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LIE IDEALS AND COMMUTATIVITY OF 2−TORSION FREE SEMIPRIME RINGS WITH GENERALIZED DERIVATION

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Abstract. In this paper, we investigate commutativity of semiprime rings with a derivation which is strongly commutativity preserving and acts as a homomorphism or as an anti-homomorphism on a nonzero Lie ideal.

1. Introduction

Throughout R will present an associative ring with center Z. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy-yx$ and the symbol xoy denotes the anti-commutator $xy + yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and R is semiprime if for $x \in R$, $xRx = 0$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. A Lie ideal U of R is said to be a square-closed Lie ideal of R if $u^2 \in U$ for all $u \in U$. Let S be a nonempty subset of R. A mapping F from R to R is called strong commutativity preserving (simply, SCP) on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

Recently, in [6], Bresar defined the following notation. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). One may observe that the concept of generalized derivations includes the concept of derivations and of the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). Hence it should be interesting to extend some results concerning these notions to generalized derivations.

The commutativity of prime rings with derivation was initiated by E. C. Posner [10]. There has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. In [7], M. N. Daif and H. E. Bell proved that if a semiprime

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ring R with a derivation d satisfies one of the conditions i) $d([x,y]) = [x,y]$ and ii) $d([x,y]) = -[x,y]$ for all $x,y \in I$, a nonzero ideal I of R, then I is central.

In [4], H. E. Bell and L. C. Kappe proved that if d is a derivation of a prime ring R which is either a homomorphism or an anti-homomorphism on a nonzero right ideal of R, then $d = 0$. In [1], A. Ali, M. Yasen and M. Anwar showed that if R is a semiprime ring, f is an endomorphism which is a strong commutativity preserving map on a non-zero ideal U of R, then f is commuting on U. In [12], M. S. Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Being inspired by these results, recently E. Koç discuss the commutativity theorems for semiprime rings involving derivations in [9].

On the other, in [2] the authors explored the commutativity of the ring R satisfying one of the following conditions: (i) $d(x)oF(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x)oF(y) = xoy$, (iv) $d(x)oF(y) + xoy = 0$, (v) $d(x)F(y) - xy \in Z$, (vi) $d(x)F(y) + xy \in Z$, (vii) $[d(x), F(y)] = [x,y]$ and (viii) $[d(x), F(y)] + [x,y] = 0$, for all $x,y \in R$ in some appropriate subset of the ring R.

In the present paper, we shall extend the above result for a nonzero Lie ideals of semiprime rings with derivation of R.

2. Preliminaries

Make some extensive use of the basic commutator identities:
$[x,yz] = y[x,z] + [x,y]z$
$[xy,z] = [x,z]y + x[y,z]$
$xo(yz) = (xoy)z - y[x,z] = y(xoz) + [x,y]z$
$(xy)oz = x(yoz) - [x,z]y = (xoz)y + x[y,z]$.
Moreover, we shall require the following lemmas.

Lemma 2.1. [5, Lemma 4] *Let R be a prime ring with characteristic not two, $a,b \in R$. If U is a noncentral Lie ideal of R and $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2. [5, Lemma 2] *Let R be a prime ring with characteristic not two. If U is a noncentral Lie ideal of R, then $C_R(U) = Z$.*

Lemma 2.3. [5, Lemma 5] *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d(U) = 0$, then $U \subseteq Z$.*

Lemma 2.4. [5, Theorem 1] *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d^2(U) = 0$, then $U \subseteq Z$.*

Lemma 2.5. [5, Theorem 2] *Let R be a prime ring with characteristic not two and U a noncentral Lie ideal of R. If d is a nonzero derivation of R, then $C_R(d(U)) = Z$.*

Lemma 2.6. [3, Theorem 7] *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[u, d(u)] \in Z$, for all $u \in U$, then $U \subseteq Z$.*

Lemma 2.7. [11, Lemma 2] *Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \not\subseteq Z$ and $a \in U$. If $aUa = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = R[U,U]R$ of R generated by $[U,U]$ such that $[K,R] \subseteq U$ and $Ka = aK = 0$.*

Corollary 2.8. [8, Corollary] *Let R be a 2-torsion free semiprime ring, U a noncentral Lie ideal of R and $a, b \in U$.*
(i) If $aUa = 0$, then $a = 0$.
(ii) If $aU = 0$(or $Ua = 0$), then $a = 0$
(iii) If U is square-closed and $aUb = 0$, then $ab = 0$ and $ba = 0$.

