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DEBTOR'S BANKRUPTCY AND UPPER BOUND ON CONSUMPTION

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Abstract. This paper obtains an explicit expression for a debtor's consumption, porfolio and time for filing for bankruptcy when the debtor is subject to upper bound on consumption rate. The utility of consumption is CRRA type and the dynamic programming principle is applied.

1. Introduction

In the United States, debtors' unsecured debts can be discharged under Chapter 7. Jeanblanc et al. [2] is the first quantitative research on personal bankruptcy based on stochastic optimization and optimal stopping time. In [2], debtor and filer don't have any restrictions on liquidity or consumption. Recently, [4] investigates the debtor's optimal consumption, portfolio, and bankruptcy strategy for the case where filer faces liquidity constraint.

This paper investigates debtor's optimal strategy when there is a upper bound on consumption rate. Literature on the optimal consumption, portfolio with restrictions on consumption includes [1], [5], [6], [7], [8], [9], [10], and others. We employ dynamic programming principle to tackle the HJB equation for the value function. In order to linearize the Bellman equation, the transform in [3] is used. The optimal time for bankruptcy is a stopping and charecterized as the first time the debtor's wealth level reaches a threshold, which is determined by smooth pasting conditions between filer's value function and debtor's value function.

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2. The Model

The financial market consists of a riskless asset (money market account) and a risky asset. The risk-free interest rate $r >$ is assumed to be a constant. The risky asset price S_t evolves the following equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by the standard Brownian motion $(B_t)_{t\geq 0}$. $\mu > r$ and σ are constants.

Denote by $\boldsymbol{\pi} \equiv (\pi_t)_{t\geq 0}$ the amount of money invested in the risky asset, and $\mathbf{c} \equiv (c_t)_{t\geq 0}$ the nonnegative rate of consumption. The debtor pays a fixed rate repayment p. If we denote by τ the time at which the debtor files for bankruptcy, the wealth level X_t evolves the following equations

$$dX_t = \begin{cases} [rX_t + \pi_t(\mu - r) - c_t - p]\, dt + \sigma\pi_t dB_t, & t \leq \tau, \\ [rX_t + \pi_t(\mu - r) - c_t]\, dt + \sigma\pi_t dB_t. & t > \tau. \end{cases}$$

The utility function is a CRRA utility function of consumption the form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},\ \gamma \neq 1,\ \gamma > 0,$$

where γ is the agent's coefficient of relative risk aversion.

3. Optimization problems and solutions

3.1. Filer's optimization problem

The filer's wealth process is given by

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t]\, dt + \sigma\pi_t dB_t,\ X_0 = x. \tag{3.1}$$

We define admissible policy for defining the filer's optimization problem as follows

Definition 1. A policy pair $(\mathbf{c}, \boldsymbol{\pi})$ is admissible at x if

(a) $\int_0^t c_s ds < \infty,$ for all $t \geq 0$ a.s.
(b) $\int_0^t \pi_s^2 ds < \infty,$ for all $t \geq 0$ a.s.
(c) X_t in (3.1) satisfies $X_t \geq 0,$ for all $t \geq 0$ a.s.

Let $\mathcal{A}_f(x)$ the set of all admissible policy pairs at x and define the filer's value function as follows

$$V_F(x) = \max_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}_f(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right] = \max_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}_f(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right],$$

where β is the subjective discount rate. The Bellman equation for V_F is given by

$$\beta V_F(x) = \max_{c,\pi}\left[\{rx + \pi(\mu - r) - c\} V_F'(x) + \frac{1}{2}\sigma^2\pi^2 V_F''(x) + \frac{c^{1-\gamma}}{1-\gamma}\right],$$

and the value function is given by

$$V_F(x) = \frac{x^{1-\gamma}}{K^\gamma(1-\gamma)},$$

where

$$K = r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2,\ \theta = \frac{\mu - r}{\sigma}.$$

Moreover, the optimal consumption c^* and portfolio π^* are given by

$$c^* = Kx \text{ and } \pi^* = \frac{\theta}{\sigma\gamma}x.$$

Note that $V_F'(x) = (c^*)^{-\gamma}$.

