East Asian Math. J.
Vol. 36 (2020), No. 1, pp. 035–053
http://dx.doi.org/10.7858/eamj.2020.004

HYERS-ULAM-RASSIAS STABILITY OF AN ADDITIVE-QUARTIC, A QUADRATIC-QUARTIC, AND A CUBIC-QUARTIC FUNCTIONAL EQUATION

Sun-Sook Jin and Yang-Hi Lee*

Abstract. In this paper, we investigate Hyers-Ulam-Rassias stability of an additive-quartic functional equation, of a quadratic-quartic functional equation, and of a cubic-quartic functional equation.

1. Introduction

Throughout this paper, let V, W be real vector spaces, X be a real normed space, Y be a real Banach space, and k be a fixed real number such that $k \notin \{0, 1, -1\}$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned}
f_o(x) &:= \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2},\\
Af(x,y) &:= f(x+y) - f(x) - f(y),\\
Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y),\\
Cf(x,y) &:= f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),\\
Q'f(x,y) &:= f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y) - 24f(y),\\
D_kf(x,y) &= f(x+ky) + f(x-ky) - k^2f(x+y) - k^2f(x-y) + 2(k^2-1)f(x)\\
&\quad - f(ky) - \frac{k^4 - 2k^2 - k}{2}f(y) - \frac{k^4 - 2k^2 + k}{2}f(-y),\\
E_kf(x,y) &= f(kx+y) + f(kx-y) - k^2f(x+y) - k^2f(x-y) - 2f(kx)\\
&\quad + 2k^2f(x) + 2(k^2-1)f(y),
\end{aligned}$$

Received Novermber 7, 2019; Accepted January 2, 2020.
2010 *Mathematics Subject Classification.* 39B82, 39B52.
Key words and phrases. stability of a functional equation; additive-quartic functional equation; quadratic-quartic functional equation; cubic-quartic functional equation; additive-quartic mappings; quadratic-quartic mappings; cubic-quartic mappings.
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

$$\begin{aligned} H_k f(x,y) =& f(kx+y) + f(kx-y) - \frac{k^2+k}{2} f(x+y) - \frac{k^2-k}{2} f(-x-y) \\ & - \frac{k^2+k}{2} f(x-y) - \frac{k^2-k}{2} f(y-x) - (k^4+k^3-k^2-k) f(x) \\ & - (k^4-k^3-k^2+k) f(-x) + (k^2-1) f(y) + (k^2-1) f(-y) \end{aligned}$$

for all $x, y \in V$. Every solution of the functional equations $Af(x,y) = 0$, $Qf(x,y) = 0$, $Cf(x,y) = 0$ and $Q'f(x,y) = 0$ are called an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping, respectively. If a mapping can be expressed by the sum of an additive mapping and quartic mapping, the sum of a quadratic and quartic mapping, and the sum of a cubic and quartic mapping, respectively, then we call the mapping an additive-quartic mapping, a quadratic-quartic mapping and a cubic-quartic mapping, respectively.

A functional equation is called an additive-quartic functional equation provided that each solution of that equation is an additive-quartic mapping and every additive-quartic mapping is a solution of that equation. Some mathematicians have investigated the stability of various types of the additive-quartic functional equations [4, 3, 6].

A functional equation is called a quadratic-quartic functional equation provided that each solution of that equation is a quadratic-quartic mapping and every quadratic-quartic mapping is a solution of that equation. M. E. Gordji etc. [10] investigated the stability of the quadratic-quartic functional equation $E_2 f(x,y) = 0$, and Abbaszadeh etc. [1], Gordji etc. [7] and Wang etc. [18] investigated the stability of the functional equation $E_k f(x,y) = 0$ on the various spaces for k is a natural number. Many mathematicians have investigated the stability of various types of the quadratic-quartic functional equations [13, 19].

A functional equation is called a cubic-quartic functional equation provided that each solution of that equation is a cubic-quartic mapping and every cubic-quartic mapping is a solution of that equation. Several mathematicians have investigated the stability of various types of the cubic-quartic functional equations [8, 9, 20].In particular, Jang et al. [12], Lee et al. [14], and Park [15] investigated the stability of the cubic-quartic functional equation $H_2 f(x,y) = 0$ on the various spaces.

A study on the stability of the functional equation starting from the Ulam's question [17] about the stability of the group homomorphisms obtained the meaningful result about the stability of the Cauchy additive function equation by Hyers [11] for the first time. Rassias then generalized Hyers' results and Găvruta [5] extended the results of Rassias. The concept of stability introduced by Rassias [16] is referred to as the functional equation 'Hyers-Ulam-Rassias stability'.

In section 2, we will show that the functional equation $D_r f(x,y) = 0$ is an additive-quartic functional equation when r is a rational number and investigate

Hyers-Ulam-Rassias stability of that functional equation $D_k f(x,y)=0$ when k is a real number.

In section 3, we will show that the functional equation $E_r f(x,y)=0$ is a quadratic-quartic functional equation when r is a rational number and investigate Hyers-Ulam-Rassias stability of that functional equation $E_k f(x,y)=0$ when k is a real number.

In section 4, we will show that the functional equation $H_r f(x,y)=0$ is a cubic-quartic functional equation when r is a rational number and investigate Hyers-Ulam-Rassias stability of that functional equation $H_k f(x,y)=0$ when k is a real number.

We need the following particular case of Baker's theorem [2] to prove that the functional equations $D_r f(x,y)=0$, $E_r f(x,y)=0$ and $H_r f(x,y)=0$ are an additive-quartic functional equation, a quadratic-quartic functional equation, a cubic-quartic functional equation, respectively.

Theorem 1.1. (Theorem 1 in [2]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0,\beta_0,\ldots,\alpha_m,\beta_m$ are scalar such that $\alpha_j\beta_l-\alpha_l\beta_j\neq 0$ whenever $0\le j<l\le m$. If $f_l: V\to W$ for $0\le l\le m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x+\beta_l y)=0$$

for all $x,y\in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m-1$.

The following corollary follows from Theorem 1.1.

Corollary 1.2. *If a mapping $f: V\to W$ satisfies one of the functional equations $D_k f(x,y)=0$, $E_k f(x,y)=0$ and $H_k f(x,y)=0$ for all $x,y\in X$, then f is a "generalized" polynomial mapping of "degree" at most 4.*

Baker [2] also states that if f is a "generalized" polynomial mapping of "degree" at most $m-1$, then f is expressed as $f(x)=x_0+\sum_{l=1}^{m-1}a_l^*(x)$ for $x\in V$, where a_l^* is a monomial mapping of degree l and f has a property $f(rx)=x_0+\sum_{l=1}^{m-1}r^l a_l^*(x)$ for $x\in V$ and $r\in\mathbb{Q}$. Notice that a_1^*, a_2^*, a_3^* and a_4^* are differently called an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping, respectively.

