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A PROBABILISTIC APPROACH FOR VALUING EXCHANGE OPTION WITH DEFAULT RISK

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Abstract. We study a probabilistic approach for valuing an exchange option with default risk. The structural model of Klein [6] is used for modeling default risk. Under the structural model, we derive the closed-form pricing formula of the exchange option with default risk. Specifically, we provide the pricing formula of the option with the bivariate normal cumulative function via a change of measure technique and a multidimensional Girsanov's theorem.

1. Introduction

An exchange option is one of the most popular exotic options with two underlying assets. Since Margrabe [7] first derived the closed-form formula of an exchange option under the Black-Scholes model, a lot of researchers have developed the valuation of the option. Geman, Karoui, and Rochet [3] used the change of numéraire method to obtain the pricing formula of the exchange option. Antonelli, Ramponi, and Scarlatti [1] considered the stochastic volatility model based on a correlation expansion to price the exchange option. Cheang and Chiarella [2] derived the a formula of the exchange option when the underlying assets have the jump-diffusion characteristics. More recently, Kim and Park [5] extended the Margrabe formula under a stochastic volatility model with fast mean-reversion.

The aim of this paper is to present the pricing formula of the exchange option with default risk using a probabilistic approach. In fact, Kim and Koo [4] derived the closed-form formula of the exchange option with default risk based on the partial differential equation (PDE) approach. They used the Mellin transforms to solve the PDE and the structure model of Klein [6] to model default risk. In contrast, we propose a simple approach for valuing the exchange option with default risk in this paper. Concretely, we provide the closed-form

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solution of the exchange option price using the change of measure technique and the property of the bivariate normal cumulative distribution.

This paper is organized as follows. In section 2, we introduce the exchange option and the model for default risk. In section 3, we derive the closed-form formula for the price of the exchange option with default risk using a probabilistic approach. For the probabilistic approach, we use a change of measure and a multidimensional Girsanov's theorem. Section 4 provides concluding remarks.

2. Model and exchange option with default

We assume that the processes of two underlying assets with correlation ρ_{12} under the risk-neutral probability P are governed by the geometric Brownian motions (GBM) as

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{1}$$
$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{2}$$

where r is the risk-free interest rate, σ_i, $i = 1, 2$ are the volatilities of asset i, and $W_1(t)$ and $W_2(t)$ are the standard Brownian motions with correlation ρ_{12}, $dW_1(t)dW_2(t) = \rho_{12}dt$. To describe the structure model of Klein, we must define the dynamics of the option issuer's asset $V(t)$. Here, according to the model of Klein, we assume that there is an asset $V(t)$ which follows GBM under the measure P:

$$dV(t) = rV(t)dt + \sigma_3 V(t)dW_3(t), \tag{3}$$

where σ_3 is the volatility of asset $V(t)$, $W_3(t)$ is the standard Brownian motion satisfying $dW_1(t)dW_3(t) = \rho_{13}dt, dW_2(t)dW_3(t) = \rho_{23}dt$. As mentioned in [6], Klein assumes that if a default of the option issuer happens, the option issuer's asset is liquidated and provides only the scrap value at the maturity T. The scrap value is defined by

$$(1-\alpha)\frac{V(T)}{D}(S_1(T) - S_2(T))^+,$$

where α is a deadweight cost related with the default, T is a time to maturity, D is a critical value (or a value of the option issuer's liability) that a default occurs if the value of the option issuer asset is lower than D. Then, from the underlying assets $S_1(t)$, $S_2(t)$ and $V(t)$, the price of an exchange option C with default risk at time 0 under the measure P is given by

$$\begin{aligned} C &= E^P\left[e^{-rT}C(S_1(T), S_2(T), V(T)) | S_1(0) = S_1, S_2(0) = S_2, V(0) = V\right] \\ &= e^{-rT}E^P\Big[(S_1(T) - S_2(T))^+ \\ &\quad \times \left(\mathbf{1}_{\{V(T)\geq D\}} + \mathbf{1}_{\{V(T)<D\}}\frac{(1-\alpha)V(T)}{D}\right) | S_1(0) = S_1, S_2(0) = S_2, V(0) = V\Big]. \end{aligned} \tag{4}$$

3. Option pricing: A probabilistic approach

In this section, we derive the closed-form formula for the price of an exchange option with default risk based on the model in the previous section. A probabilistic approach is used to price the exchange option with default risk. The following theorem presents the valuation formula of the option.