3.2. Debtor's optimization problem

The debtor's wealth level X_t follows

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t - p]\,dt + \sigma\pi_t dB_t,\ X_0 = x,\ t \le \tau. \tag{3.2}$$

The debtor should pay a fixed cost $F >$ at the time of bankruptcy for filing bankruptcy and retain the rate α of the remaining wealth $X_\tau - F$, i.e.,

$$X_{\tau_+} = \alpha(X_\tau - F),\ F > 0,\ 0 < \alpha < 1.$$

If $(X_\tau - F) > 0$, α stands for the exemption rate but if $(X_\tau - F) < 0$, $(1-\alpha)$ is the rate of the value of discharged asset to the lump sum debt $|(X_\tau - F)|$ (apart from the debt repayment stream p). The debtor's consumption rate has a upper bound $R > 0$, i.e.,

$$c_t \le R,\ t \ge 0. \tag{3.3}$$

The inequality (3.3) implies that the debtor cannot consume more than R per unit time.

Let $\mathcal{S}_{[0,T]}$ the set of all $\mathcal{F}_T$-stopping times for a fixed $T > 0$ and define the set of stopping times when $T \to \infty$ by $\mathcal{S}$. We define

Definition 2. A policy triple $(\mathbf{c}, \boldsymbol{\pi}, \tau)$ is asmissible at x if

(a) $\int_0^t c_s ds < \infty,\ c_t \le R,$ for all $t \ge 0$ a.s.
(b) $\int_0^t \pi_s^2 ds < \infty,$ for all $t \ge 0$ a.s.
(c) X_t in (3.2) satisfies $X_t \ge \frac{p}{r},$ for all $t \ge 0$ a.s.
(d) $\tau \in \mathcal{S}$.

Let $\mathcal{A}(x)$ the set of all admissible policy triples at x and define the debtor's value function as follows

$$\begin{aligned} V(x) &= \max_{(\mathbf{c},\boldsymbol{\pi},\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\beta t}u(c_t)dt + e^{-\beta\tau}V_F\left(\alpha\left(X_\tau - F\right)\right)\right] \\ &= \max_{(\mathbf{c},\boldsymbol{\pi},\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V_F\left(\alpha\left(X_\tau - F\right)\right)\right]. \end{aligned}$$

There exists a wealth threshold $\tilde{x}$ such that debtor's consumpmtion rate is constantly R when the debtor's wealth level is at or above $\tilde{x}$. We assume that the debtor does not file for bankruptcy while consuming the full consumption rate. If we denote by $\bar{x}$ the bankruptcy wealth level, the previous assumption implies

$$X_\tau = \bar{x} < \tilde{x}.$$

For $\bar{x} < x < \tilde{x}$, the Bellman equation of $V(x)$ is given by

$$\beta V(x) = \max_{c,\pi}\left[\left\{rx + \pi(\mu - r) - c - p\right\}V'(x) + \frac{1}{2}\sigma^2\pi^2V''(x) + \frac{c^{1-\gamma}}{1-\gamma}\right],$$

while $V(x)$ satisfies the following Bellman equation

$$\beta V(x) = \max_{c,\pi}\left[\left\{rx + \pi(\mu - r) - R - p\right\}V'(x) + \frac{1}{2}\sigma^2\pi^2V''(x) + \frac{R^{1-\gamma}}{1-\gamma}\right],$$

for $x \geq \tilde{x}$. The first order conditions are given by

$$\pi^* = -\frac{\mu - r}{\sigma^2}\frac{V'(x)}{V''(x)},\ \bar{x} < x, \tag{3.4}$$

$$c^* = \begin{cases} (V'(x))^{-\frac{1}{\gamma}},\ \bar{x} < x < \tilde{x}, \\ R,\ x \geq \tilde{x}. \end{cases} \tag{3.5}$$

Therefore, the Bellman equation can be rewritten as follows

$$V(x) = \begin{cases} (rx - p)\,V'(x) - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \beta V(x) + \frac{\gamma}{1-\gamma}\left(V'(x)\right)^{-\frac{1-\gamma}{\gamma}} = 0,\ \bar{x} < x < \tilde{x}, \\ (rx - R - p)\,V'(x) - \frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)} - \beta V(x) + \frac{R^{1-\gamma}}{1-\gamma} = 0,\ x \geq \tilde{x}. \end{cases} \tag{3.6}$$

Remark 1. The equation

$$\frac{1}{2}\theta^2n^2 + \left(\beta - r + \frac{1}{2}\theta^2\right)n - r = 0,$$

has a negative root n_2 such that $n_2 < -1$. The equation

$$rm^2 - \left(\beta + r + \frac{1}{2}\theta^2\right)m + \beta = 0,$$

has two roots $m_1,\ m_2$, such that $0 < m_2 < 1 < m_1$.