3. Lie Ideals and Derivations in Semiprime Rings

The following theorem gives a generalization of theorems in [12] and [4].

Theorem 3.1. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \not\subseteq Z(R)$. Suppose that R admits a derivation d such that $d(x) \in U$, for all $x \in U$. If d is strong commutativity preserving on U, then d is commuting on U.*

Proof. Suppose that
$$[d(x), d(y)] = [x, y], \text{ for all } x, y \in U.$$
Replacing y by yz, $z \in U$, we obtain that
$$[d(x), d(y)z + yd(z)] = [x, yz].$$
Applying the hypothesis, we have
$$d(y)[d(x), z] + [d(x), y]d(z) = 0.$$
Taking z by $d(x)$ in the above equation, we find that
$$[d(x), y]d^2(x) = 0, \text{ for all } x, y \in U.$$
Writing y by $d(y)$, we get
$$[d(x), d(y)]d^2(x) = 0, \text{ for all } x, y \in U.$$
Using the hypothesis, we see that
$$[x, y]d^2(x) = 0, \text{ for all } x, y \in U. \tag{1}$$
Substituting yr for y, $r \in U$ in (1) and using (1), we have
$$[x, y]rd^2(x) = 0 \text{ for all } x, y, r \in U. \tag{2}$$
Multiplying (2) on the right by $[x, y]$ and the left by $d^2(x)$, we get
$$d^2(x)[x, y]Ud^2(x)[x, y] = 0, \text{ for all } x, y \in U.$$
By Corollary 1, we obtain
$$d^2(x)[x, y] = 0, \text{ for all } x, y \in U. \tag{3}$$

Replacing y by ry in the last equation, we see that

$$d^2(x)\, r\, [x,y] = 0, \text{ for all } x,y,r \in U. \tag{4}$$

Writing $x+z$ by x, $z \in U$ in (1) and using (1), we have

$$[x,y]\, d^2(z) + [z,y]\, d^2(x) = 0$$

and so

$$[x,y]\, d^2(z) = -[z,y]\, d^2(x), \text{ for all } x,y,z \in U. \tag{5}$$

Moreover, equation (5) implies that, we arrive at

$$[x,y]\, d^2(z)\, r\, [x,y]\, d^2(z) = -[x,y]\, d^2(z)\, r\, [z,y]\, d^2(x)$$

Using (4), we find that

$$[x,y]\, d^2(z)\, r\, [x,y]\, d^2(z) = 0, \text{ for all } x,y,z,r \in U.$$

By Corollary 1, we have

$$[x,y]\, d^2(z) = 0, \text{ for all } x,y,z \in U. \tag{6}$$

Taking yr instead of y in (6) and using (6), we have

$$[x,y]\, rd^2(z) = 0, \text{ for all } x,y,z,r \in U. \tag{7}$$

Replacing z by zt, $t \in U$ in (6), we get

$$[x,y]\, d^2(z)\, t + 2\, [x,y]\, d(z)\, d(t) + [x,y]\, zd^2(t) = 0, \text{ for all } x,y,z,t \in U.$$

Using equations (6) and (7) and R is 2-torsion free ring, we have

$$[x,y]\, d(z)\, d(t) = 0, \text{ for all } x,y,z,t \in U.$$

Writing t by tz in the last equation, we have

$$[x,y]\, d(z)\, td(z) = 0, \text{ for all } x,y,z,t \in U.$$

Replacing t by $t\,[x,y]$, we have

$$[x,y]\, d(z)\, t\, [x,y]\, d(z) = 0, \text{ for all } x,y,z,t \in U.$$

By Corollary 1, we have

$$[x,y]\, d(z) = 0, \text{ for all } x,y,z \in U.$$

and so

$$[x,y]\, d(x) = 0, \text{ for all } x,y \in U. \tag{8}$$

Writting $d(x)y$ for y in (8) and using (8), we obtain that

$$[x,d(x)]yd(x) = 0, \text{ for all } x,y \in U. \tag{9}$$

Replacing y by yx in (9), we find that

$$[x,d(x)]yxd(x) = 0, \text{ for all } x,y \in U. \tag{10}$$

Multiplying (9) on the right by x, we have

$$[x,d(x)]yd(x)\, x = 0, \text{ for all } x,y \in U. \tag{11}$$

Subtracting (11) from (10), we arrive at

$$[x, d(x)]y[x, d(x)] = 0, \text{ for all } x, y \in U.$$

By Corollary 1, we conclude that $[x, d(x)] = 0$, for all $x \in R$. We conclude that d is commuting map on U. $\square$