Remark 1. Suppose that $f_1,f_2,f_3,f_4: V\to W$ are generalized polynomial mapping of degree at most 4 and r is a rational number such that $r\notin\{0,1,-1\}$. It is easily obtained that if the equalities $f_1(rx)=rf_1(x)$, $f_2(rx)=r^2f_2(x)$, $f_3(rx)=r^3f_3(x)$ and $f_4(rx)=r^4f_4(x)$ hold for all $x\in V$, then f_1, f_2, f_3 and f_4 are an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping, respectively.

2. Stability of an additive-quartic functional equation

We will prove that the functional equation $D_r f(x,y) = 0$ is an additive-quartic functional equation when r is a rational number.

Theorem 2.1. *Let r be a rational number such that $r \notin \{0,1,-1\}$. A mapping f satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$ if and only if f_o is an additive mapping and f_e is a quartic mapping.*

Proof. Assume that a mapping $f : V \to W$ satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$. Then $f(0) = \frac{-D_r f(0,0)}{(r^2-1)^2} = 0$. The equalities $f_o(rx) = r f_o(x)$ and $f_e(rx) = r^4 f_e(x)$ follow from the equalities $f_o(rx) - r f_o(x) = -D_r f_o(0,x)$ and $f_e(rx) - r^4 f_e(x) = D_r f_e(0,x)$ for all $x \in V$. According to Corollary 1.2 and Remark 1, f_o and f_e are an additive mapping and a quartic mapping, respectively.

Conversely, assume that f_o is an additive mapping and f_e is a quartic mapping, i.e. f is an additive-quartic mapping. Notice that equalities $f_o(rx) = r f_o(x)$, $f_o(x) = -f_o(-x)$, $f_e(rx) = r^4 f_e(x)$, $f_e(x) = f_e(-x)$, and $f(x) = f_o(x) + f_e(x)$ hold for all $x \in V$ and $r \in \mathbb{Q}$.

First the equality $D_r f_o(x,y) = 0$ follows from the equality

$$D_r f_o(x,y) = -A f_o(x+ry, x-ry) + r^2 A f_o(x+y, x-y),$$

for all $x, y \in V$. Using mathematical induction, we obtain

$$D_n f_e(x,y) = 0$$

from the equalities

$$\begin{aligned} D_2 f_e(x,y) =& Q f_e(x,y), \\ D_3 f_e(x,y) =& D_2 f_e(x+y,y) + D_2 f_e(x-y,y) + 4 D_2 f_e(x,y), \\ D_n f_e(x,y) =& D_{n-1} f_e(x+y,y) + D_{n-1} f_e(x-y,y) - D_{n-2} f_e(x,y) \\ &+ (n-1)^2 Q f_e(x,y) \end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Notice that if $r \in \mathbb{Q}$, then there exist $m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $D_{\frac{n}{m}} f_e(x,y) = 0$ and $D_{\frac{-n}{m}} f_e(x,y) = 0$ follow from the equalities

$$\begin{aligned} D_{\frac{n}{m}} f_e(x,y) =& D_n f_e\left(x, \frac{y}{m}\right) - \frac{n^2}{m^2} D_m f_e\left(x, \frac{y}{m}\right), \\ D_{\frac{-n}{m}} f_e(x,y) =& D_{\frac{n}{m}} f_e(x,y) \end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $D_r f_e(x,y) = 0$ for all $x, y \in V$ and $r \in \mathbb{Q}$. From the equality $D_r f(x,y) = D_r f_e(x,y) + D_r f_o(x,y)$, we obtain $D_r f(x,y) = 0$ for all $x, y \in V$. □

For a given mapping $f : X \to Y$, let $J_n f : X \to Y$ be the mappings defined by
$J_n f(x) =$

$$\begin{cases} \frac{1}{2}k^n\big(f(k^{-n}x) - f(-k^{-n}x)\big) + \frac{1}{2}k^{4n}\big(f(k^{-n}x) + f(-k^{-n}x)\big) & \text{if } \ p > 4, \\ \frac{1}{2}k^n\big(f(k^{-n}x) - f(-k^{-n}x)\big) + \frac{1}{2}k^{-4n}\big(f(k^{n}x) + f(-k^{n}x)\big) & \text{if } \ 1 < p < 4, \\ \frac{1}{2}k^{-n}\big(f(k^{n}x) - f(k^{-n}x)\big) + \frac{1}{2}k^{-4n}\big(f(k^{n}x) + f(-k^{n}x)\big) & \text{if } \ 0 \le p < 1 \end{cases}$$

when $|k| > 1$ and
$J_n f(x) =$

$$\begin{cases} \frac{1}{2}k^n\big(f(k^{-n}x) - f(-k^{-n}x)\big) + \frac{1}{2}k^{4n}\big(f(k^{-n}x) + f(-k^{-n}x)\big) & \text{if } \ 0 \le p < 1, \\ \frac{1}{2}k^n\big(f(k^{-n}x) - f(-k^{-n}x)\big) + \frac{1}{2}k^{-4n}\big(f(k^{n}x) + f(-k^{n}x)\big) & \text{if } \ 1 < p < 4, \\ \frac{1}{2}k^{-n}\big(f(k^{n}x) - f(k^{-n}x)\big) + \frac{1}{2}k^{-4n}\big(f(k^{n}x) + f(-k^{n}x)\big) & \text{if } \ p > 4 \end{cases}$$

for all $x \in X$ and all nonnegative integers n when $|k| < 1$. From this, if $f(0) = 0$, then
$J_n f(x) - J_{n+1} f(x) =$

$$(1) \qquad \begin{cases} \frac{k^{4n}+k^n}{2} D_k f(0, -k^{-n-1}x) + \frac{k^{4n}-k^n}{2} D_k f(0, k^{-n-1}x) & \text{if } p > 4, \\ \frac{k^n}{2} D_k f(0, -k^{-n-1}x) - \frac{k^n}{2} D_k f(0, k^{-n-1}x) & \\ -\frac{1}{2k^{4n+4}} D_k f(0, -k^n x) - \frac{1}{2k^{4n+4}} D_k f(0, k^n x) & \text{if } 1 < p < 4, \\ -\frac{1+k^{3n+3}}{2k^{4n+4}} D_k f(0, -k^n x) - \frac{1-k^{3n+3}}{2k^{4n+4}} D_k f(0, k^n x) & \text{if } 0 \le p < 1 \end{cases}$$

when $|k| > 1$ and
$J_n f(x) - J_{n+1} f(x) =$

$$(2) \quad \begin{cases} \frac{k^{4n}+k^n}{2} D_k f(0, -k^{-n-1}x) + \frac{k^{4n}-k^n}{2} D_k f(0, k^{-n-1}x) & \text{if } 0 \le p < 1, \\ \frac{k^{4n}}{2} D_k f(0, -k^{-n-1}x) + \frac{k^{4n}}{2} D_k f(0, k^{-n-1}x) & \\ +\frac{1}{2k^{n+1}} D_k f(0, -k^n x) - \frac{1}{2k^{n+1}} D_k f(0, k^n x) & \text{if } 1 < p < 4, \\ -\frac{1+k^{3n+3}}{2k^{4n+4}} D_k f(0, -k^n x) - \frac{1-k^{3n+3}}{2k^{4n+4}} D_k f(0, k^n x) & \text{if } p > 4 \end{cases}$$

when $|k| < 1$. The following lemma follows from the above equality and the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1} (J_i f(x) - J_{i+1} f(x))$ for all $x \in X$.