Theorem 3.1. *The debtor's value function $V(x)$ is given by*

$$V(x)=\begin{cases}\dfrac{r-\frac{1}{2}\theta^2 n_1}{\beta}D_1 z^{-\gamma(n_1+1)}+\dfrac{r-\frac{1}{2}\theta^2 n_2}{\beta}D_2 z^{-\gamma(n_2+1)}+\dfrac{z^{1-\gamma}}{K(1-\gamma)},\ \bar{x}<x<\tilde{x},\\[2ex] C_2\left(x-\dfrac{R+p}{r}\right)^{m_2}+\dfrac{R^{1-\gamma}}{\beta(1-\gamma)},\ x\ge\tilde{x},\end{cases}$$

where

$$D_1=\frac{\Phi}{\Sigma}R^{\gamma n_1+1},$$

$$\Phi=\left(\frac{1}{r}+\frac{1-\gamma-m_2}{\gamma K}\right)\frac{1}{1-(m_2-1)n_2}\frac{m_2\left(r-\frac{1}{2}\theta^2 n_2\right)-\beta}{m_2\beta}-\frac{r-K}{m_2K\gamma}-\frac{1}{1-\gamma}\frac{K-\beta}{\beta K},$$

$$\Sigma=\frac{m_2\left(r-\frac{1}{2}\theta^2 n_2\right)-\beta}{m_2\beta}\frac{1-(m_2-1)n_1}{1-(m_2-1)n_2}-\frac{m_2\left(r-\frac{1}{2}\theta^2 n_1\right)-\beta}{m_2\beta},$$

$$D_2=-\frac{1-(m_2-1)n_1}{1-(m_2-1)n_2}D_1R^{\gamma(n_2-n_1)}+\frac{\frac{1}{r}+\frac{1-\gamma-m_2}{\gamma K}}{1-(m_2-1)n_2}R^{\gamma n_2+1},$$

$$\tilde{x}=D_1R^{-\gamma n_1}+D_2R^{-\gamma n_2}+\frac{1}{K}R+\frac{p}{r},$$

$$C_2=\frac{R^{-\gamma}}{m_2}\left(\tilde{x}-\frac{R+p}{r}\right)^{1-m_2}.$$

Define

$$X(z;D_1,D_2)\triangleq D_1z^{-\gamma n_1}+D_2z^{-\gamma n_2}+\frac{1}{K}z+\frac{p}{r}.$$

Then,

$$\bar{x}=X(\bar{c};D_1,D_2),$$

where $\bar{c}$ solves

$$X(\bar{c};D_1,D_2)-\frac{\alpha^{\frac{1-\gamma}{\gamma}}}{K}\bar{c}-F=0.$$

For $\bar{x}<x<\tilde{x}$, z follows from the relation

$$x=X(z;D_1,D_2).$$

Proof. For $\bar{x}<x<\tilde{x}$, suppose that the optimal consumption c can be expressed as an invertible function of wealth level x, i.e.,

$$c=C(x).$$

If we denote by $X(\cdot)$ the inverse function of $C(\cdot)$ then

$$X(c)=X(C(x))=x,$$

and from the first order condition (3.5),

$$V'(x)=C^{-\gamma}(x),\ V''(x)=-\gamma\frac{C^{-\gamma-1}(x)}{X'(c)}. \tag{3.7}$$

Substituting (3.7) into (3.6) we obtain

$$(rX(c)-p)\,c^{-\gamma}+\frac{1}{2\gamma}\theta^2c^{1-\gamma}X'(c)-\beta V(X(c))+\frac{\gamma}{1-\gamma}c^{1-\gamma}=0. \tag{3.8}$$

If we differentiate (3.8) with respect to c, we have

$$\frac{1}{2\gamma}\theta^2c^2X''(c)+\left(r-\beta+\frac{1-\gamma}{2\gamma}\theta^2\right)cX'(c)-r\gamma X(c)+\gamma c+\gamma p=0. \tag{3.9}$$

The homogeneous solution $X_h(c)$ to the equation (3.9) is given by

$$X_h(c)=D_1c^{-\gamma n_1}+D_2c^{-\gamma n_2}.$$

Trying a particular solution to the equation (3.9) to obtain

$$X_p(c)=\frac{1}{K}c+\frac{p}{r}.$$

Thus, we have

$$X(c)=X(c;D_1,D_2)=X_h(c)+X_p(c)=D_1c^{-\gamma n_1}+D_2c^{-\gamma n_2}+\frac{1}{K}c+\frac{p}{r}. \tag{3.10}$$