Corollary 3.2. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and d be a derivation of R. If d is strong commutativity preserving on U, then $U \subseteq Z$.*

Proof. Assume that

$$[d(x), d(y)] = [x, y], \text{ for all } x, y \in U.$$

Replacing y by yz, $z \in U$ and using the this equation, we obtain

$$d(y)[d(x), z] + [d(x), y]d(z) = 0.$$

Taking z by $d(x)$ and y by $d(y)$, $x, y \in [U, U]$ in the above equation, we find that

$$[d(x), d(y)]d^2(x) = 0, \text{ for all } x, y \in [U, U].$$

Using the hypothesis, we see that

$$[x, y]d^2(x) = 0, \text{ for all } x, y \in [U, U]. \tag{12}$$

Replacing y by yz, $z \in [U, U]$ in the above equation, we have

$$[x, y]zd^2(x) = 0, \text{ for all } x, y, z \in [U, U].$$

By Lemma 1, we get either $[x, y] = 0$ or $d(x) = 0$, for each $x \in [U, U]$. We set $K = \{x \in [U, U] \mid [x, y] = 0, \text{ for all } y \in [U, U]\}$ and $L = \left\{x \in [U, U] \mid d^2(x) = 0\right\}$. Clearly each of K and L is additive subgroup of $[U, U]$. Moreover, $[U, U]$ is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = [U, U]$ or $L = [U, U]$. In the first case, we have $U \subseteq Z$ by Lemma 2. In the latter case, we have $U \subseteq Z$ by Lemma 2 and Lemma 4. This completes the proof. $\square$

Theorem 3.3. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z$. Suppose that R admits a derivation d such that $d(x) \in U$, for all $x \in U$. If $d(x) o d(y) = xoy$, for all $x, y \in U$, then d is commuting on U.*

Proof. By the hypothesis, we get

$$d(x) o d(y) = xoy, \text{ for all } x, y \in U.$$

Replacing x by xz, $z \in U$ in the hypothesis, we obtain

$$(d(x) o d(y))z + d(x)[z, d(y)] + x(d(z) o d(y)) - [x, d(y)]d(z) = (xoy)z + x[z, y].$$

Applying the hypothesis, we have

$$d(x)[z, d(y)] + x(zoy) - [x, d(y)]d(z) = x[z, y].$$

That is

$$d(x)[z,d(y)]+xzy+xyz-[x,d(y)]d(z)=xzy-xyz$$

and so

$$d(x)[z,d(y)]-[x,d(y)]d(z)+2xyz=0. \tag{13}$$

Substituting zx for z in (13) and using (13), we have

$$d(x)z[x,d(y)]=[x,d(y)]zd(x), \text{ for all } x,y,z\in U.$$

Writing z by $z[x,d(y)]$ in this equation and using this equation, we find that

$$[x,d(y)]zd(x)[x,d(y)]=[x,d(y)]z[x,d(y)]d(x) \text{ for all } x,y,z\in U$$

and so

$$[x,d(y)]z[d(x),[x,d(y)]]=0, \text{ for all } x,y,z\in U. \tag{14}$$

Multiplying (14) on the left by $d(x)$, we have

$$d(x)[x,d(y)]z[d(x),[x,d(y)]]=0, \text{ for all } x,y,z\in U. \tag{15}$$

Taking $d(x)z$ instead of z in (14), we find that

$$[x,d(y)]d(x)z[d(x),[x,d(y)]]=0, \text{ for all } x,y,z\in U. \tag{16}$$

Subtracting (16) from (15), we see that

$$[d(x),[x,d(y)]]z[d(x),[x,d(y)]]=0, \text{ for all } x,y,z\in U.$$

By Corollary 1, we arrive at

$$[d(x),[x,d(y)]]=0, \text{ for all } x,y\in U.$$

Moreover, replacing z by x in (13) and using the last equation, we see that

$$d(x)[x,d(y)]-[x,d(y)]d(x)+2xyx=0.$$

That is $2xyx=0$, for all $x,y\in U$. Since R is a $2-$torsion free ring, we obtain $xyx=0$, for all $x,y\in U$. By Corollary 1, we conclude that $x=0$. Hence, d is commuting on U. We complate the proof. □

