathcal{P}e_8 = e_4, \mathcal{P}e_7 = e_3, \mathcal{P}e_6 = e_2$$

(3.3) \[ \mathcal{P}e_5 = e_1, \mathcal{g} = \sum_{i=1}^{4} (dx_i)^2 - \sum_{j=5}^{8} (dx_j)^2. \]

Here, $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a local orthonormal frame for $\Gamma(TM)$, given by

$$e_i = \frac{\partial}{\partial x_i}, e_j = \frac{\partial}{\partial x_j}.$$

Let $M$ be an isometrically immersed pseudo-Riemannian submanifold in a para-Kähler manifold $\overline{M}$ defined by

$$\Omega(x_1, x_2, x_3, x_4) = \left( x_1 \cos(\alpha), x_3 \cosh(\alpha), x_4 \sinh(\alpha), x_1 \sin(\alpha), x_2, x_4 \cosh(\alpha), x_3 \sinh(\alpha), k \right),$$

where $\alpha \in (0, \pi/2)$ and $k \neq 0$. Then the tangent bundle $\Gamma(TM)$ of $M$ is spanned by the vector fields

$$X_1 = \cos(\alpha)e_1 + \sin(\alpha)e_4, \quad X_2 = e_5,$$

$$X_3 = \cosh(\alpha)e_2 + \sinh(\alpha)e_7,$$

$$X_4 = \sinh(\alpha)e_3 + \cosh(\alpha)e_6,$$

(3.4)

where $X_1, X_3$ are spacelike and $X_2, X_4$ are timelike vector fields tangent to $M$.

Now, the space $\mathcal{P}(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure $(\mathcal{P}, \mathcal{g})$ of $\overline{M}$ becomes

$$\mathcal{P}(X_1) = \cos(\alpha)e_5 + \sin(\alpha)e_8, \quad \mathcal{P}(X_2) = e_1,$$

$$\mathcal{P}(X_3) = \cosh(\alpha)e_6 + \sinh(\alpha)e_3,$$

$$\mathcal{P}(X_4) = \sinh(\alpha)e_7 + \cosh(\alpha)e_2.$$

(3.5)

Therefore from equations (5.2), (5.3) and (5.1), we obtain that $\mathcal{D}_T$ and $\mathcal{D}_{\lambda}$, are the distributions defined by $\text{span}\{X_3, X_4\}$, and $\text{span}\{X_1, X_2\}$ respectively, where $\mathcal{D}_T$ is an invariant distribution and $\mathcal{D}_{\lambda}$ is a slant distribution with slant coefficient $\lambda = |\cos(\alpha)|$. Thus $M$ becomes a proper $\mathcal{PR}$-semi-slant submanifold $M$ of a para-Kähler manifold $\overline{M}$.
Next, we prove the following result as a proposition for later use.

**Proposition 3.6.** Let $M$ be a slant submanifold of an almost para-Hermitian manifold $M(\mathcal{P}, \mathcal{G})$. Then

\[(3.6)\quad g(tX, tY) = \lambda\{-g(X, Y)\} = \lambda g(\mathcal{P}X, \mathcal{P}Y),\]

\[(3.7)\quad g(nX, nY) = (1 - \lambda)\{-g(X, Y)\} = (1 - \lambda)g(\mathcal{P}X, \mathcal{P}Y),\]

for any $X, Y \in \Gamma(TM)$.

**Proof.** We have from equations (2.1), (2.3) and (2.11), that $g(tX, tY) = -g(X, t^2Y) = -\lambda g(X, \mathcal{P}^2Y) = \lambda g(\mathcal{P}X, \mathcal{P}Y)$. Therefore, by the use of equation (2.2) and the Definition 3.2(a), we get equation (3.6). Equation (3.7) follows from equations (2.9) and (3.6). This completes the proof of the proposition.

Here, we obtain some necessary and sufficient conditions for the foliation determined by distributions allied with the definition of $\mathcal{P}\mathcal{R}$-semi-slant submanifolds in para-Kähler manifolds $\overline{M}$ to be involutive and totally geodesic.