Lemma 2.2. *If $f : X \to Y$ is a mapping such that*

$$D_k f(x, y) = 0$$

for all $x, y \in X$, then

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n.

From Theorem 2.1 and Lemma 2.2, we can prove the following stability theorem, where k is a real number with $k \notin \{0, 1, -1\}$.

Theorem 2.3. *Let $p \notin \{1, 4\}$ be a nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|D_k f(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{3}$$

for all $x, y \in X$ and $f(0) = 0$. Then there exists a unique solution mapping F of the functional equation $D_k F(x, y) = 0$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{\theta\|x\|^p}{||k|^4 - |k|^p|} & \text{if } p > 4, \\ \left(\frac{1}{||k| - |k|^p|} + \frac{1}{||k|^4 - |k|^p|}\right)\theta\|x\|^p & \text{if } 1 < p < 4, \\ \frac{\theta\|x\|^p}{||k| - |k|^p|} & \text{if } 0 \le p < 1 \end{cases} \tag{4}$$

for all $x \in X$.

Proof. The proof of this theorem will be divided into two cases, either $|k| > 1$ or $|k| < 1$.

Case 1. Let $|k| > 1$. It follows from (1) and (3) that

$$\|J_n f(x) - J_{n+1} f(x)\| \le \begin{cases} \frac{|k|^{4n}\theta\|x\|^p}{|k|^{(n+1)p}} & \text{if } p > 4, \\ \frac{|k|^{np}\theta\|x\|^p}{|k|^{4(n+1)}} + \frac{|k|^{n}\theta\|x\|^p}{|k|^{(n+1)p}} & \text{if } 1 < p < 4, \\ \frac{|k|^{np}\theta\|x\|^p}{|k|^{n+1}} & \text{if } 0 \le p < 1 \end{cases}$$

for all $x \in X$. Together with the equality $J_n f(x) - J_{n+m} f(x) = \sum_{i=n}^{n+m-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we get

$$\|J_n f(x) - J_{n+m} f(x)\| \le \begin{cases} \sum_{i=n}^{n+m-1} \frac{|k|^{4i}\theta\|x\|^p}{|k|^{(i+1)p}} & \text{if } p > 4, \\ \sum_{i=n}^{n+m-1} \frac{|k|^{ip}\theta\|x\|^p}{|k|^{4(i+1)}} + \frac{|k|^{i}\theta\|x\|^p}{|k|^{(i+1)p}} & \text{if } 1 < p < 4, \\ \sum_{i=n}^{n+m-1} \frac{|k|^{ip}\theta\|x\|^p}{|k|^{i+1}} & \text{if } 0 \le p < 1 \end{cases} \tag{5}$$

for all $x \in X$. From (5), it follows that the sequence $\{J_n f(x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{J_n f(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ given by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (5) we get (4). For the case $p > 4$, from the definition of F, we easily get

$$\begin{aligned}\|D_kF(x,y)\| &= \lim_{n\to\infty}\Big\| \frac{k^n}{2}\left(D_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) - D_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\\ &\quad + \frac{k^{4n}}{2}\left(D_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) + D_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\Big\|\\ &\le \lim_{n\to\infty}(|k|^n + |k|^{4n})\frac{\theta(\|x\|^p+\|y\|^p)}{|k|^{np}}\\ &=0\end{aligned}$$

for all $x, y \in X$. For the other cases, we also easily show that $D_kF(x,y) = 0$ by the similar method. Now let $F' : X \to Y$ be another solution mapping satisfying (4). By Theorem 2.1 and Lemma 2.2, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $p > 4$, we have

$$\begin{aligned}\|J_nf(x) - F'(x)\| =& \|J_nf(x) - J_nF'(x)\|\\ \le& \frac{k^n}{2}\left(\|(f-F')(k^{-n}x)\| + \|(f-F')(-k^{-n}x)\|\right)\\ &+ \frac{k^{4n}}{2}\left(\|(f-F')(k^{-n}x)\| + \|(f-F')(-k^{-n}x)\|\right)\\ \le& \frac{|k|^n+|k|^{4n}}{|k|^{np}}\left(\frac{1}{||k|-|k|^p|} + \frac{1}{||k|^4-|k|^p|}\right)\theta\|x\|^p\end{aligned}$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

Case 2. Let $|k| < 1$. It follows from (2) and (3) that

$$\|J_nf(x) - J_{n+1}f(x)\| \le \begin{cases} \frac{|k|^n\theta\|x\|^p}{|k|^{(n+1)p}} & \text{if } 0 \le p < 1,\\ \frac{|k|^{4n}\theta\|x\|^p}{|k|^{(n+1)p}} + \frac{|k|^{np}\theta\|x\|^p}{|k|^{(n+1)}} & \text{if } 1 < p < 4,\\ \frac{|k|^{np}\theta\|x\|^p}{|k|^{4n+4}} & \text{if } p > 4\end{cases}$$

for all $x \in X$. Together with the equality $J_nf(x) - J_{n+m}f(x) = \sum_{i=n}^{n+m-1} J_if(x) - J_{i+1}f(x)$ for all $x \in X$, we get

(6)

$$\|J_nf(x) - J_{n+m}f(x)\| \le \begin{cases} \sum_{i=n}^{n+m-1}\frac{|k|^{ip}\theta\|x\|^p}{|k|^{4(i+1)}} & \text{if } p > 4,\\ \sum_{i=n}^{n+m-1}\frac{|k|^{4i}\theta\|x\|^p}{|k|^{(i+1)p}} + \frac{|k|^{ip}\theta\|x\|^p}{|k|^{(i+1)}} & \text{if } 1 < p < 4,\\ \sum_{i=n}^{n+m-1}\frac{|k|^{i}\theta\|x\|^p}{|k|^{(i+1)p}} & \text{if } 0 \le p < 1\end{cases}$$

for all $x \in X$. From (6), it follows that the sequence $\{J_nf(x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{J_nf(x)\}$ converges for all $x \in X$.