From (3.8) and (3.10) the value function $V(x)$ is given by

$$V(x)=V(X(z))=\frac{r-\frac{1}{2}\theta^2n_1}{\beta}D_1z^{-\gamma(n_1+1)}+\frac{r-\frac{1}{2}\theta^2n_2}{\beta}D_2z^{-\gamma(n_2+1)}+\frac{z^{1-\gamma}}{K(1-\gamma)},$$

where z is determined by the relation

$$x=X(z;D_1,D_2).$$

For $x\geq\tilde{x}$, we try a homogeneous solution $\left(x-\dfrac{R+p}{r}\right)^m$ to the equation (3.6) then we are lead to the equation

$$rm^2-\left(\beta+r+\frac{1}{2}\theta^2\right)m+\beta=0,$$

which has two roots $m_1,\ m_2$, such that $0<m_2<1<m_1$. The particular solution is given by $\dfrac{R^{1-\gamma}}{\beta(1-\gamma)}$. Therefore, the value function is given by

$$V(x)=C_2\left(x-\frac{R+p}{r}\right)^{m_2}+\frac{R^{1-\gamma}}{\beta(1-\gamma)},$$

where we set $C_1=0$.

Let $\bar{c}$ and $\bar{c}_l$ the optimal consumtion rates such that

$$X_\tau=\bar{x}=X(\bar{c};D_1,D_2), \tag{3.11}$$

$$X_{\tau_+}=\alpha(X_\tau-F)=\frac{1}{K}\bar{c}_l. \tag{3.12}$$

Firstly, we use the smooth pasting condition at $x=\bar{x}$

$$V'(\bar{x})=\frac{d}{dx}V_F\left(\alpha(x-F)\right)\Big|_{x=\bar{x}}=\alpha V_F'\left(\alpha(\bar{x}-F)\right),$$

which yields

$$\bar{c}_l = \alpha^{\frac{1}{\gamma}}\bar{c}. \tag{3.13}$$

The value matching condition at $x = \bar{x}$,

$$V(\bar{x}) = V_F(\alpha(\bar{x} - F)),$$

gives us

$$\frac{r - \frac{1}{2}\theta^2 n_1}{\beta} D_1 \bar{c}^{-\gamma(n_1+1)} + \frac{r - \frac{1}{2}\theta^2 n_2}{\beta} D_2 \bar{c}^{-\gamma(n_2+1)} + \frac{\bar{c}^{1-\gamma}}{K(1-\gamma)}$$
$$= \frac{\alpha^{\frac{1-\gamma}{\gamma}}}{K(1-\gamma)}\bar{c}^{1-\gamma}.$$

Combining (3.11), (3.12), and (3.13) yields

$$D_1\bar{c}^{-\gamma n_1} + D_2\bar{c}^{-\gamma n_2} + \frac{1}{K}\bar{c} + \frac{p}{r} - F = \frac{\alpha^{\frac{1-\gamma}{\gamma}}}{K}\bar{c}.$$

At $x = \tilde{x}$,

$$\tilde{x} = D_1 R^{-\gamma n_1} + D_2 R^{-\gamma n_2} + \frac{1}{K}R + \frac{p}{r}, \tag{3.14}$$

$$V(\tilde{x}) = \frac{r - \frac{1}{2}\theta^2 n_1}{\beta} D_1 R^{-\gamma(n_1+1)} + \frac{r - \frac{1}{2}\theta^2 n_2}{\beta} D_2 R^{-\gamma(n_2+1)} + \frac{R^{1-\gamma}}{K(1-\gamma)}$$
$$= C_2\left(\tilde{x} - \frac{R+p}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\beta(1-\gamma)}, \tag{3.15}$$

$$V'(\widetilde{x}) = m_2 C_2 \left(\widetilde{x} - \frac{R+p}{r}\right)^{m_2-1} = R^{-\gamma}, \tag{3.16}$$

and

$$V''(\widetilde{x}) = m_2(m_2-1)C_2\left(\widetilde{x} - \frac{R+p}{r}\right)^{m_2-2} = -\gamma\frac{R^{-\gamma-1}}{X'(R; D_1, D_2)}. \tag{3.17}$$

From (3.16) and (3.17) we have

$$\tilde{x} = (m_2-1)n_1 D_1 R^{-\gamma n_1} + (m_2-1)n_2 D_2 R^{-\gamma n_2} - \frac{m_2-1}{\gamma K}R + \frac{R+p}{r}. \tag{3.18}$$

