AN EVALUATION FORMULA FOR A GENERALIZED CONDITIONAL EXPECTATION WITH TRANSLATION THEOREMS OVER PATHS

DONG HYUN CHO

Abstract. Let $C[0,T]$ denote an analogue of Wiener space, the space of real-valued continuous functions on the interval $[0,T]$. For a partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ of $[0,T]$, define $X_n : C[0,T] \to \mathbb{R}^{n+1}$ by $X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. In this paper we derive a simple evaluation formula for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ with the conditioning function $X_n$ which has a drift and does not contain the present position of paths. As applications of the formula with $X_n$, we evaluate the Radon-Nikodym derivatives of the functions $\int_0^T [x(t)]^m d\lambda(t)$ ($m \in \mathbb{N}$) and $[\int_0^T x(t) d\lambda(t)]^2$ on $C[0,T]$, where $\lambda$ is a complex-valued Borel measure on $[0,T]$. Finally we derive two translation theorems for the Radon-Nikodym derivatives of the functions on $C[0,T]$.

1. Introduction and an analogue of the Wiener space

Let $C_0[0,T]$ denote the Wiener space, the space of real-valued continuous functions $x$ on the interval $[0,T]$ with $x(0) = 0$. Calculations involving the conditional Wiener integrals of the functions on $C_0[0,T]$ are important in the study of Feynman integral. In particular, when $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ is a partition of $[0,T]$ and $\xi_j \in \mathbb{R}$ for $j = 0,1,\ldots,n$, the conditional Wiener integral of a time integral in which the paths pass through the point $\xi_j$ at each time $t_j$ for $j = 0,1,\ldots,n$, where $t_j$ is not the present time $T$, is very useful in the Feynman integration theory. But, in general, the Wiener integral and the conditional Wiener integral on $C_0[0,T]$ is not invariant under translation [1,11]. In [7], Park and Skoug derived a simple formula for conditional Wiener integrals containing the time integral with the conditioning function $(x(t_1),\ldots,x(t_n),x(t_{n+1}))$ for $x \in C_0[0,T]$ which contains the present positions of the paths in $C_0[0,T]$. In their simple formula, they expressed the
conditional Wiener integral directly in terms of ordinary Wiener integral and
derived a translation theorem for the conditional Wiener integrals on $C_0[0,T]$.

More generally, let $C[0,T]$ denote the space of continuous real-valued functions
on $[0,T]$. Im and Ryu [6, 9] introduced a finite positive measure $w_\varphi$ on
$C[0,T]$ which generalizes the Wiener space $C_0[0,T]$. When $w_\varphi$ is a probability
measure, the author [2] and Ryu [8] derived separately the same simple formula
for a generalized conditional Wiener integral of the functions on $C[0,T]$ with
the conditioning function $X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1}))$
for $x \in C[0,T]$. They then evaluated the generalized conditional Wiener integrals
of various functions which are interested in both Feynman integral and quantum
mechanics. In particular, Ryu [10] derived a translation theorem for the

In this paper, using the formula with $X_{n+1}$ in [5], we derive another simple evaluation formula for Radon-Nikodym derivatives similar to the
generalized conditional Wiener integrals of functions on $C[0,T]$ with the conditioning function $X_{n+1}$ which has a drift and an initial weight. Using the
formula, he evaluated various Radon-Nikodym derivatives of the functions on $C[0,T]$ containing the time integral.