Hence we can define a mapping $F : X \to Y$ given by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (6) we get (4). For the case $p < 1$, from the definition of F, we easily get

$$\begin{aligned}\|D_k F(x,y)\| = &\lim_{n\to\infty} \Big\| \frac{k^n}{2}\Big(D_k f\Big(\frac{x}{k^n},\frac{y}{k^n}\Big) - D_k f\Big(-\frac{x}{k^n},-\frac{y}{k^n}\Big)\Big) \\ &+ \frac{k^{4n}}{2}\Big(D_k f\Big(\frac{x}{k^n},\frac{y}{k^n}\Big) + D_k f\Big(-\frac{x}{k^n},-\frac{y}{k^n}\Big)\Big)\Big\| \\ \le &\lim_{n\to\infty}(|k|^n + |k|^{4n})\frac{\theta(\|x\|^p + \|y\|^p)}{|k|^{np}} \\ =&0\end{aligned}$$

for all $x, y \in X$. For the other cases, we also easily show that $D_k F(x,y) = 0$ by the similar method. Now let $F' : X \to Y$ be another solution mapping satisfying (4). By Theorem 2.1 and Lemma 2.2, the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. For the case $p < 1$, we have

$$\begin{aligned}\|J_n f(x) - F'(x)\| =&\|J_n f(x) - J_n F'(x)\| \\ \le&\frac{k^n}{2}\big(\|(f-F')(k^{-n}x)\| + \|(f-F')(-k^{-n}x)\|\big) \\ &+ \frac{k^{4n}}{2}\big(\|(f-F')(k^{-n}x)\| + \|(f-F')(-k^{-n}x)\|\big) \\ \le&\frac{|k|^n + |k|^{4n}}{|k|^{np}}\left(\frac{1}{||k| - |k|^p|} + \frac{1}{||k|^4 - |k|^p|}\right)\theta\|x\|^p\end{aligned}$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_n f(x)$ for all $x \in X$. For the other cases, we also easily show that $F'(x) = \lim_{n\to\infty} J_n f(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$. □

3. Stability of a quadratic-quartic functional equation

Throughout this section, for a given mapping $f : V \to W$, we use the following abbreviation:

$$\begin{aligned}\Delta f(x) :=&\frac{1}{k^4 - k^2}\big(- E_k f_e(x,(k+2)x) - E_k f_e(x,(k-2)x) - 4E_k f_e(x,(k+1)x) \\ &- 4E_k f_e(x,(k-1)x) + 10E_k f_e(x,kx) + E_k f_e(2x,2x) + 4E_k f_e(2x,x) \\ &- k^2 E_k f_e(x,3x) - 2(k^2+1)E_k f_e(x,2x) + (17k^2 - 8)E_k f_e(x,x)\big) \\ &+ \frac{E_k f(0,4x) - 20E_k f(0,2x) + 64E_k f(0,x)}{2(k^2-1)} - \frac{(28k^2-10)E_k f(0,0)}{2k^2(k^2-1)}\end{aligned} \tag{7}$$

for all $x, y \in V$.

Theorem 3.1. *Let k be a real number such that $k \notin \{0, 1, -1\}$. If a mapping f satisfies the functional equation $E_k f(x, y) = 0$ for all $x, y \in V$, then f is a quadratic-quartic mapping.*

Proof. Assume that a mapping $f : V \to W$ satisfies the functional equation $E_k f(x, y) = 0$ for all $x, y \in V$. Let g, h be the mappings defined by $g(x) = \frac{-f(2x)+16f(x)}{12}$ and $h(x) = \frac{f(2x)-4f(x)}{12}$, respectively. Then $f = g+h$, $E_k g(x, y) = 0$, $E_k h(x, y) = 0$, and $\Delta f(x) = 0$ for all $x, y \in V$, where $\Delta f(x)$ is the mapping defined in (7). The mappings g and h are generalized polynomial mappings of degree at most 4 by Corollary 1.2. Through tedious calculations, we get the equation

$$f(4x) - 20f(2x) + 64f(x) = \Delta f(x) \tag{8}$$

for all $x \in V$. So $f(4x) - 20f(2x) + 64f(x) = 0$, $g(2x) = 4g(x)$, and h satisfies $h(2x) = 2^4 h(x)$ for all $x \in V$. According to Remark 1, g is a quadratic mapping and h is a quartic mapping, i.e. f is a quadratic-quartic mapping. □

We now show that the functional equation $E_r f(x, y) = 0$ is a quadratic-quartic functional equation in the following theorem.

Theorem 3.2. *Let r be a rational number such that $r \notin \{0, 1, -1\}$. A mapping f satisfies the functional equation $E_r f(x, y) = 0$ for all $x, y \in V$ if and only if f is a quadratic-quartic mapping.*

Proof. If a mapping $f : V \to W$ satisfies the functional equation $E_r f(x, y) = 0$ for all $x, y \in V$, then f is a quadratic-quartic mapping by Theorem 3.1.

Conversely, assume that f is a quadratic-quartic mapping, i.e. there exist a quadratic mapping g and a quartic mapping h such that $f = g + h$. Notice that the equalities $g(rx) = r^2 g(x)$, $g(x) = g(-x)$, $h(rx) = r^4 h(x)$, and $h(x) = h(-x)$ for all $x \in V$ and $r \in \mathbb{Q}$. Since $E_r g(x, y) = 0$ is obtained from

$$E_r g(x, y) = Qg(rx, y) - r^2 Qg(x, y)$$

for all $x, y \in V$, we now prove that $E_r h(x, y) = 0$ for all $x, y \in V$. Let us first see that $E_n h(x, y) = 0$ is true for any natural number $n \neq 1$. Using mathematical induction, the equality $E_n h(x, y) = 0$ is derived from the equalities

$$\begin{aligned}
E_2 h(x, y) =& Q' h(x, y), \\
E_3 h(x, y) =& E_2 h(x, x + y) + E_2 h(x, y - x) + 4E_2 h(x, y), \\
E_n h(x, y) =& E_{n-1} h(x, x + y) + E_{n-1} h(x, y - x) - E_{n-2} h(x, y) \\
&+ (n - 1)^2 E_2 h(x, y)
\end{aligned}$$

for all $x, y \in V$. Let us now prove $E_r h(x, y) = 0$ if r is a rational number such that $r \notin \{0, 1, -1\}$. Notice that if $r \in \mathbb{Q}$, then there exist $m, n \in \mathbb{N}$ such that

$r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $E_{\frac{n}{m}} f(x,y) = 0$ and $E_{\frac{-n}{m}} f(x,y) = 0$ are obtained from the equalities

$$\begin{aligned} E_{\frac{n}{m}} h(x,y) =& E_n h\left(\frac{x}{m}, y\right) - \frac{n^2}{m^2} E_m h\left(\frac{x}{m}, y\right), \\ E_{\frac{-n}{m}} h(x,y) =& E_{\frac{n}{m}} h(x,y) \end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $E_r h(x,y) = 0$ for all $x, y \in V$. □

For a given mapping $f : X \to Y$ and a fixed positive real number $p \notin \{2, 4\}$, let $J_n f : X \to Y$ be the mappings defined by

$$J_n f(x) = \begin{cases} \frac{4^{2n+1}-4^n}{3} f(2^{-n}x) - \frac{4^{2n+2}-4^{n+2}}{3} f(2^{-n-1}x)\big) & \text{if } \ p > 4, \\ -\frac{4^{n-1}}{3}\big(f(2^{-n+1}x) - 16 f(2^{-n}x)\big) & \text{if } \ 2 < p < 4, \\ \frac{16 f(2^n x) - f(2^{n+1}x)}{12 \cdot 4^n} + \frac{f(2^{n+1}x) - 4 f(2^n x)}{12 \cdot 16^n} & \text{if } \ 0 < p < 2 \end{cases}$$

for all $x \in X$ and all nonnegative integers n. Then, by the definition of $J_n f$ and (8), the equality