Plugging (3.16) into (3.15) to obtain

$$\left(\frac{r - \frac{1}{2}\theta^2 n_1}{\beta} - \frac{1}{m_2}\right) D_1 R^{-\gamma n_1 - 1} + \left(\frac{r - \frac{1}{2}\theta^2 n_2}{\beta} - \frac{1}{m_2}\right) D_2 R^{-\gamma n_2 - 1}$$
$$= \frac{1}{m_2}\left(\frac{1}{K} - \frac{1}{r}\right) + \frac{1}{1-\gamma}\left(\frac{1}{\beta} - \frac{1}{K}\right). \tag{3.19}$$

Equating (3.14) and (3.18) we have

$$\{1 - (m_2-1)n_1\} D_1 R^{-\gamma n_1 - 1} + \{1 - (m_2-1)n_2\} D_2 R^{-\gamma n_2 - 1}$$
$$= \frac{1}{r} + \frac{1-\gamma-m_2}{\gamma K}. \tag{3.20}$$

From (3.19) and (3.20) we obtain

$$D_1 = \frac{\Phi}{\Sigma} R^{\gamma n_1 + 1}.$$

$\square$

Theorem 3.2. *Debtor's optimal policy is given by*

$$c_t^* = \begin{cases} Z_t, & \text{for } \bar{x} < X_t < \tilde{x}, \\ R, & \text{for } X_t \geq \tilde{x}. \end{cases}$$

$$\pi_t^* = \begin{cases} \dfrac{\theta}{\gamma\sigma}\left(-\gamma n_1 D_1 Z_t^{-\gamma n_1} - \gamma n_2 D_2 Z_t^{-\gamma n_2} + \dfrac{Z_t}{K}\right), & \text{for } \bar{x} < X_t < \tilde{x}, \\ \dfrac{\theta}{\sigma}\dfrac{1}{1-m_2}\left(X_t - \dfrac{R+p}{r}\right), & \text{for } X_t \geq \tilde{x}, \end{cases}$$

where, Z_t follows from the relation

$$X_t = X(Z_t; D_1, D_2),$$

and the optimal time for filling bankruptcy τ^* is given by

$$\tau^* = \inf\{t \geq 0 : X_t \leq \bar{x}\}.$$

Proof. The first order conditions (3.4) and (3.5) leads us to the optimal consumption c_t^* and portfolio π_t^*. The derivation of the optimal stopping time τ^* is obtained by following similar lines to the proof of Theorem 3.1 in [2]. $\square$

References

[1] N. Gong, T. Li, *Role of index bonds in an optimal dynamic asset allocation model with real subsistence consumption*, Applied Mathematics and Computation (2006), no. 174, 710-731.

[2] M. Jeanblanc, P. Lakner, and A. Kadam, *Optimal bankruptcy time and consumption/investment policies on an infinite horizon with a continuous debt repayment until bankruptcy*, Mathematics of Operations Research (2004), no. 29, 649-671.

[3] I. Karatzas, J.P. Lehoczky, S.P. Sethi, and S.E. Shreve, *Explicit solution of a general consumption/investment problem*, Mathematics of Operations Research (1986), no. 11, 261-294.

[4] H. Lee, B.H. Lim, *Strategic default and post-bankruptcy liquidity constraint*, Working paper (2020).

[5] B.H. Lim, H. Lee, and Y.H. Shin, *The effects of pre-/post-retirement downside consumption constraints on optimal consumption, portfolio, and retirement*, Finance Research Letters (2018), no. 25, 213-221.

[6] P. Lakner, L.M. Nygren, *Portfolio optimization with downside constraints*, Mathematical Finance (2006), no. 16, 283-299.

[7] H.-S. Lee, Y.H. Shin, *An optimal investment, consumption-leisure and voluntary retirement choice problem with subsistence consumption constraints*, Japan Journal of Industrial and Applied Mathematics (2016), no. 33, 297-320.

[8] B.H. Lim, Y.H. Shin, and U.J. Choi, *Optimal investment, consumption and retirement choice problem with disutility and subsistence consumption constraints*, Journal of Mathematical Analysis and Applications 3 (2008), no. 345, 109-122.

[9] Y.H. Shin, B.H. Lim, *Comparison of optimal portfolios with and without subsistence consumption constraints*, Nonlinear Analysis (2011), no. 74, 50-58.

[10] Y.H. Shin, B.H. Lim, and U.J. Choi, *Optimal consumption and portfolio selection problem with downside consumption constraints*, Applied Mathematics and Computation (2007), no. 188, 1801-1811.

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