**Theorem 3.7.** Let $M$ be a proper $\mathcal{P}\mathcal{R}$-semi-slant submanifold $\overline{M}$ of a para-Kähler manifold $\overline{M}$. Then the distribution $(\mathcal{D}_T)$

1. is involutive if and only if $h(X, tY) = h(tX, Y)$;
2. defines totally geodesic foliation if and only if $A_{nZ}tY = A_{ntZ}Y$;

for any $X, Y \in \Gamma(\mathcal{D}_T)$ and $Z \in \Gamma(\mathcal{D}_\lambda)$.

**Proof.** For $M$ to be a proper $\mathcal{P}\mathcal{R}$-semi-slant submanifold $\overline{M}$ of a para-Kähler manifold, we have

\[(3.8)\quad g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z).\]

Now, using equations (2.2) and (2.6), we achieve that

\[(3.9)\quad g([X, Y], Z) = -\overline{g}(\mathcal{P}\nabla_X Y, \mathcal{P}Z) + \overline{g}(\mathcal{P}\nabla_Y X, \mathcal{P}Z).\]

Employing equations (2.5), (2.8), (2.9), (3.2) and Gauss-Weingarten formula in equation (3.9), we obtain that

\[(3.10)\quad g([X, Y], Z) = \lambda g([X, Y], Z) + \overline{g}(h(X, \mathcal{P}Y), nZ) - \overline{g}(h(Y, \mathcal{P}X), nZ).\]

We can rewrite above equation as follows;

\[(3.11)\quad (1 - \lambda)g([X, Y], Z) = \overline{g}(h(X, \mathcal{P}Y), nZ) - \overline{g}(h(Y, \mathcal{P}X), nZ).\]

Since $\lambda = 1$ is impossible and $X, Y, Z$ are all non-degenerate vector fields, hence from equation (3.11), we can easily deduce that the distribution $(\mathcal{D}_T)$ is involutive if and only if $h(X, tY) = h(tX, Y)$. Furthermore, for totally geodesic $M$ in $\overline{M}$, we obtain that $g(\nabla_X Y, Z) = \overline{g}(\nabla_X Y, Z)$. Now in light of equations (2.2), (2.5), (2.9),
(3.2) and the fact that the distributions \((\mathcal{D}_T, \mathcal{D}_\lambda)\) are orthogonal, we derive the required condition for \((\mathcal{D}_T)\) to be totally geodesic. This completes the proof of the theorem.

**Theorem 3.8.** Let \(M\) be a proper \(PR\)-semi-slant submanifold \(M\) of a para-Kähler manifold \(\overline{M}\). Then the distribution \((\mathcal{D}_\lambda)\)

1. is involutive if and only if \(g(A_nWZ - A_nZW, tX) = g(A_nZW - A_nWZ, X)\);
2. defines totally geodesic if and only if \(A_nWZ = A_nZW\),

for any \(Z, W \in \Gamma(\mathcal{D}_\lambda)\) and \(X, Y \in \Gamma(\mathcal{D}_T)\).

**Proof.** The proof of this theorem can be obtained by following the steps of the Theorem 3.7.

4. \(PR\)-semi-slant Warped Product Submanifolds

In this section, we investigate the existence or non-existence of non-trivial \(PR\)-semi-slant warped product submanifolds of the forms \(M_T \times_f M_\lambda, M_\lambda \times_f M_T\) in para-Kähler manifolds \(\overline{M}\), where \(M_T\) and \(M_\lambda\) are invariant and slant submanifolds of \(\overline{M}\), respectively.

Now, we derive the following major results.

**Theorem 4.1.** Let \(M \rightarrow \overline{M}\) be an isometric immersion of a proper \(PR\)-semi-slant submanifold \(M\) into a para-Kähler manifold \(\overline{M}\). Then there does not exist any non-trivial \(PR\)-semi-slant warped product submanifold \(M = M_T \times_f M_\lambda\) in \(\overline{M}\).