(9)

$$J_n f(x) - J_{n+1} f(x) = \begin{cases} \frac{4 \cdot 16^n}{3} \Delta f\big(2^{-n-2}x\big) - \frac{4^n}{3} \Delta f\big(2^{-n-2}x\big) & \text{if } p > 4, \\ -\frac{1}{192 \cdot 16^n} \Delta f\big(2^n x\big) - \frac{4^{n-1}}{3} \Delta f\big(2^{-n-1}x\big) & \text{if } 2 < p < 4, \\ \frac{1}{48 \cdot 4^n} \Delta f\big(2^n x\big) - \frac{1}{192 \cdot 16^n} \Delta f\big(2^n x\big) & \text{if } 0 < p < 2 \end{cases}$$

for all $x \in X$ and all nonnegative integers n. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1} (J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we obtain the following lemma.

Lemma 3.3. *If $f : X \to Y$ is a mapping such that*

$$E_k f(x,y) = 0$$

for all $x, y \in X$, then

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n.

We can prove the main theorem, 'Hyers-Ulam-Rassias stability of the functional equation $E_k f(x,y) = 0$' as the following theorem, where k is a real number with $k \notin \{0, 1, -1\}$.

Theorem 3.4. *Let X be a normed space and p a positive real number with $p \notin \{2, 4\}$. Suppose that $f : X \to Y$ is a mapping such that*

$$\|E_k f(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{10}$$

for all $x, y \in X$. Then there exists a unique solution mapping F of the functional equation $E_kF(x,y) = 0$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{K\theta\|x\|^p}{3\cdot 2^p}\left(\frac{4}{2^p-16} - \frac{1}{2^p-4}\right) & \textit{if} \quad p > 4, \\ \frac{K\theta\|x\|^p}{12}\left(\frac{1}{16-2^p} + \frac{1}{2^p-4}\right) & \textit{if} \quad 2 < p < 4, \\ \frac{K\theta\|x\|^p}{12}\left(\frac{1}{16-2^p} + \frac{1}{4-2^p}\right) & \textit{if} \quad 0 < p < 2 \end{cases} \tag{11}$$

for all $x \in X$, where

$$\begin{aligned} K =& \frac{69k^2 + 42 + (12k^2 + 8)2^p + k^2 3^p + \frac{k^2}{2}4^p}{|k^4 - k^2|} \\ &+ \frac{10|k|^p + 4|k-1|^p + 4|k+1|^p + |k-2|^p + |k+2|^p}{|k^4 - k^2|}. \end{aligned}$$

Proof. From (7) and (10), we have

$$\begin{aligned} \|\Delta f(x)\| =& \Bigg\| \frac{1}{k^4 - k^2}\big(- E_{1,k}f_e(x,(k+2)x) - E_kf_e(x,(k-2)x) \\ &- 4E_kf_e(x,(k+1)x) - 4E_kf_e(x,(k-1)x) + 10E_kf_e(x,kx) \\ &+ E_kf_e(2x,2x) + 4E_kf_e(2x,x) - 2(k^2+1)E_kf_e(x,2x) \\ &- k^2E_kf_e(x,3x) + (17k^2-8)E_kf_e(x,x)\big) - \frac{(28k^2-10)E_kf(0,0)}{2k^2(k^2-1)} \\ &+ \frac{E_kf(0,4x) - 20E_kf(0,2x) + 64E_kf(0,x)}{2(k^2-1)} \Bigg\| \\ \leq& K\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (9) and (10) that

$$\|J_nf(x) - J_{n+1}f(x)\| \leq \begin{cases} \frac{4^n(4^{n+1}-1)}{3\cdot 2^{(n+2)p}}K\theta\|x\|^p & \text{if } p > 4, \\ \left(\frac{2^{np}}{12\cdot 16^{n+1}} + \frac{4^{n-1}}{3\cdot 2^{(n+1)p}}\right)K\theta\|x\|^p & \text{if } 2 < p < 4, \\ \frac{(4^{n+1}-1)2^{np}}{3\cdot 4^{2n+1}}K\theta\|x\|^p & \text{if } 0 < p < 2 \end{cases}$$

for all $x \in X$. Since the equality $J_nf(x) - J_{n+m}f(x) = \sum_{i=n}^{n+m-1}(J_if(x) - J_{i+1}f(x))$ holds for all $x \in X$, we get
$\|J_nf(x) - J_{n+m}f(x)\| \leq$

$$\begin{cases} \sum_{i=n}^{n+m-1} \frac{4^i(4^{i+1}-1)}{3\cdot 2^{(i+2)p}}K\theta\|x\|^p & \text{if } p > 4, \\ \sum_{i=n}^{n+m-1} \left(\frac{2^{ip}}{12\cdot 16^{i+1}} + \frac{4^{i-1}}{3\cdot 2^{(i+1)p}}\right)K\theta\|x\|^p & \text{if } 2 < p < 4, \\ \sum_{i=n}^{n+m-1} \frac{(4^{i+1}-1)2^{ip}}{3\cdot 4^{2i+1}}K\theta\|x\|^p & \text{if } 0 < p < 2 \end{cases} \tag{12}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (12) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence

$\{J_nf(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (12) we get the inequality (11). For the case $2 < p < 4$, from the definition of F, we easily get

$$\begin{aligned}\|E_kF(x,y)\| &= \lim_{n\to\infty}\Big\|\frac{4^n}{12}\left(-E_kf\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+16E_kf\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right) \\ &\qquad + \frac{E_kf\left(2^{n+1}x,2^{n+1}y\right)-4E_kf\left(2^nx,2^ny\right)}{12\cdot 16^n}\Big\| \\ &\le \lim_{n\to\infty}\left(\frac{4^n(2^p+16)}{12\cdot 2^{np}}+\frac{2^{np}(2^p+4)}{12\cdot 16^n}\right)\theta(\|x\|^p+\|y\|^p) \\ &=0\end{aligned}$$

for all $x, y \in X$. Also we easily show that $E_kF(x,y) = 0$ by the similar method for the other cases, either $0 < p < 2$ or $4 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (11). Instead of the condition (11), it is sufficient to show that there is a unique mapping that satisfies condition $\|f(x)-F(x)\| \le \frac{K\theta\|x\|^p}{12}\left(\frac{1}{|16-2^p|}+\frac{1}{|4-2^p|}\right)$ simply. By Lemma 3.3, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $p > 4$, we have

$$\begin{aligned}&\|J_nf(x)-F'(x)\| \\ &=\|J_nf(x)-J_nF'(x)\| \\ &\le\frac{4^{2n+1}-4^n}{3}\|(f-F')(2^{-n}x)\|+\frac{4^{2n+2}-4^{n+2}}{3}\|(f-F')(2^{-n-1}x)\| \\ &\le\left(\frac{4^{2n+1}-4^n}{3\cdot 2^{np}}+\frac{4^{2n+2}-4^{n+2}}{3\cdot 2^{(n+1)p}}\right)\frac{K\theta\|x\|^p}{12}\left(\frac{1}{|16-2^p|}+\frac{1}{|4-2^p|}\right) \\ &\le\frac{4^{2n+2}}{3\cdot 2^{np}}\frac{K\theta\|x\|^p}{12}\left(\frac{1}{|16-2^p|}+\frac{1}{|4-2^p|}\right)\end{aligned}$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, either $0 < p < 2$ or $2 < p < 4$, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

□

4. Stability of a cubic-quartic functional equation

Now we will show that the functional equation $H_rf(x,y) = 0$ is a cubic-quartic functional equation when r is a rational number such that $r \notin \{0, 1, -1\}$.