Theorem 3.1. *Let C be the price of an exchange option with default risk at time 0 with the maturity T. Then, the closed-form formula of C is given by*

$$\begin{aligned} C &= S_1(0)\Phi_2(a_1,a_2,\theta) - S_2(0)\Phi_2(b_1,b_2,\theta) \\ &\quad + \frac{1-\alpha}{D} S_1(0)V(0)e^{(\sigma_1\sigma_3\rho_{13})T}\Phi_2(c_1,c_2,-\theta) - \frac{1-\alpha}{D} S_2(0)V(0)e^{r+\sigma_2\sigma_3\rho_{23}}\Phi_2(d_1,d_2,-\theta) \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{1}{\sigma\sqrt{T}}\ln\frac{S_1(0)}{S_2(0)} + \frac{\sigma}{2}\sqrt{T}, \\ a_2 &= \frac{1}{\sigma_3\sqrt{T}}\ln\frac{V(0)}{D} + \left(\frac{\rho_{13}\sigma_1\sigma_3 + r}{\sigma_3} - \frac{\sigma_3}{2}\right)\sqrt{T}, \\ b_1 &= \frac{1}{\sigma\sqrt{T}}\ln\frac{S_1(0)}{S_2(0)} - \frac{\sigma}{2}\sqrt{T-t}, \\ b_2 &= \frac{1}{\sigma_3\sqrt{T}}\ln\frac{V(0)}{D} + \left(\frac{\rho_{23}\sigma_2\sigma_3 + r}{\sigma_3} - \frac{\sigma_3}{2}\right)\sqrt{T}, \\ c_1 &= \frac{1}{\sigma\sqrt{T}}\ln\frac{S_1(0)}{S_2(0)} + \left(\frac{\rho_{13}\sigma_1\sigma_3 - \rho_{23}\sigma_2\sigma_3}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T}, \\ c_2 &= \frac{1}{\sigma_3\sqrt{T}}\ln\frac{D}{V(0)} - \left(\frac{\rho_{13}\sigma_1\sigma_3 + r}{\sigma_3} + \frac{\sigma_3}{2}\right)\sqrt{T}, \\ d_1 &= \frac{1}{\sigma\sqrt{T}}\ln\frac{S_1(0)}{S_2(0)} + \left(\frac{\rho_{13}\sigma_1\sigma_3 - \rho_{23}\sigma_2\sigma_3}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}, \\ d_2 &= \frac{1}{\sigma_3\sqrt{T}}\ln\frac{D}{V(0)} - \left(\frac{\rho_{23}\sigma_2\sigma_3 + r}{\sigma_3} + \frac{\sigma_3}{2}\right)\sqrt{T}, \\ \theta &= \frac{\rho_{13}\sigma_1\sigma_3 - \rho_{23}\sigma_2\sigma_3}{\sigma\sigma_3}, \end{aligned}$$

with $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$ and the bivariate standard normal cumulative density function

$$\Phi_2(n_1,n_2,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{n_1}\int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}(x^2-2xy\rho+y^2)}dydx.$$

Proof. From Eq. (4), the price of an exchange option with default risk at time 0 under the risk-neural measure P is simply given by

$$C = e^{-rT}E^P\left[(S_1(T)-S_2(T))^+\left(\mathbf{1}_{\{V(T)\geq D\}} + \mathbf{1}_{\{V(T)<D\}}\frac{(1-\alpha)V(T)}{D}\right)\right] \quad (5)$$

Eq. (5) is divided into 4 terms as follows.

$$\begin{aligned}
&e^{-rT}E^P\left[(S_1(T)-S_2(T))^+\left(\mathbf{1}_{\{V(T)\geq D\}}+\delta(T)\mathbf{1}_{\{V(T)<D\}}\right)\right]\\
&=e^{-rT}E^P\left[S_1(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right]\\
&\quad -e^{-rT}E^P\left[S_2(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right]\\
&\quad +e^{-rT}E^P\left[S_1(T)\delta(T)1_{\{S_1(T)>S_2(T),V(T)<D\}}\right]\\
&\quad -e^{-rT}E^P\left[S_2(T)\delta(T)1_{\{S_1(T)>S_2(T),V(T)<D\}}\right],
\end{aligned}$$

where $\delta(T)=\frac{(1-\alpha)V(T)}{D}$. For simplifying the notations, we denote

$$\begin{aligned}
I_1&=e^{-rT}E^P\left[S_1(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right],\\
I_2&=e^{-rT}E^P\left[S_2(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right],\\
I_3&=e^{-rT}E^P\left[S_1(T)\delta(T)1_{\{S_1(T)>S_2(T),V(T)<D\}}\right],\\
I_4&=e^{-rT}E^P\left[S_2(T)\delta(T)1_{\{S_1(T)>S_2(T),V(T)<D\}}\right].
\end{aligned}$$

