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# ON SYMMETRIC BI-$f$-DERIVATIONS OF LATTICE IMPLICATION ALGEBRAS

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Abstract. In this paper, we introduce the notion of symmetric bi-$f$-derivation of lattice implication algebra and investigated some related properties. Also, we prove that if $D$ is a symmetric bi-$f$-derivation of $L$, then $D(x \to y, z) = f(x) \to D(y, z)$ for all $x, y, z \in L$.

## 1. Introduction

The concept of lattice implication algebra was proposed by Y. Xu [11], in order to establish an alternative logic knowledge representation. Also, in [12], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [13] introduced the notion of filters in a lattice implication, and investigated their properties. The present author [5, 14] introduced the notion of derivation and $f$-derivation in lattice implications algebras and obtained some related results. In this paper, we introduce the notion of symmetric bi-$f$-derivation of lattice implication algebra and investigated some related properties. Also, we prove that if $D$ is a symmetric bi-$f$-derivation of $L$, then $D(x \to y, z) = f(x) \to D(y, z)$ for all $x, y, z \in L$.

## 2. Preliminary

A *lattice implication algebra* is an algebra $(L; \wedge, \vee, ', \to, 0, 1)$ of type $(2, 2, 1, 2, 0, 0)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\to$ " is a binary operation, satisfying the following axioms, for all $x, y, z \in L$,

(L1) $x \to (y \to z) = y \to (x \to z)$,
(L2) $x \to x = 1$,
(L3) $x \to y = y' \to x'$,
(L4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(L5) $(x \to y) \to y = (y \to x) \to x$,

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(L6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

If $L$ satisfies conditions (L1) – (L5), we say that $L$ is a *quasi lattice implication algebra*. A lattice implication algebra $L$ is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\to$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra $L$ by $x \leq y$ if and only if $x \to y = 1$ for all $x, y \in L$.

In a lattice implication algebra $L$, the following hold (see [11]),

(u1) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$,
(u2) $x \to y \leq (y \to z) \to (x \to z)$,
(u3) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$,
(u4) $x' = x \to 0$.
(u5) $x \vee y = (x \to y) \to y$,
(u6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$,
(u7) $x \leq (x \to y) \to y$.

for all $x, y, z \in L$.

**Definition 1.** In a lattice H implication algebra $L$, the following hold, for all $x, y, z \in L$,

(u8) $x \to (x \to y) = x \to y$,
(u9) $x \to (y \to z) = (x \to y) \to (x \to z)$.

**Definition 2.** A subset $F$ of a lattice implication algebra $L$ is called a *filter* of $L$ it satisfies,

(F1) $1 \in F$,
(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$.

**Definition 3.** Let $L_1$ and $L_2$ be lattice implication algebras. A mapping $f : L_1 \to L_2$ is an *implication homomorphism* if

$$f(x \to y) = f(x) \to f(y)$$

for all $x, y \in L_1$. Moreover, if $f : L_1 \to L_2$ satisfies the conditions

$$f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y), f(x') = f(x)'$$

for all $x, y \in L_1$, we say that $f$ is a *lattice implication homomorphism* on $L_1$.

**Definition 4.** Let $L$ be a lattice implication algebra. A mapping $D(.,.) : L \times L \to L$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in L$.

**Definition 5.** Let $L$ be a lattice implication algebra and $x \in L$. A mapping $d(x) = D(x, x)$ is called *trace* of $D(.,.)$, where $D(.,.) : L \times L \to L$ is a symmetric mapping on $L$.

## 3. Symmetric bi-$f$-derivations of lattice implication algebras

In what follows, let $L$ denote a lattice implication algebra and let $f$ be an implication homomorphism on $L$ unless otherwise specified.

**Definition 6.** Let $L$ be a lattice implication algebra and let $f$ be an implication homomorphism on $L$. A symmetric map $D : L \times L \to L$ is called a *symmetric bi-$f$-derivation* of $L$ if the following condition holds

$$D(x \to y, z) = (f(x) \to D(y, z)) \vee (D(x, z) \to f(y))$$

for all $x, y, z \in L$.