Theorem 4.1. *Let r be a rational number such that $r \notin \{0, 1, -1\}$. A mapping $f : V \to W$ satisfies the functional equation $H_r f(x, y) = 0$ for all $x, y \in V$ if and only if f_o is a cubic mapping and f_e is a quartic mapping.*

Proof. Assume that a mapping $f : V \to W$ satisfies the functional equation $H_r f(x, y) = 0$ for all $x, y \in V$. The equalities $f(0) = 0$, $f_o(rx) = r^3 f_o(x)$ and $f_e(rx) = r^4 f_e(x)$ follow from the equalities

$$\begin{aligned} f(0) =& \frac{-H_r f(0,0)}{2r^2(r^2-1)}, \\ f_o(rx) - r^3 f_o(x) =& \frac{H_r f(x,0) - H_r f(-x,0)}{4}, \\ f_e(rx) - r^4 f_e(x) =& \frac{H_r f(x,0) + H_r f(-x,0)}{4} \end{aligned}$$

for all $x \in V$. The mappings f_o and f_e are generalized polynomial mappings of degree at most 4 by Corollary 1.2, so f_o is a cubic mapping and f_e is a quartic mapping by Remark 1.

Conversely, assume that f_o is a cubic mapping and f_e is a quartic mapping, i.e., f is a cubic-quartic mapping. Notice that the equalities $f_o(rx) = r^3 f_o(x)$, $f_o(x) = -f_o(-x)$, $f_e(rx) = r^4 f_e(x)$, $f_e(x) = f_e(-x)$, and $f(x) = f_o(x) + f_e(x)$ for all $x \in V$ and $r \in \mathbb{Q}$. Also we know that

$$\begin{aligned} H_r f(x,y) =& H_r f_e(x,y) + H_r f_o(x,y), \\ H_r f_o(x,y) =& f_o(rx+y) + f_o(rx-y) - r f_o(x+y) - r f_o(x-y) - 2(r^3 - r) f_o(x), \\ H_r f_e(x,y) =& f_e(rx+y) + f_e(rx-y) - r^2 f_e(x+y) \\ & - r^2 f_e(x-y) - 2(r^4 - r^2) f_e(x) + 2(r^2-1) f_e(y) \end{aligned}$$

for all $x, y \in V$.

Let us first prove $H_n f(x, y) = 0$ if n is a natural number. Using mathematical induction, the equalities $H_n f_o(x, y) = 0$ and $H_n f_e(x, y) = 0$ follow from the equalities

$$\begin{aligned} H_2 f_o(x,y) =& C f_o(y,x) + C f_o(-y,x), \\ H_3 f_o(x,y) =& C f_o(y-x, 2x), \\ H_n f_o(x,y) =& H_{n-1} f_o(x, x+y) + H_{n-1} f_o(x, x-y) - H_{n-2} f_o(x,y) \\ & + (n-1) H_2 f_o(x,y), \\ H_2 f_e(x,y) =& Q f_e(y,x), \\ H_3 f_e(x,y) =& H_2 f_e(x, x+y) + H_2 f_e(x, x-y) + 4 H_2 f_e(x,y), \\ H_n f_e(x,y) =& H_{n-1} f_e(x, x+y) + H_{n-1} f_e(x, x-y) - H_{n-2} f_e(x,y) \\ & + (n-1)^2 H_2 f_e(x,y) \end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Let us now prove $H_r f(x, y) = 0$ if r is a rational number such that $r \notin \{0, 1, -1\}$. Notice that if $r \in \mathbb{Q}$, then there exist

$m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $H_{\frac{n}{m}} f(x,y) = 0$ and $H_{\frac{-n}{m}} f(x,y) = 0$ follow from the equalities

$$\begin{aligned} H_{\frac{n}{m}} f_o(x,y) =& H_n f_o\left(\frac{x}{m}, y\right) - \frac{n}{m} H_m f_o\left(\frac{x}{m}, y\right), \\ H_{\frac{n}{m}} f_e(x,y) =& H_n f_e\left(\frac{x}{m}, y\right) - \frac{n^2}{m^2} H_m f_e\left(\frac{x}{m}, y\right), \\ H_{\frac{-n}{m}} f_o(x,y) =& - H_{\frac{n}{m}} f_o(x,y), \\ H_{\frac{-n}{m}} f_e(x,y) =& H_{\frac{n}{m}} f_e(x,y) \end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $H_r f(x,y) = 0$ for all $x, y \in V$. □

For a given mapping $f : X \to Y$ and a fixed positive real number $p \notin \{3, 4\}$, let $J_n f : X \to Y$ be the mappings defined by
$J_n f(x) =$

$$\begin{cases} \frac{1}{2} k^{3n} \big(f(k^{-n}x) - f(-k^{-n}x) \big) + \frac{1}{2} k^{4n} \big(f(k^{-n}x) + f(-k^{-n}x) \big) & \text{if } p > 4, \\ \frac{1}{2} k^{3n} \big(f(k^{-n}x) - f(-k^{-n}x) \big) + \frac{1}{2} k^{-4n} \big(f(k^{n}x) + f(-k^{n}x) \big) & \text{if } 3 < p < 4, \\ \frac{1}{2} k^{-3n} \big(f(k^{n}x) - f(k^{-n}x) \big) + \frac{1}{2} k^{-4n} \big(f(k^{n}x) + f(-k^{n}x) \big) & \text{if } 0 < p < 3 \end{cases}$$

for all $x \in X$ and all nonnegative integers n when $|k| > 1$ and
$J_n f(x) =$

$$\begin{cases} \frac{1}{2} k^{3n} \big(f(k^{-n}x) - f(-k^{-n}x) \big) + \frac{1}{2} k^{4n} \big(f(k^{-n}x) + f(-k^{-n}x) \big) & \text{if } 0 < p < 3, \\ \frac{1}{2} k^{-3n} \big(f(k^{n}x) - f(-k^{n}x) \big) + \frac{1}{2} k^{4n} \big(f(k^{-n}x) + f(-k^{-n}x) \big) & \text{if } 3 < p < 4, \\ \frac{1}{2} k^{-3n} \big(f(k^{n}x) - f(k^{-n}x) \big) + \frac{1}{2} k^{-4n} \big(f(k^{n}x) + f(-k^{n}x) \big) & \text{if } p > 4 \end{cases}$$