Corollary 3.4. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and d be a derivation of R. If $d(x)od(y)=xoy$, for all $x,y\in U$, then $U\subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 2, we obtain that

$$d(x)[z,d(y)]-[x,d(y)]d(z)+2xyz=0, \quad \text{for all } x,y,z\in U.$$

Replacing x by $d(y)$, $y\in[U,U]$ in this equation, we have

$$d^2(y)[z,d(y)]+2d(y)yz=0, \quad \text{for all } z\in U,\ y\in[U,U].$$

Taking z by xz, $x\in U$ in the last equation, we have

$$d^2(y)x[z,d(y)]=0 \text{ for all } x,z\in U,\ y\in[U,U].$$

By Lemma 1, we get either $d^2(y)=0$ or $[z,d(y)]=0$, for each $y\in[U,U]$. We set $K=\{y\in[U,U]\mid d^2(y)=0\}$ and $L=\{y\in[U,U]\mid [z,d(y)]=0, \text{ for all } z\in U\}$. Clearly each of K and L is additive subgroup of $[U,U]$. Moreover, $[U,U]$ is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = [U, U]$ or $L = [U, U]$. In the first case, we have $U \subseteq Z$ by Lemma 4 and Lemma 2. In the latter case, we have $U \subseteq Z$ by Lemma 5 and Lemma 2. This completes the proof. $\square$

Theorem 3.5. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z$. Suppose that R admits a derivation d such that $d(x) \in U$, for all $x \in U$. If d acts as a homomorphism on U, then d is commuting map on U.*

Proof. Assume that d acts as a homomorphism on U. We get

$$d(xy) = d(x)y + xd(y) = d(x)d(y), \text{ for all } x, y \in U.$$

Replacing y by yz, $z \in U$ in above equation, we get

$$d(xy)z + xyd(z) = d(x)d(y)z + d(x)yd(z).$$

Using the hypothesis, the last relation gives

$$d(xy)z + xyd(z) = d(xy)z + d(x)yd(z),$$

and so

$$xyd(z) = d(x)yd(z).$$

That is

$$(d(x) - x)yd(z) = 0, \text{ for all } x, y, z \in U. \tag{17}$$

Writing y by $d(y)$ in (17), we get

$$(d(x) - x)d(y)d(z) = 0, \text{ for all } x, y, z \in U.$$

By the hypothesis, we obtain

$$(d(x) - x)d(yz) = (d(x) - x)d(y)z + (d(x) - x)yd(z) = 0.$$

Using (17), we have

$$(d(x) - x)d(y)z = 0.$$

Expanding this equation, we get

$$d(x)d(y)z = xd(y)z$$
$$d(xy)z = d(x)yz + xd(y)z = xd(y)z.$$

Hence we get $d(x)yz = 0$, for all $x, y, z \in U$. Replacing z by $[x, d(x)]$ in the last equation, we get $d(x)y[x, d(x)] = 0$, for all $x, y \in U$. Replacing y by yx in this equation, we find that

$$[x, d(x)]yxd(x) = 0, \text{ for all } x, y \in U. \tag{18}$$

Multiplying the above equation on the right by x, we have

$$[x, d(x)]yd(x)x = 0, \text{ for all } x, y \in U. \tag{19}$$

Subtracting (18) from (19), we arrive at

$$[x, d(x)]y[x, d(x)] = 0, \text{ for all } x, y \in U.$$

By Corollary 1, we get $[x, d(x)] = 0$, for all $x \in U$. This completes the proof. $\square$

Corollary 3.6. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and d be a derivation of R. If d acts as a homomorphism on U, then $U \subseteq Z$.*

Proof. By the same techniques in the beginning of the proof of Theorem 3, we obtain that

$$xyd(z) = d(x)\, yd(z)\,, \text{ for all } x, y, z \in U.$$

Replacing x by xt, $t \in U$ in the last equation, we have

$$d(x)\, tyd(z) = 0, \text{ for all } x, y, z, t \in U.$$

Using Lemma 1, we get $d(z) = 0$, for all $z \in U$, and so, $U \subseteq Z$ by Lemma 3. □