The mapping $d : L \to L$ defined by $d(x) = D(x, x)$ is called the *trace* of symmetric bi-$f$-derivation $D$. Obviously, a symmetric bi-$f$-derivation $D$ on $L$ satisfies the relation

$$D(x, y \to z) = (D(x, y) \to f(z)) \vee (f(y) \to D(x, z))$$

for all $x, y, z \in L$.

**Example 1.** *Let $L := \{0, a, b, 1\}$ be a set with the Cayley table.*

| $x$ | $x'$ |
|---|---|
| $0$ | $1$ |
| $a$ | $b$ |
| $b$ | $a$ |
| $1$ | $0$ |

| $\to$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $b$ | $1$ | $1$ | $1$ |
| $b$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

*For any $x \in L$, we have $x' = x \to 0$. The operations $\wedge$ and $\vee$ on $L$ are defined as follows:*

$$x \vee y = (x \to y) \to y, \qquad x \wedge y = ((x' \to y') \to y')'.$$

*Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $D : L \times L \to L$ by*

$$D(x, y) = \begin{cases} a & \text{if } (x, y) = (0, 0) \\ b & \text{if } (x, y) = (0, a) \text{ or } (x, y) = (a, 0) \\ 1, & \text{otherwise} \end{cases}$$

*and define an endomorphism $f : L \to L$ by*

$$f(x) = \begin{cases} a & \text{if } x = 0, a \\ 1 & \text{if } x = 1 \\ b & \text{if } x = b \end{cases}$$

*Then it is easily checked that $D$ is a symmetric bi-$f$-derivation of lattice implication algebra $L$.*

**Proposition 3.1.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation on $L$. Then the mapping $f_1(x) = D(x, z)$ is a $f$-derivation on $L$.*

*Proof.* Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation on $L$. Then

$$\begin{aligned} f_1(x \to y) = D(x \to y, z) &= (f(x) \to D(y,z)) \vee (D(x,z) \to f(y)) \\ &= (f(x) \to f_1(y)) \vee (f_1(x) \to f(y)) \end{aligned}$$

for every $x, y, z \in L$. This implies that $f_1$ is a $f$-derivation on $L$. □

**Proposition 3.2.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(1,x) = D(x,1) = 1$ for all $x \in L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Since $f(1) = 1$, we have

$$\begin{aligned} D(1,x) &= D(1 \to 1, x) \\ &= (f(1) \to D(1,x)) \vee (D(1,x) \to f(1)) \\ &= (1 \to D(1,x) \vee (D(1,x) \to 1) \\ &= D(1,x) \vee 1 = 1 \end{aligned}$$

for every $x \in L$. Similarly, $D(x,1) = 1$ for every $x \in L$. □

**Corollary 3.3.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(1,1) = 1$.*

**Proposition 3.4.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(x,y) = D(x,y) \vee f(x)$ for all $x, y \in L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} D(x,y) &= D(1 \to x, y) \\ &= (f(1) \vee D(x,y)) \vee (D(1,y) \to f(x)) \\ &= (1 \to D(x,y)) \vee (1 \to f(x)) \\ &= D(x,y) \vee f(x) \end{aligned}$$

for all $x, y \in L$. □

**Proposition 3.5.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. If $d$ is a trace of $D$, then $d(x) = d(x) \vee f(x)$ for all $x \in L$.*

*Proof.* Let $d$ be a trace of symmetric bi-$f$-derivation $D$ of $L$. Then we have

$$\begin{aligned} d(x) = D(x,x) &= D(1 \to x, x) \\ &= (f(1) \to D(x,x)) \vee (D(1,x) \to f(x)) \quad = (1 \to d(x)) \vee (1 \to f(x)) \\ &= d(x) \vee f(x) \end{aligned}$$

for all $x \in L$. This completes the proof. □

**Corollary 3.6.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. If $d$ is a trace of $D$, then $f(x) \le d(x)$ for all $x \in L$.*