for all $x \in X$ and all nonnegative integers n when $|k| < 1$. From the definition of $J_n f$, if $f(0) = 0$, the equality
$J_n f(x) - J_{n+1} f(x) =$

$$(13) \begin{cases} \frac{k^{4n}+k^{3n}}{4} H_k f(k^{-n-1}x, 0) + \frac{k^{4n}-k^{3n}}{4} H_k f(-k^{-n-1}x, 0) & \text{if } p > 4, \\ \frac{k^{3n}}{4} H_k f(k^{-n-1}x, 0) - \frac{k^{3n}}{4} H_k f(-k^{-n-1}x, 0) \\ \quad - \frac{1}{4k^{4n+4}} H_k f(k^{n}x, 0) - \frac{1}{4k^{4n+4}} H_k f(-k^{n}x, 0) & \text{if } 3 < p < 4, \\ - \frac{1+k^{n+1}}{4k^{4n+4}} H_k f(k^{n}x, 0) - \frac{1-k^{n+1}}{4k^{4n+4}} H_k f(-k^{n}x, 0) & \text{if } 0 < p < 3 \end{cases}$$

holds for all $x \in X$ and all nonnegative integers n when $|k| > 1$ and
$J_n f(x) - J_{n+1} f(x) =$

$$(14) \begin{cases} \frac{k^{4n}+k^{3n}}{4} H_k f(k^{-n-1}x, 0) + \frac{k^{4n}-k^{3n}}{4} H_k f(-k^{-n-1}x, 0) & \text{if } 0 < p < 3, \\ \frac{k^{4n}}{4} H_k f(k^{-n-1}x, 0) + \frac{k^{4n}}{4} H_k f(-k^{-n-1}x, 0) \\ \quad - \frac{1}{4k^{3n+3}} H_k f(k^n x, 0) + \frac{1}{4k^{3n+3}} H_k f(-k^n x, 0) & \text{if } 3 < p < 4, \\ -\frac{1+k^{n+1}}{4k^{4n+4}} H_k f(k^n x, 0) - \frac{1-k^{n+1}}{4k^{4n+4}} H_k f(-k^n x, 0) & \text{if } p > 4 \end{cases}$$

holds for all $x \in X$ and all nonnegative integers n when $|k| < 1$. From the above equality and the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1} (J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we obtain the following lemma.

Lemma 4.2. *If $f : X \to Y$ is a mapping such that*

$$H_k f(x, y) = 0$$

for all $x, y \in X$, then

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n.

From Theorem 4.1-Lemma 4.2, we can prove the following stability theorem, where k is a real number with $k \notin \{0, 1, -1\}$.

Theorem 4.3. *Let $p \notin \{3, 4\}$ be a fixed positive real number. Suppose that $f : X \to Y$ is a mapping such that*

$$(15) \qquad \|H_k f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ (and $f(0) = 0$ when $p = 0$). Then there exists a unique solution mapping F of the functional equation $H_k F(x, y) = 0$ such that

(16)

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta\|x\|^p}{2||k|^4 - |k|^p|} & \text{if} \quad p > 4, \\ \left(\frac{1}{2||k|^3 - |k|^p|} + \frac{1}{2||k|^4 - |k|^p|}\right) \theta\|x\|^p & \text{if} \quad 3 < p < 4, \\ \frac{\theta\|x\|^p}{2||k|^3 - |k|^p|} & \text{if} \quad 0 < p < 3 \end{cases}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ follows from $\|2(k^4 - k^2) f(0)\| = \|H_k f(0, 0)\| \leq 0$. The proof of this theorem will be divided into two cases, either $|k| > 1$ or $|k| < 1$.

Case 1. Let $|k| > 1$. It follows from (13) and (15) that

$$\|J_nf(x) - J_{n+1}f(x)\| \leq \begin{cases} \frac{|k|^{4n}\theta\|x\|^p}{2|k|^{(n+1)p}} & \text{if } p > 4, \\ \frac{|k|^{np}\theta\|x\|^p}{2|k|^{4(n+1)}} + \frac{|k|^{3n}\theta\|x\|^p}{2|k|^{(n+1)p}} & \text{if } 3 < p < 4, \\ \frac{|k|^{np}\theta\|x\|^p}{2|k|^{3n+3}} & \text{if } 0 < p < 3 \end{cases}$$

for all $x \in X$. Together with the equality $J_nf(x)-J_{n+m}f(x) = \sum_{i=n}^{n+m-1}(J_if(x)-J_{i+1}f(x))$ for all $x \in X$, we get

(17)
$$\|J_nf(x) - J_{n+m}f(x)\| \leq \begin{cases} \sum_{i=n}^{n+m-1} \frac{|k|^{4i}\theta\|x\|^p}{2|k|^{(i+1)p}} & \text{if } p > 4, \\ \sum_{i=n}^{n+m-1} \left(\frac{|k|^{ip}\theta\|x\|^p}{2|k|^{4(i+1)}} + \frac{|k|^{3i}\theta\|x\|^p}{2|k|^{(i+1)p}}\right) & \text{if } 3 < p < 4, \\ \sum_{i=n}^{n+m-1} \frac{|k|^{ip}\theta\|x\|^p}{2|k|^{3i+3}} & \text{if } 0 < p < 3 \end{cases}$$

for all $x \in X$. It follows from (17) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_nf(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (17) we get the inequality (16). For the case $3 < p < 4$, from the definition of F, we easily get

$$\begin{aligned} \|H_kF(x, y)\| &= \lim_{n\to\infty} \Big\| \frac{k^{3n}}{2}\left(H_kf\left(\frac{x}{k^n}, \frac{y}{k^n}\right) - H_kf\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right) \\ &\qquad + \frac{H_kf(k^nx, k^ny) + H_kf(-k^nx, -k^ny)}{2k^{4n}} \Big\| \\ &\leq \lim_{n\to\infty} \left(\frac{|k|^{3n}}{|k|^{np}} + \frac{|k|^{np}}{|k|^{4n}}\right)\theta(\|x\|^p + \|y\|^p) \\ &=0 \end{aligned}$$

for all $x, y \in X$. For the other cases, we also easily show that $H_kF(x, y) = 0$ by the similar method. Now let $F' : X \to Y$ be another solution mapping satisfying (16). Instead of condition (16), it is sufficient to show that there is a unique mapping that satisfies condition $\|f(x) - F(x)\| \leq \left(\frac{1}{2||k|^3-|k|^p|} + \frac{1}{2||k|^4-|k|^p|}\right)\theta\|x\|^p$ simply. By Lemma 4.2, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $p > 4$, we have