Theorem 3.7. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a derivation d such that $d(x) \in U$, for all $x \in U$. If d acts as an anti-homomorphism on U, then d is commuting map on U.*

Proof. By the hypothesis, we have

$$d(xy) = d(x)\, y + xd(y) = d(y)\, d(x)\,.$$

Replacing y by xy in the last relation and using d is a derivation of U, we arrive at

$$d(x)\, xy + xd(xy) = d(x)\, yd(x) + xd(y)\, d(x)\,.$$

By the hypothesis, we get

$$d(x)\, xy + xd(xy) = d(x)\, yd(x) + xd(xy)$$

and so

$$d(x)\, xy = d(x)\, yd(x)\,, \text{ for all } x, y \in U. \tag{20}$$

Writing yx by y in (20) and using this equation, we have

$$d(x)\, y\, [d(x)\,, x] = 0, \text{ for all } x, y \in U.$$

Using the same arguments after the equation (9) in the proof of Theorem 1, we find that $[d(x)\,, x] = 0$. Hence d is commuting on U. □

Corollary 3.8. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and d be a derivation of R. If d acts as an anti-homomorphism on U, then $U \subseteq Z$.*

Proof. Appliying the same techniques in the proof of Theorem 4, we obtain that d is commuting on U. We conclude that $U \subseteq Z$ by Lemma 6. □

4. Lie Ideals and Generalized Derivations in Semiprime Rings

We shall extend the commutativity conditions in [2] for a Lie ideals of semiprime rings.

Theorem 4.1. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \in U$, for all $x \in U$. If $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in U$, then d is commuting map on U.*

Proof. By the hypothesis, we have

$$[d(x), F(y)] \pm [x, y] = 0, \text{ for all } x, y \in U. \tag{21}$$

Taking xz instead of x in (21) and appliying (21), we see that

$$d(x)[z, F(y)] + [x, F(y)]\, d(z) = 0, \text{ for all } x, y, z \in U.$$

If we write $d(x)$ and $d(z)$ in order, instead of x and z

$$d^2(x)[d(z), F(y)] + [d(x), F(y)]\, d^2(z) = 0, \text{ for all } x, y, z \in U.$$

Using the hypothesis in the above relation, it follows that

$$d^2(x)[z, y] + [x, y]\, d^2(z) = 0, \text{ for all } x, y, z \in U.$$

Substituting x for y, we find that

$$d^2(x)[z, x] = 0, \text{ for all } x, z \in U.$$

It can be proved using the same techniques after the equation (1) in Theorem 1. We obtain that d is commuting map on U. □

Corollary 4.2. *Let R be a $2-$torsion free semiprime ring, U a nonzero square-closed Lie ideal of R and d and g be two derivations of R such that $d(x) \in U$, for all $x \in U$. If i) $[d(x), g(y)] = [x, y]$, for all $x, y \in U$, or ii) $[d(x), d(y)] = [x, y]$ for all $x, y \in U$, then d is commuting map on U.*

Corollary 4.3. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and (F, d) be a generalized derivation of R. If $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in U$, then $U \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 5, we obtain that

$$d^2(x)[z, x] = 0, \text{ for all } x, z \in [U, U].$$

This equation is same as (12) in the proof of Corollary 2. Hence, using the same arguments in there, we get the required result. □

Theorem 4.4. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \in U$, for all $x \in U$. If $d(x)F(y) = \pm[x, y]$, for all $x, y \in U$, then d is commuting map on U.*

Proof. We obtain that

$$d(x)F(y) \pm [x,y] = 0, \text{ for all } x, y \in U. \tag{22}$$

Replacing x by $xz, z \in U$ in (22), we get

$$d(x)\, zF(y) + xd(z)\, F(y) \pm x[z,y] \pm [x,y]\, z = 0 \text{ for all } x, y, z \in U.$$

Using (22), we see that

$$d(x)\, zF(y) \pm [x,y]\, z = 0, \text{ for all } x, y, z \in U. \tag{23}$$

Substituting y for x in (23), we have

$$d(x)\, zF(x) = 0, \text{ for all } x, z \in U.$$

Replacing z by $d(z)$, we get

$$d(x)\, d(z)F(x) = 0, \text{ for all } x, z \in U.$$

By the hypothesis, we have

$$d(x)[z,x] = 0, \text{ for all } x, z \in U.$$

Using the same arguments after the equation (8) in the proof of Theorem 1, we find that $[d(x), x] = 0$. Hence d is commuting on U. □