**Proposition 3.7.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(x,y) \ge f(x)$ and $D(x,y) \ge f(y)$ for all $x, y \in L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} D(x,y) &= D(1 \to x, y) = (f(1) \to D(x,y)) \vee (D(1,y) \to f(x)) \\ &= (1 \to D(x,y) \vee (1 \to f(x)) \\ &= D(x,y) \vee f(x) = (D(x,y) \to f(x)) \to f(x) \\ &= (f(x) \to D(x,y)) \to D(x,y) \geq f(x) \end{aligned}$$

for all $x, y \in L$ by (u7). Similarly, we have $f(y) \leq D(x,y)$ for all $x, y \in L$. This completes the proof. □

**Proposition 3.8.** *Let $D$ be a symmetric bi-$f$-derivation of $L$ and let $d$ be a trace of $D$. Then $d(x) \to f(y) \leq f(x) \to f(y) \leq f(x) \to d(y)$ for all $x, y \in L$.*

*Proof.* By Corollary 3.6, we have $f(x) \leq d(x)$ and $f(y) \leq d(y)$ for all $x, y \in L$. Hence we obtain

$$d(x) \to f(y) \leq f(x) \to f(y) \leq f(x) \to d(y)$$

for all $x, y \in L$, by (u3). □

**Theorem 3.9.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(x \to y, z) = f(x) \to D(y,z)$ for all $x, y, z \in L$.*

*Proof.* Since $f(x) \leq D(x,z)$ and $f(y) \leq D(y,z)$ by Proposition 3.9, we have

$$D(x,z) \to f(y) \leq f(x) \to f(y) \leq f(x) \to D(y,z)$$

for all $x, y, z \in L$. Hence we get

$$\begin{aligned} D(x \to y, z) &= (f(x) \to D(x,y)) \vee (D(x,z) \to f(y)) \\ &= (((f(x) \to D(x,y)) \to (D(x,z) \to f(y)))) \to (D(x,z) \to f(y)) \\ &= (((D(x,z) \to f(y)) \to (f(x) \to D(y,z)))) \to (f(x) \to D(y,z)) \\ &= f(x) \to D(y,z) \end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

**Proposition 3.10.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(x, y \to z) = f(y) \to D(x,z)$ for all $x, y, z \in L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} D(x, y \to z) &= D(y \to z, x) = f(y) \to D(z,x) \\ &= f(y) \to D(x,z) \end{aligned}$$

for all $x, y, z \in L$ by Theorem 3.9. This completes the proof. □

**Proposition 3.11.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $D(x,y) = f(x') \to (f(y') \to D(0,0))$ for all $x, y \in L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} D(x,y) &= D(x'',y'') = D(x' \to 0, y' \to 0) \\ &= f(x') \to D(0, y' \to 0) = f(x') \to (f(y') \to D(0,0)) \end{aligned}$$

for all $x, y \in L$ by Theorem 3.9. This completes the proof. □

**Proposition 3.12.** *Let $D$ be a symmetric bi-$f$-derivation of $L$ and let $d$ be a trace of $D$. Then $d(x \to y) = f(x) \to (f(x) \to d(y))$ for all $x, y \in L$.*

*Proof.* Let $d$ be a trace of symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) \\ &= f(x) \to D(y, x \to y) = f(x) \to D(x \to y, y) \\ &= f(x) \to (f(x) \to D(y,y)) = f(x) \to (f(x) \to d(y)) \end{aligned}$$

for all $x, y \in L$ by Theorem 3.9. This completes the proof. □

**Proposition 3.13.** *Let $D$ be a symmetric bi-$f$-derivation of $L$ and let $d$ be a trace of $D$. Then $d(x \vee y) = f(x \to y) \to (f(x \to y) \to d(y))$ for all $x, y \in L$.*

*Proof.* Let $d$ be a trace of symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y) = D((x \to y) \to y, (x \to y) \to y) \\ &= f(x \to y) \to D(y, (x \to y) \to y) = f(x \to y) \to D((x \to y) \to y, y) \\ &= f(x \to y) \to (f(x \to y) \to D(y,y)) = f(x \to y) \to (f(x \to y) \to d(y)) \end{aligned}$$

for all $x, y \in L$ by Theorem 3.9. This completes the proof. □

**Corollary 3.14.** *Let $D$ be a symmetric bi-$f$-derivation of $L$ and let $d$ be a trace of $D$. If $x \le y$, then $d(x \vee y) = d(y)$ for all $x, y \in L$.*

Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. For a fixed element $a \in L$, define a map $d_a : L \to L$ by $d_a(x) = D(x, a)$ for all $x \in L$.