**Proof.** Let us assume that \(M = M_T \times_f M_\lambda\) is a non-trivial \(PR\)-semi-slant warped product submanifold in a para-Kähler manifold \(\overline{M}\). Then by using equations (2.5) and (2.6), we obtain that

\[
\nabla_Z \mathcal{P}X = \mathcal{P}\nabla_Z X
\]

for any non-degenerate vector fields \(X \in \Gamma(M_T)\) and \(Z \in \Gamma(M_\lambda)\). By putting together equations (2.9), (2.10) and Proposition 2.2 in (4.1), we get that

\[
\nabla_Z tX + h(Z, tX) = X(ln f)tZ + X(ln f)nZ + t'h(X, Z) + n'h(X, Z).
\]

By equating the tangential and normal parts of equation (4.2), we get

\[
tX(ln f)Z = X(ln f)tZ + t'h(X, Z),
\]

\[
h(Z, tX) = X(ln f)nZ + n'h(X, Z).
\]

On the other hand, by virtue of Lemma 2.4 and the fact that \(\Gamma(M_T)\) is a invariant submanifold, we have

\[
\overline{g}(h(X, Z), nZ) = \overline{g}(h(Z, X), nX) = 0,
\]
for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Moreover, we can write from equations (2.3), (2.10) and (4.5) that

\begin{equation}
(4.6) \quad g(h(X, Z), \mathcal{P}Z) = g(t'h(X, Z), Z) + g(n'h(X, Z), Z) = 0.
\end{equation}

We attain from equation (4.6), that

\begin{equation}
(4.7) \quad g(t'h(X, Z), Z) = 0 \implies t'h(X, Z) = 0.
\end{equation}

Now, from above expression and the fact that the distributions are orthogonal, we conclude that

\begin{equation}
(4.8) \quad tX(\ln f)g(Z, tZ) = X(\ln f)g(Z, Z) + g(t'h(X, Z), tZ).
\end{equation}

Then using equations (3.6) and (4.7) in (4.8), we achieve that

\begin{equation}
(4.9) \quad tX(\ln f)g(Z, tZ) = -\lambda X(\ln f)g(Z, Z).
\end{equation}

Furthermore, employing equations (2.2), (2.6) and (2.7), we get

\begin{equation}
(4.10) \quad X(\ln f)g(Z, Z) = (tX, \nabla_Z tZ) = g(tX, A_n Z).
\end{equation}

The above equation with the help of equations (2.8), (4.5) and Proposition 2.2(iii), becomes

\begin{equation}
(4.11) \quad X(\ln f)g(Z, Z) = -g(tX, g(Z, tZ)\text{grad}(\ln f)) = -tX(\ln f)g(Z, tZ).
\end{equation}

Now, from equations (4.9) and (4.11), we derive that

\begin{equation}
(4.12) \quad X(\ln f)g(Z, Z) = \lambda X(\ln f)g(Z, Z),
\end{equation}

for all non-degenerate vector fields $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Therefore from equation (4.12), we conclude that either $\lambda = 1$ or $f$ is a constant function on $M_T$. But $\lambda = 1$ is impossible since $M$ is a proper $\overline{PR}$-semi-slant submanifold in $\overline{M}$. Hence $f$ must be constant on $M_T$. Which is contradiction to our assumption. This completes the proof of the theorem. \hfill \Box

**Theorem 4.2.** Let $M$ be an isometrically immersed $\overline{PR}$-semi-slant submanifold of a para-Kähler manifold $\overline{M}$. Then there doesn't exist a non-trivial $\overline{PR}$-semi-slant warped product submanifold $M = M_\lambda \times_f M_T$ in $\overline{M}$.

**Proof.** Let us assume that $M = M_\lambda \times_f M_T$ be a non-trivial $\overline{PR}$-semi-slant warped product submanifold in a para-Kähler manifold $\overline{M}$. Then we can write from Proposition 2.2, Gauss formula and the connection property for $\nabla$, that

\begin{equation}
(4.13) \quad \overline{g}(\nabla_X Z, X) = -g(X, \nabla_X Z) = -Z(\ln f)g(X, X),
\end{equation}
for any $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Also from equations (2.2), (2.6), (2.9) and the fact that structure is para-Kähler, we have that

$$
\mathcal{g}(\nabla_X Z, Z) = -g(\nabla_X P_{X} Z, tZ) - \mathcal{g}(h(X, P_{X} Z), nZ).
$$

(4.14)