Corollary 4.5. *Let R be a $2-$torsion free semiprime ring, U a nonzero square-closed Lie ideal of R and d and g be two derivations of R such that $d(x) \in U$, for all $x \in U$. If i) $d(x)g(y) = \pm[x,y]$, for all $x, y \in U$ or ii) $d(x)d(y) = \pm[x,y]$, for all $x, y \in U$, then d is commuting map on U.*

Corollary 4.6. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and (F,d) be a generalized derivation of R. If $d(x)F(y) = \pm[x,y]$, for all $x, y \in U$, then $U \subseteq Z$.*

Proof. Using the same methods in the beginning of the proof of Theorem 6, we have

$$d(x)\, zF(x) = 0, \text{ for all } x, z \in U.$$

Taking z by $d(z)$, $z \in [U,U]$, we see that

$$d(x)\, d(z)F(x) = 0, \text{ for all } x \in U, \ z \in [U,U].$$

Appliying the hypothesis, we get

$$d(x)[z,x] = 0, \text{ for all } x \in U, \ z \in [U,U].$$

Replacing z by yz, $z \in [U,U]$ in the above equation, we have

$$d(x)y[z,x] = 0, \text{ for all } x \in U, \ y, z \in [U,U].$$

By Lemma 1, we get either $[z,x] = 0$ or $d(x) = 0$, for each $x \in U$. We set $K = \{x \in U \mid [x,y] = 0, \text{ for all } y \in [U,U]\}$ and $L = \{x \in U \mid d(x) = 0\}$. Appliying Brauer's Trick, we get $K = U$ or $L = U$. In the first case, we have $U \subseteq Z$ by Lemma 2. In the latter case, we have $U \subseteq Z$ by Lemma 3. This completes the proof. □

Theorem 4.7. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \in U$, for all $x \in U$. If $d(x)F(y) = \pm xoy$, for all $x, y \in U$, then d is commuting map on U.*

Proof. We have

$$d(x)F(y) = \pm xoy, \text{ for all } x, y \in U. \tag{24}$$

Writing $xz, z \in U$ by x in (24) and since R is 2-torsion free, we find that

$$d(x)\, zF(y) + xd(z)\, F(y) = \pm(x\,(zoy) - [x, y]\, z) = 0, \text{ for all } x, y, z \in U.$$

Using (24), we obtain

$$d(x)\, zF(y) = \mp [x, y]\, z, \text{ for all } x, y, z \in U. \tag{25}$$

Replacing z by $d(t)z$ in (25), we have

$$d(x)\, d(t)zF(y) = \mp [x, y]\, d(t)z, \text{ for all } x, y, z, t \in U.$$

Using equation (25), we get

$$\pm d(x)\, [t, y]\, z = \mp [x, y]\, d(t)z, \text{ for all } x, y, z, t \in U.$$

Taking t by y in this equation, we have

$$[x, y]\, d(y)z = 0, \text{ for all } x, y, z \in U.$$

Multiplying the last equation on the right by $[x, y]\, d(y)$, we have

$$[x, y]\, d(y)z\, [x, y]\, d(y) = 0, \text{ for all } x, y, z \in U.$$

By Corollary 1, we obtain that

$$[x, y]\, d(y) = 0, \text{ for all } x, y, z \in U.$$

Using the same arguments after the equation (8) in the proof of Theorem 1, we find that $[d(x), x] = 0$. This completes the proof. □

Corollary 4.8. *Let R be a $2-$torsion free semiprime ring, U a nonzero square-closed Lie ideal of R and d and g be two derivations of R such that $d(x) \in U$, for all $x \in U$. If i) $d(x)g(y) = \pm xoy$, for all $x, y \in U$ or ii) $d(x)d(y) = \pm xoy$, for all $x, y \in R$, then d is commuting map on U.*

Corollary 4.9. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and (F, d) be a generalized derivation of R. If $d(x)F(y) = \pm xoy$, for all $x, y \in U$, then $U \subseteq Z$.*