**Proposition 3.15.** *Let $D$ be a symmetric bi-$f$-derivation of $L$. Then $d_a$ is a $f$-derivation of $L$.*

*Proof.* Let $D$ be a symmetric bi-$f$-derivation of $L$. Then we have

$$\begin{aligned} d_a(x \to y) &= D(x \to y, a) = (f(x) \to D(y,a)) \vee (D(x,a) \to f(y)) \\ &= (f(x) \to d_a(y)) \vee (d_a(x) \to f(y)) \end{aligned}$$

for all $x, y \in L$. This completes the proof. □

**Proposition 3.16.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. Then the following conditions hold:*

(1) *$d_a(x) = d_a(x) \vee f(x)$ for every $x \in L$.*
(2) *$d_a(x \vee y) = f(x \to y) \vee d_a(y)$ for every $x, y \in L$.*
(3) *If $x \le y$, then $d_a(x \vee y) = d_a(y) \vee f(y)$ for $x, y \in L$.*

*Proof.* (1) For every $x \in L$, we have

$$\begin{aligned} d_a(x) &= D(x,a) = D(1 \to x, a) \\ &= (f(1) \to D(x,a)) \vee (D(1,a) \to f(x)) \\ &= (1 \to d_a(x)) \vee (1 \to f(x)) \\ &= d_a(x) \vee f(x). \end{aligned}$$

(2) For every $x, y \in L$, we have

$$\begin{aligned} d_a(x \vee y) &= D(x \vee y, a) = D((x \to y) \to y, a) \\ &= f(x \to y) \to D(y,a) = f(x \to y) \to d_a(y) \end{aligned}$$

(3) Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$. Hence

$$\begin{aligned} d_a(x \vee y) &= D(x \vee y, a) = D((x \to y) \to y, a) \\ &= (f(x \to y) \to D(y,a)) \vee (D(x \to y, a) \to f(y)) \\ &= (1 \to D(y,a) \vee (D(1,a) \to f(y)) = d_a(y) \vee f(y). \end{aligned}$$

□

**Proposition 3.17.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. Then $d_a(1) = 1$.*

*Proof.* Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. By Theorem 3.9, we get

$$\begin{aligned} d_a(1) &= D(1,a) = D(1 \to 1, a) \\ &= f(1) \to D(1,a) = 1 \vee 1 = 1. \end{aligned}$$

This completes the proof. □

**Definition 7.** Let $L$ be a lattice implication algebra and let $D$ be a symmetric mapping on $L$. The mapping $D : L \to L$ satisfying $D(x \to y, z) = D(x,z) \to D(y,z)$ for all $x, y, z \in L$, is called a *joinitive mapping*.

**Proposition 3.18.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric mapping on $L$. If $D$ is a joinitive mapping, then $d_a$ is isotone.*

*Proof.* Let $D$ be a symmetric mapping on $L$ and $x \leq y$. Then

$$d_a(x) \to d_a(y) = D(x,a) \to D(y,a) = D(x \to y, a) = D(1,a) = 1,$$

which implies $d_a(x) \leq d_a(y)$ for all $x, y \in L$. This completes the proof. □

Let $L$ be a lattice implication algebra and let $d$ be a symmetric bi-$f$-derivation of $L$. Define a set $Fix_d(L)$ by

$$Fix_d(L) = \{x \in L \mid d(x) = D(x,x) = f(x)\}.$$

**Proposition 3.19.** *Let $L$ be a lattice $H$ implication algebra and let $d$ be a trace of symmetric bi-$f$-derivation $D$ of $L$. If $x \in L$ and $y \in Fix_d(L)$, then $x \vee y \in Fix_d(L)$.*