The above equation by the use of connection property for $\nabla$ and Proposition 2.2, reduces to

$$
\mathcal{g}(\nabla_X Z, Z) = tZ(\ln f)g(P_{X} Z, X) - \mathcal{g}(h(X, P_{X} Z), nZ).
$$

(4.15)

Now by the virtue of equations (4.13) and (4.15), we obtain for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$ that

$$
-Z(\ln f)g(X, Z) = tZ(\ln f)g(tX, X) - \mathcal{g}(h(X, tX), nZ).
$$

On the other hand, using equations (2.5), (2.9), (2.10) and the fact that $M_T$ is an invariant submanifold, we derive that

$$
h(Z, tX) = t'h(Z, X) + n'h(Z, X).
$$

(4.17)

By comparing the tangential and normal parts from the above expression, we get

$$
h(Z, tX) = n'h(Z, X) \quad \text{and} \quad t'h(Z, X) = 0,
$$

(4.18)

for all $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Analogous to equation (4.17), we obtain by simplifying equation $\nabla_X P_{Z} = P_{X} \nabla_X Z$ and using equation (4.18) that

$$
tZ(\ln f)X + h(X, tZ) - A_{nZ} X + \nabla^k_X nZ = Z(\ln f)tX + n'h(X, Z).
$$

(4.19)

Now, equating the tangential parts of equation (4.19) and interchanging $X$ by $tX$, we achieve that

$$
tZ(\ln f)tX - A_{nZ} tX = Z(\ln f)t^2 X
$$

(4.20)

Taking inner product of equation (4.20) with $X$ and then using equations (2.8), (2.11), we deduce that

$$
tZ(\ln f)g(X, tX) - \mathcal{g}(h(X, tX), nZ) = -Z(\ln f)g(tX, tX).
$$

(4.21)

Employing equation (4.16) in equation (4.21), we find that

$$
Z(\ln f)g(X, X) = Z(\ln f)g(tX, tX).
$$

(4.22)

Furthermore by applying Proposition 3.6 and the fact that $M_T$ is an invariant submanifold in the above equation, we get

$$
Z(\ln f)g(X, X) = 0
$$

(4.23)
for any non-degenerate vector fields $X \in \Gamma(M_T)$ and $Z \in \Gamma(M_\lambda)$. Now, from equation (4.23), we deduce that $f$ must be constant on $M_\lambda$. Which is contradiction to our assumption that $M = M_\lambda \times_f M_T$ is a non-trivial PR-semi-slant warped product submanifold in a para-Kähler manifold $\overline{M}$. Hence the proof of the theorem completed.

5. Example

In contrast to the results in Section 4, we hereby give a numerical example to illustrates the existence of a PR-warped product submanifold $M$ of the form $M_T \times_f M_\lambda$ with vanishing slant coefficient (i.e. $M_T \times_f M_\perp$) in a para-Kähler manifold.

Let $\overline{M} = \mathbb{R}^8$ be a 8-dimensional manifold with the standard Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$. Define the para-Kähler pseudo-Riemannian metric structure $(\mathcal{P}, \mathcal{g})$ on $\overline{M}$ by

$\mathcal{P}e_1 = e_5$, $\mathcal{P}e_2 = e_6$, $\mathcal{P}e_3 = e_7$, $\mathcal{P}e_4 = e_8$, $\mathcal{P}e_8 = e_4$, $\mathcal{P}e_7 = e_3$, $\mathcal{P}e_6 = e_2$

(5.1) $\mathcal{P}e_5 = e_1$, $\mathcal{g} = \sum_{i=1}^{4} (dx_i)^2 - \sum_{j=5}^{8} (dy_j)^2$,

here $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a local orthonormal frame for $\Gamma(TM)$, given by

$e_i = \frac{\partial}{\partial x_i}$, $e_j = \frac{\partial}{\partial y_j}$.