$$\begin{aligned}\|J_nf(x) - F'(x)\| =&\|J_nf(x) - J_nF'(x)\|\\ \leq&\frac{k^{3n}}{2}\left(\|(f-F')(k^{-n}x)\| + \|(f-F')(-k^{-n}x)\|\right)\\ &+ \frac{1}{2k^{4n}}\left(\|(f-F')(k^{n}x)\| + \|(f-F')(-k^{n}x)\|\right)\\ \leq&\left(\frac{|k|^{3n}}{|k|^{np}} + \frac{|k|^{np}}{|k|^{4n}}\right)\left(\frac{1}{2||k|^3-|k|^p|} + \frac{1}{2||k|^4-|k|^p|}\right)\theta\|x\|^p\end{aligned}$$

for all $x \in X$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

Case 2. Let $|k| < 1$. It follows from (14) and (15) that

$$\|J_nf(x) - J_{n+1}f(x)\| \leq \begin{cases} \frac{|k|^{3n}\theta\|x\|^p}{2|k|^{(n+1)p}} & \text{if } 0 < p < 3,\\ \frac{|k|^{4n}\theta\|x\|^p}{2|k|^{(n+1)p}} + \frac{|k|^{np}\theta\|x\|^p}{2|k|^{3(n+1)}} & \text{if } 3 < p < 4,\\ \frac{|k|^{np}\theta\|x\|^p}{2|k|^{4n+4}} & \text{if } p > 4\end{cases}$$

for all $x \in X$. Together with the equality $J_nf(x) - J_{n+m}f(x) = \sum_{i=n}^{n+m-1}(J_if(x) - J_{i+1}f(x))$ for all $x \in X$, we get

(18)
$$\|J_nf(x) - J_{n+m}f(x)\| \leq \begin{cases} \sum_{i=n}^{n+m-1} \frac{|k|^{ip}\theta\|x\|^p}{2|k|^{4(i+1)}} & \text{if } p > 4,\\ \sum_{i=n}^{n+m-1} \frac{|k|^{4i}\theta\|x\|^p}{2|k|^{(i+1)p}} + \frac{|k|^{ip}\theta\|x\|^p}{2|k|^{3(i+1)}} & \text{if } 3 < p < 4,\\ \sum_{i=n}^{n+m-1} \frac{|k|^{3i}\theta\|x\|^p}{2|k|^{(i+1)p}} & \text{if } 0 < p < 3\end{cases}$$

for all $x \in X$. It follows from (18) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for any $x \in X$. Since Y is complete, the sequence $\{J_nf(x)\}$ converges for any $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (18), we get (16). For the case $p < 3$, from the definition of F, we easily get

$$\begin{aligned}\|H_kF(x,y)\| =& \lim_{n\to\infty}\left\|\frac{k^{3n}}{2}\left(H_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) - H_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\right.\\ &\left.+\frac{k^{4n}}{2}\left(H_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) + H_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\right\|\\ \leq& \lim_{n\to\infty}(|k|^{3n} + |k|^{4n})\frac{\theta(\|x\|^p + \|y\|^p)}{|k|^{np}}\\ =&0\end{aligned}$$

for all $x, y \in X$. For the other cases, we also easily show that $H_kF(x,y) = 0$ by the similar method. Now let $F' : X \to Y$ be another solution mapping satisfying (16). By Lemma 4.2, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $0 < p < 3$, we have

$$\begin{aligned}\|J_nf(x) - F'(x)\| =& \|J_nf(x) - J_nF'(x)\| \\ \le& \frac{k^{3n}}{2}\left(\|(f - F')(k^{-n}x)\| + \|(f - F')(-k^{-n}x)\|\right) \\ &+ \frac{k^{4n}}{2}\left(\|(f - F')(k^{-n}x)\| + \|(f - F')(-k^{-n}x)\|\right) \\ \le& \frac{|k|^{3n} + |k|^{4n}}{|k|^{np}}\left(\frac{1}{2||k|^3 - |k|^p|} + \frac{1}{2||k|^4 - |k|^p|}\right)\theta\|x\|^p\end{aligned}$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$. □

References

[1] S. Abbaszadeh, *Intutionistic fuzzy stability of a quadratic and quartic functional equation*, Int. J. Nonlinear Anal. Appl. **1** (2010), 100–124.

[2] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci. **133(6)** (2005), 1657–1664.

[3] A. Bodaghi, *Stability of a mixed type additive and quartic functional equation*, Filomat, **28(8)** (2014) 1629–1640.

[4] A. Bodaghi, *Approximate mixed type additive and quartic functional equation*, Bol. Soc. Paran. Mat. **35(1)** (2017), 43–56.

[5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[6] M. E. Gordji, *Stability of a functional equation deriving from quartic and additive functions*, arXiv preprint arXiv:0812.5025 (2008).

[7] M. E. Gordji, S. Abbaszadeh, and C. Park, *On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces*, J. Inequal. Appl. **2009** Article ID 153084.

[8] M. E. Gordji, A. Ebadian, and S. Zolfaghari, *Stability of a functional equation deriving from cubic and quartic functions*, Abstr. Appl. Anal. **2008** Article ID 801904.

[9] M. E. Gordji, M. B. Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-archimedean spaces*, Appl. Math. Lett. **23** (2010), 1198–1202.

[10] M. E. Gordji, M. B. Savadkouhi, and C. Park, *Quadratic-quartic functional equations in RN-spaces*, J. Inequal. Appl. **2009** Article ID 868423.

[11] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] S.-Y. Jang, C.-K. Park, and D. Y. Shin, *Fuzzy stability of a cubic-quartic functional equation: A fixed point approach*, Bull. Korean Math. Soc. **48** (2011), 491–503.

[13] H.-M. Kim, *On the stability problem for a mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl. **324** (2006), 358–372.

[14] J.-R. Lee, C.-K. Park, Y.-J. Cho and D.-Y. Shin, *Orthogonal stability of a cubic-quartic functional equation in non-archimedean spaces*, J. Comput. Anal. Appl. **15** (2013), 572–583.

[15] C.-K. Park, *Orthogonal stability of a cubic-quartic functional equation*, Journal of Nonlinear Sciences & Applications (JNSA) **5** (2012), 28–36.
[16] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[17] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.
[18] Z. Wang and P. K. Sahoo, *Stability of the generalized quadratic and quartic type functional equation in non-archimedean fuzzy normed spaces*, J. Appl. Anal. Comput. **6** (2016), 917–938.
[19] T. Z. Xu, J. M. Rassias, and W. X. Xu, *A generalized mixed quadratic-quartic functional equation*, Bull. Malays. Math. Sci. Soc. **35(3)** (2012), 633–649.
[20] X. Zhao, X. Yang, and C. T. Pang, *Solution and stability of a general mixed type cubic and quartic functional equation*, J. Funct. Spaces Appl. **2013** Article ID 673810.

Sun-Sook Jin
Department of Mathematics Education, Gongju National University of Education, Gongju 32553, Republic of Korea
E-mail address: `ssjin@ gjue.ac.kr`

Yang-Hi Lee
Department of Mathematics Education, Gongju National University of Education, Gongju 32553, Republic of Korea
E-mail address: `yanghi2@hanmail.net`