*Proof.* Let $x \in L$ and $y \in Fix_d(L)$. Then we obtain

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y) = D((x \to y) \to y, (x \to y) \to y) \\ &= f(x \to y) \to D(y, (x \to y) \to y) = f(x \to y) \to D((x \to y) \to y, y) \\ &= f(x \to y) \to (f(x \to y) \to D(y, y)) = f(x \to y) \to (f(x \to y) \to d(y)) \\ &= f(x \to y) \to (f(x \to y) \to f(y)) = f(x \to y) \to f((x \to y) \to y) \\ &= f((x \to y) \to (x \to y) \to y)) = f(((x \to y) \to (x \to y)) \to (x \to y) \to y) \\ &= f(1 \to (x \to y) \to y) = f((x \to y) \to y) \\ &= f(x \vee y). \end{aligned}$$

This implies $x \vee y \in Fix_d(L)$. This completes the proof. □

**Proposition 3.20.** *Let $L$ be a lattice $H$ implication algebra and let $d$ be a trace of symmetric bi-$f$-derivation $D$ of $L$. If $x \in L$ and $y \in Fix_d(L)$, then $x \to y \in Fix_d(L)$.*

*Proof.* Let $d$ be a symmetric bi-$f$-derivation of $L$. By Theorem 3.9 and by (u9), we obtain

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) = f(x) \to D(x \to y, y) \\ &= f(x) \to (f(x) \to d(y)) = f(x) \to (f(x) \to f(y)) \\ &= (f(x) \to f(x)) \to (f(x) \to f(y)) = 1 \to f(x \to y) \\ &= f(x \to y). \end{aligned}$$

This completes the proof. □

**Definition 8.** Let $L$ be a lattice implication algebra. A non-empty set $F$ of $L$ is called a *normal filter* if it satisfies the following conditions:

(1) $1 \in F$,
(2) $x \in L$ and $y \in F$ imply $x \to y \in F$.

**Theorem 3.21.** *Let $L$ be a lattice $H$ implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. Then $Fix_d(L)$ is a normal filter of $L$.*

*Proof.* Clearly, $1 \in Fix_d(L)$. Let $x \in L$ and $y \in Fix_d(L)$. Then we have

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) = f(x) \to D(x \to y, y) \\ &= f(x) \to (f(x) \to d(y)) = f(x) \to (f(x) \to f(y)) \\ &= (f(x) \to f(x)) \to (f(x) \to f(y)) = 1 \to f(x \to y) \\ &= f(x \to y). \end{aligned}$$

Therefore, this implies that $Fix_d(L)$ is normal filter of $L$. This completes the proof. □

Let $D$ be a symmetric bi-$f$-derivation of $L$ and let $d$ be a trace of $D$. Define a set $Kerd$ by

$$Kerd = \{x \in L \mid D(x, x) = d(x) = 1\}.$$

**Proposition 3.22.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. If $x \in L$ and $y \in Kerd$, then $x \to y \in Kerd$.*

*Proof.* Let $x \in L$ and $y \in Fix_d(L)$. Then

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) = f(x) \to D(y, x \to y) \\ &= f(x) \to D(x \to y, y) = f(x) \to (f(x) \to D(y, y)) \\ &= f(x) \to (f(x) \to d(y)) = f(x) \to (f(x) \to 1) \\ &= f(x) \to 1 = 1. \end{aligned}$$

Hence $x \to y \in Kerd$. This completes the proof. □

**Proposition 3.23.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. If $x \in L$ and $y \in Kerd$, then $x \vee y \in Kerd$.*

*Proof.* Let $x \in L$ and $y \in Kerd$. Then we obtain $d(y) = 1$. Hence

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y) = D((x \to y) \to y, (x \to y) \to y) \\ &= (f(x \to y) \to D(y, (x \to y) \to y) = (f(x \to y) \to (D((x \to y) \to y, y)) \\ &= (f(x \to y) \to (f(x \to y) \to D(y, y)) = (f(x \to y) \to (f(x \to y) \to d(y)) \\ &= (f(x \to y) \to (f(x \to y) \to 1) = f(x \to y) \to 1 = 1. \end{aligned}$$