Let $M$ be an isometrically immersed pseudo-Riemannian submanifold in a para-Kähler manifold $\overline{M}$ defined by

$\Omega(u, v, \gamma, \beta) = \left( u \cosh(\gamma), u \cosh(\beta), v \sinh(\gamma), v \sinh(\beta), v \cosh(\gamma), v \cosh(\beta), u \sinh(\gamma), u \sinh(\beta) \right)$,

where $\gamma, \beta \in (0, \pi/2)$ and either $u < v \in \mathbb{R}_+$ or $v < u \in \mathbb{R}_-$. Then the tangent bundle $\Gamma(TM)$ of $M$ is spanned by the vectors

$X_u = \cosh(\gamma)e_1 + \cosh(\beta)e_2 + \sinh(\gamma)e_7 + \sinh(\beta)e_8$,

$X_v = \sinh(\gamma)e_3 + \sinh(\beta)e_4 + \cosh(\gamma)e_5 + \cosh(\beta)e_6$,

(5.2) $X_\gamma = u \sinh(\gamma)e_1 + v \cosh(\gamma)e_4 + v \sinh(\gamma)e_5 + u \cosh(\gamma)e_7$,

$X_\beta = u \sinh(\beta)e_2 + v \cosh(\beta)e_4 + v \sinh(\beta)e_6 + u \cosh(\beta)e_8$,

where $X_u, X_\gamma, X_\beta$ are spacelike vector fields and $X_v$ is a timelike vector field tangent to $M$. Now the space $\mathcal{P}(TM)$ with respect to the para-Kähler pseudo-Riemannian
metric structure \((\mathcal{P}, \mathcal{g})\) of \(\overline{M}\) becomes
\[
\mathcal{P}(X_u) = \sinh(\gamma) e_3 + \sinh(\beta) e_4 + \cosh(\gamma) e_5 + \cosh(\beta) e_6,
\]
\[
\mathcal{P}(X_v) = \cosh(\gamma) e_1 + \cosh(\beta) e_2 + \sinh(\gamma) e_7 + \sinh(\beta) e_8,
\]
\[
\mathcal{P}(X_\gamma) = v \sinh(\gamma) e_1 + u \cosh(\gamma) e_3 + v \sinh(\gamma) e_5 + v \cosh(\gamma) e_7,
\]
\[
\mathcal{P}(X_\beta) = v \sinh(\beta) e_2 + u \cosh(\beta) e_4 + u \sinh(\beta) e_6 + v \cosh(\beta) e_8.
\]
(5.3)

From equations (5.2) and (5.3), we obtain that \(\mathcal{P}(X_\gamma)\) and \(\mathcal{P}(X_\beta)\) are orthogonal to \(\Gamma(\mathcal{T}M)\), \(\mathcal{P}(X_u)\) and \(\mathcal{P}(X_v)\) are tangent to \(M\). This means \(\mathcal{D}_T\) and \(\mathcal{D}_\lambda\) can be taken as a subspace span\(\{X_u, X_v\}\) and a subspace span\(\{X_\gamma, X_\beta\}\), respectively. Consequently, \(M\) becomes a \(\mathcal{PR}\)-semi-slant submanifold. Furthermore, we can say from Theorem 3.7 and Theorem 3.8 that \(\mathcal{D}_T\) and \(\mathcal{D}_\lambda\) are integrable. We denote the integral manifolds of \(\mathcal{D}_T\) and \(\mathcal{D}_\lambda\) by \(M_T\) and \(M_\lambda\), respectively, then the induced pseudo-Riemannian metric tensor \(g\) of \(M\) is given by

\[
[g(e_i, e_j)] = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & v^2 - u^2 & 0 \\
0 & 0 & 0 & v^2 - u^2
\end{bmatrix},
\]

that is,
\[
g = 2du^2 - 2dv^2 + (v^2 - u^2)(d\gamma^2 + d\beta^2) = g_{M_T} + (v^2 - u^2) g_{M_\lambda}.
\]

Hence, \(M\) is a 4-dimensional \(\mathcal{PR}\)-semi-slant warped product submanifold in \(\overline{M}\) with warping function \(f = \sqrt{(v^2 - u^2)}\) and slant coefficient \(\lambda\) for \(\mathcal{D}_\lambda\) is 0. Thus, \(M\) is a non-trivial \(\mathcal{PR}\)-semi-slant warped product submanifold with improper slant coefficient in \(M\) (in particular, a \(\mathcal{PR}\)-warped product submanifold in a para-Kähler manifold [11]).

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References


Non Existence of PR-semi-slant Warped Product Submanifolds