We first consider I_1 to obtain the option price C. To calculate I_1, we define a new measure $\widetilde{P}$ equivalent to P by the following Radon-Nikodym derivative

$$\frac{d\widetilde{P}}{dP}=\exp\left[\sigma_1W_1(T)-\frac{1}{2}\sigma_1^2T\right].$$

Then, by Girsanov's theorem,

$$\widetilde{W_1}(T)=W_1(T)-\sigma_1T,\widetilde{W_2}(T)=W_2(T)-\rho_{12}\sigma_1T,\widetilde{W_3}(T)=W_3(T)-\sigma_1\rho_{13}T$$

are the standard Brownian motions under the measure $\widetilde{P}$. I_1 is represented by

$$\begin{aligned}
I_1&=e^{-rT}E^P\left[S_1(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right]\\
&=e^{-rT}E^{\widetilde{P}}\left[\frac{dP}{d\widetilde{P}}S_1(T)1_{\{S_1(T)>S_2(T),V(T)\geq D\}}\right]\\
&=S_1(0)\widetilde{P}\left(S_1(T)>S_2(T),V(T)\geq D\right)
\end{aligned}$$

Under the measure $\widetilde{P}$,

$$\begin{aligned}
&\widetilde{P}\left(S_1(T)>S_2(T),V(T)\geq D\right)\\
&=\widetilde{P}\Big(S_1(0)e^{(r+\frac{1}{2}\sigma_1^2)T+\sigma_1\widetilde{W}_1(T)}>S_2(0)e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2\widetilde{W}_2(T)+\sigma_1\sigma_2\rho_{12}T},\\
&\qquad V(0)e^{(r-\frac{1}{2}\sigma_3^2)T+\sigma_3\widetilde{W}_3(T)+\sigma_1\sigma_3\rho_{13}T}>D\Big)\\
&=\widetilde{P}\Big(\sigma_2\widetilde{W}_2(T)-\sigma_1\widetilde{W}_1(T)<\ln\frac{S_1(0)}{S_2(0)}+\frac{1}{2}\sigma^2T,\\
&\qquad -\sigma_3\widetilde{W}_3(T)<\ln\frac{V(0)}{D}+(\sigma_1\sigma_3\rho_{13}+r-\frac{1}{2}\sigma_3^2)T\Big)
\end{aligned}$$

Since $\sigma_2\widetilde{W}_2(T) - \sigma_1\widetilde{W}_1(T) \sim \mathcal{N}(0,\sigma^2 T)$, $-\sigma_3\widetilde{W}_3(T) \sim \mathcal{N}(0,\sigma_3^2 T)$ and

$$\begin{aligned} & \mathrm{Cov}\left(\sigma_2\widetilde{W}_2(T) - \sigma_1\widetilde{W}_1(T), -\sigma_3\widetilde{W}_3(T)\right) \\ = & \mathrm{Cov}\left(\sigma_2\widetilde{W}, -\sigma_3\widetilde{W}_3(T)\right) + \mathrm{Cov}\left(-\sigma_1\widetilde{W}_1(T), -\sigma_3\widetilde{W}_3(T)\right) \\ = & -\sigma_2\sigma_3\rho_{23}T + \sigma_1\sigma_3\rho_{13}T. \end{aligned}$$

Therefore, we have

$$I_1 = S_1(0)\Phi_2(a_1, a_2, \theta_3).$$

For I_2, we define a new measure $\widehat{P}$ equivalent to P by the following Radon-Nikodym derivative

$$\frac{d\widehat{P}}{dP} = \exp\left[\sigma_2 W_2(T) - \frac{1}{2}\sigma_2^2 T\right].$$

Then, by Girsanov's theorem,

$$\widehat{W}_1(T) = W_1(T) - \rho_{12}\sigma_2 T, \ \ \widehat{W}_2(T) = W_2(T) - \sigma_2 T, \ \ \widehat{W}_3(T) = W_3(T) - \sigma_2\rho_{23}T$$

are the standard Brownian motions under the measure $\widehat{P}$.