Therefore, $x \vee y \in Kerd$. This completes the proof. □

**Proposition 3.24.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. If $x \leq y$ and $x \in Kerd$, then $y \in Kerd$.*

*Proof.* Let $x \in Kerd$ and $x \leq y$. Then

$$\begin{aligned} d(y) &= D(y, y) = D((x \to y) \to y, (x \to y) \to y)) = D(y \to x) \to x, (y \to x) \to x) \\ &= f(y \to x) \to D(x, (y \to x) \to x)) = f(y \to x) \to D((y \to x) \to x, x) \\ &= f(y \to x) \to (f(y \to x) \to D(x, x)) = f(y \to x) \to (f(y \to x) \to 1) \\ &= f(y \to x) \to (f(y \to x) \to 1) = f(y \to x) \to 1 = 1. \end{aligned}$$

Therefore, this implies that $y \in Kerd$. This completes the proof. □

**Proposition 3.25.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation on $L$ If $d$ is isotone, then $Kerd$ is a down closed set, that is, $x \leq y$ and $y \in Kerd(L)$ implies $x \in Kerd$.*

*Proof.* Let $x, y$ be such that $x \leq y$ and $x \in Kerd$. Then we have $d(x) \leq d(y) = 1$, which implies $d(x) = 1$, that is, $x \in Kerd$. Hence $x \in Fix_d(L)$. This completes the proof. □

**Theorem 3.26.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation of $L$. Then $Kerd$ is a normal filter of $L$.*

*Proof.* Clearly, $1 \in Kerd$. Let $x \in L$ and $y \in Kerd$. Then we have

$$\begin{aligned} d(x \to y) &= D(x \to y, x \to y) = f(x) \to D(x \to y, y) \\ &= f(x) \to (f(x) \to d(y)) = f(x) \to (f(x) \to 1) \\ &= 1. \end{aligned}$$

Therefore, this implies that $Kerd$ is a normal filter of $L$. This completes the proof. □

**Theorem 3.27.** *Let $L$ be a lattice implication algebra, $D$ be a symmetric bi-$f$-derivation on $L$ and $f$ be a lattice implication homomorphism on $L$. Then $D(x \vee y, z) = D(f(x), z) \vee D(f(y), z)$ and $D(x \wedge y, z) = D(f(x), z) \wedge D(f(y), z)$ for all $x, y, z \in L$.*

*Proof.* Let $x, y, z \in L$. Then we have

$$\begin{aligned} D(x \vee y, z) &= D(x'' \vee y'', z) = D((x' \wedge y') \to 0, z) \\ &= f(x' \wedge y') \to D(0, z) = (f(x') \to D(0, z)) \vee ((y') \to D(0, z)) \\ &= D(f(x') \to 0, z) \vee D(f(y') \to 0, z) = D(f(x')', z) \vee D(f(y')', z) \\ &= D(f(x), z) \vee D(f(y), z). \end{aligned}$$

We can prove the case of meet operation in the similar way. □

**Proposition 3.28.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation on $L$. If $f(x) = x$, then $D(x', x) = D(x, x') = 1$ for every $x \in L$.*

*Proof.* For every $x \in L$, we have

$$\begin{aligned} D(x', x) &= D(x \to 0, x) = f(x) \to D(0, x) \\ &= f(x) \to D(x, 0) = x \to D(x, 0) = D(x \to x, 0) = D(1, 0) = 1. \end{aligned}$$

□

**Proposition 3.29.** *Let $L$ be a lattice implication algebra and let $D$ be a symmetric bi-$f$-derivation on $L$. If $x' \leq y$ for every $x, y \in L$ and $f(x) = x$ for all $x \in L$, we have $D(y, x) = 1$.*

*Proof.* For every $x, y \in L$, we know that $x' \leq y$ implies $x' \vee y = y$. Hence

$$\begin{aligned} D(y, x) &= D(x' \vee y, x) = D(f(x'), x) \vee D(f(y), x) \\ &= D(x', x) \vee D(y, x) = 1 \vee D(y, x) = 1. \end{aligned}$$

□

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