Proof. Appliying Theorem 7, we obtain that

$$d(x)\, zF(y) = \mp [x, y]\, z, \text{ for all } x, y, z \in U.$$

Replacing z by $d(y)z$, $y \in [U, U]$ in the above equation and using this equation, we have

$$[x, y]\, d(y)z = 0, \text{ for all } x, z \in U,\ y \in [U, U].$$

By Lemma 1, we see that

$$[x,y]\,d(y)=0,\ \text{for all } x,z\in U,\ y\in[U,U].$$

Taking xz instead of x in this equation and appliying it, we see that

$$[x,y]\,zd(y)=0,\ \text{for all } x,z\in U,\ y\in[U,U].$$

Again using Lemma 1, we get either $[x,y]=0$ or $d(y)=0$, for each $y\in U$. Using the same arguments as used Corollary 9, we conclude that $U\subseteq Z$. □

Theorem 4.10. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R such that $U\nsubseteq Z(R)$. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that $d(x)\in U$, for all $x\in U$. If $d(x)F(y)\pm xy\in Z$, for all $x,y\in U$, then d is commuting map on U.*

Proof. By the hypothesis, we have

$$d(x)F(y)-xy\in Z,\ \text{for all } x,y\in U. \tag{26}$$

Replacing y by $yz, z\in U$ in (26), we get

$$(d(x)F(y)-xy)z+d(x)yd(z)\in Z,\ \text{for all } x,y,z\in U. \tag{27}$$

Commutting this term with z and using the hyphothesis, we obtain

$$[d(x)yd(z),z]=0,\ \text{for all } x,y,z\in U. \tag{28}$$

Substituting xz for x in (28) and using this equation, we get

$$[xd(z)yd(z),z]=0,\ \text{for all } x,y,z\in U.$$

Taking x by $d(z)x$ in this eqution, we have

$$[d(z),z]xd(z)yd(z)=0,\ \text{for all } x,y,z\in U.$$

Replacing y by $y[d(z),z]x$ in the last eqution, we get

$$[d(z),z]xd(z)y[d(z),z]xd(z)=0,\ \text{for all } x,y,z\in U.$$

By Corollary 1, we conclude that

$$[d(z),z]xd(z)=0,\ \text{for all } x,y,z\in U. \tag{29}$$

Replacing x by xz in (29), we find that

$$[d(z),z]xzd(z)=0,\ \text{for all } x,y,z\in U. \tag{30}$$

Multiplying (29) on the right by z, we have

$$[d(z),z]xd\,(z)\,z=0,\ \text{for all } x,y,z\in U. \tag{31}$$

Subtracting (30) from (31), we arrive at

$$[d(z),z]x[d(z),z]=0,\ \text{for all } x,y,z\in U.$$

By Corollary 1, we conclude that $[d(z),z]=0$, for all $z\in U$, and so, d is commuting map on U.

In a similar manner, we can prove that the same conclusion holds for $d(x)F(y)+xy \in Z$, for all $x, y \in U$. □

Corollary 4.11. *Let R be a $2-$torsion free semiprime ring, U a nonzero square-closed Lie ideal of R and d and g be two derivations of R such that $d(x) \in U$, for all $x \in U$. If i) $d(x)g(y) \pm xy \in Z$, for all $x, y \in U$, or ii) $d(x)d(y) \pm xy \in Z$, for all $x, y \in U$, then d is commuting map on U.*

Corollary 4.12. *Let R be a $2-$torsion free prime ring, U a square-closed Lie ideal of R and (F, d) be a generalized derivation of R. If $d(x)F(y) \pm xy \in Z$, for all $x, y \in U$, then $U \subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 8, we obtain that

$$[xd(z)yd(z), z] = 0, \text{ for all } x, y, z \in U.$$

Replacing x by $d(z)x$, $z \in [U, U]$ in the last equation, we get

$$[d(z), z]xd(z)yd(z) = 0, \text{ for all } x, y \in U, \ z \in [U, U].$$

By Lemma 1, we get either $[d(z), z] = 0$ or $d(z) = 0$, for each $z \in [U, U]$. If $d(z) = 0$, then $[d(z), z] = 0$. Hence, as a result of both cases $[d(z), z] = 0$ for all $z \in [U, U]$. Appliying Lemma 6 and Lemma 2, we get the required result. □

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