Under the measure $\widehat{P}$, I_2 can be calculated in a similar way to I_1.

For I_3, we define a new measure Q equivalent to P by the following Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left[\sigma_1 W_1(T) - \frac{1}{2}\sigma_1^2 T + \sigma_3 W_3(T) - \frac{1}{2}\sigma_3^2 T - \rho_{13}\sigma_1\sigma_3 T\right].$$

Then, by the multidimensional Girsanov's theorem, the standard Brownian motions under the measure Q are defined by

$$\begin{aligned} W_1^Q(T) &= W_1(T) - \sigma_1 T - \sigma_3\rho_{13}T, \\ W_2^Q(T) &= W_2(T) - \rho_{12}\sigma_1 T - \sigma_3\rho_{23}T, \\ W_3^Q(T) &= W_3(T) - \sigma_3 T - \sigma_1\rho_{13}T. \end{aligned}$$

Under the measure Q, I_3 is represented by

$$\begin{aligned} I_3 &= e^{-rT}E^P\left[S_1(T)\delta(T)1_{\{S_1(T)>S_2(T), V(T)<D\}}\right] \\ &= \frac{1-\alpha}{D}S_1(0)V(0)E^Q\left[\frac{dP}{dQ}e^{rT}e^{\sigma_1\sigma_3\rho_{13}T}1_{\{S_1(T)>S_2(T), V(T)<D\}}\right] \\ &= \frac{1-\alpha}{D}S_1(0)V(0)e^{(r+\sigma_1\sigma_3\rho_{13})T}P^Q\left(S_1(T)>S_2(T), V(T)<D\right). \end{aligned}$$

Since

$$\begin{aligned} & P^Q\left(S_1(T)>S_2(T), V(T)<D\right) \\ & = P^Q\Bigg(\sigma_2 W_2^Q(T) - \sigma_1 W_1^Q(T) < \ln\frac{S_1(0)}{S_2(0)} + (\frac{1}{2}\sigma^2 + \sigma_1\sigma_3\rho_{13} - \sigma_2\sigma_3\rho_{23})T, \\ & \qquad \sigma_3 W_3^Q(T) < \ln\frac{D}{V(0)} - (\frac{1}{2}\sigma_3^2 + r + \sigma_1\sigma_3\rho_{13})T\Bigg), \end{aligned}$$

we have

$$I_3 = \frac{1-\alpha}{D} S_1(0)V(0)e^{(r+\sigma_1\sigma_3\rho_{13})T}\Phi_2(c_1, c_2, -\theta).$$

For I_4, we define a new measure $\widetilde{Q}$ equivalent to P by the following Radon-Nikodym derivative

$$\frac{d\widetilde{Q}}{dP} = \exp\left[\sigma_2 W_2(T) - \frac{1}{2}\sigma_2^2 T + \sigma_3 W_3(T) - \frac{1}{2}\sigma_3^2 T - \rho_{23}\sigma_2\sigma_3 T\right].$$

Then, by the multidimensional Girsanov's theorem, the standard Brownian motions under the measure $\widetilde{Q}$ are defined by

$$\begin{aligned} W_1^{\widetilde{Q}}(T) &= W_1(T) - \rho_{12}\sigma_2 T - \sigma_3\rho_{13}T, \\ W_2^{\widetilde{Q}}(T) &= W_2(T) - \sigma_2 T - \sigma_3\rho_{23}T, \\ W_3^{\widetilde{Q}}(T) &= W_3(T) - \sigma_3 T - \sigma_2\rho_{23}T. \end{aligned}$$

Under the measure $\widetilde{Q}$, I_4 can be derived in a similar way to I_3. This completes the proof. □

4. Concluding remarks

In this paper, we study the price of an exchange option with default risk based on the model of Klein [6]. More specifically, we derive the closed-form solution for the arbitrage free price of the exchange option using a probabilistic approach. To obtain the pricing formula of the exchange option, we use the multidimensional Girsanov's theorem and the properties of the bivariate standard normal distribution. As a result, we provide a simper approach for the pricing formula of the exchange option with default risk.

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