We now introduce a finite measure over paths with its properties. Let $\alpha, \beta : [0,T] \to \mathbb{R}$ be two functions, where $\beta$ is continuous and strictly increasing.
Let $\varphi$ be a positive finite measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ and $m_L$ be
the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. For $\beta = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \leq T$, let $J_{\beta} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by $J_{\beta}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. For $\prod_{j=0}^n B_j \in \mathcal{B}(\mathbb{R}^{n+1})$, the subset $J_{\beta}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called a cylinder set $I$ and let $\mathcal{I}$ be the set of all such cylinder sets $I$. 
Define a pre-measure \( m_{\alpha,\beta;\varphi} \) on \( \mathcal{I} \) by
\[
m_{\alpha,\beta;\varphi}(I) = \int_{B_0} \int_{\prod_{j=1}^{n} B_j} W_n(\vec{t}_n, \vec{u}_n, u_0)dm_n^\varphi(\vec{u}_n)d\varphi(u_0),
\]
where for \( \vec{u}_n = (u_1, \ldots, u_n) \in \mathbb{R}^n \) and \( u_0 \in \mathbb{R} \),
\[
W_n(\vec{t}_n, \vec{u}_n, u_0) = \left[ \prod_{j=1}^{n} \frac{1}{2\pi} \frac{1}{\beta(t_j) - \beta(t_{j-1})} \right]^{\frac{1}{2}} \times \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{\left[ u_j - \alpha(t_j) - u_{j-1} + \alpha(t_{j-1}) \right]^2}{\beta(t_j) - \beta(t_{j-1})} \right\}.
\]

Let \( \mathcal{B}(C[0,T]) \) denote the Borel \( \sigma \)-algebra of \( C[0,T] \) with the supremum norm. Then \( \mathcal{B}(C[0,T]) \) coincides with the smallest \( \sigma \)-algebra generated by \( \mathcal{I} \) and there exists a unique positive finite measure \( w_{\alpha,\beta;\varphi} \) on \( \mathcal{B}(C[0,T]) \) with \( w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I) \) for all \( I \in \mathcal{I} \). This measure \( w_{\alpha,\beta;\varphi} \) is called an analogue of a generalized Wiener measure on \( (C[0,T], \mathcal{B}(C[0,T])) \) according to \( \varphi \) [9,10]. From the definition of \( w_{\alpha,\beta;\varphi} \), we have the following theorem which is useful in the next sections [6].

**Theorem 1.1.** If \( f : \mathbb{R}^{n+1} \rightarrow \mathbb{C} \) is a Borel measurable function, then
\[
\int_{C[0,T]} f(x(t_0), x(t_1), \ldots, x(t_n))d\varphi(x) = \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \ldots, u_n)W_n(\vec{t}_n, \vec{u}_n, u_0)dm_n^\varphi(\vec{u}_n)d\varphi(u_0),
\]
where \( \int \) means that if either side exists, then both sides exist and they are equal.

Let \( m \) be a positive integer, let \( X : C[0,T] \rightarrow \mathbb{R}^m \) be Borel measurable, let \( F : C[0,T] \rightarrow \mathbb{C} \) be integrable and let \( m_X \) be the measure on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^m) \) of \( \mathbb{R}^m \) induced by \( X \). Let \( \mathcal{D} = \{ X^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^m) \} \) and let \( w_{\varphi}(E) = w_{\alpha,\beta;\varphi}(E) \) for each \( E \in \mathcal{D} \). In view of the Radon-Nikodym theorem, there exists a \( \mathcal{D} \)-measurable and \( w_{\varphi} \)-integrable function \( \Psi \) on \( C[0,T] \) which is unique up to \( w_{\varphi} \) a.e. such that the relation \( \int_E \Psi(x)d\varphi(x) = \int_E F(x)d\varphi(x) \) holds for every \( E \in \mathcal{D} \). Moreover, there exists a Borel measurable and \( m_X \)-integrable function \( \varphi \) on \( \mathbb{R}^m \) which is unique up to \( m_X \) a.e. such that \( \Psi(x) = (\varphi \circ X)(x) \) for \( w_{\varphi} \) a.e. \( x \) in \( C[0,T] \). \( \varphi \) is called a generalized conditional expectation of \( F \) given \( X \) and it is denoted by \( GE[F|X] \). We note that if \( \varphi \) is a probability measure on \( \mathbb{R} \), then \( m_X \) is also a probability measure on \( \mathbb{R}^m \), so that \( GE[F|X] \) is in fact the conditional expectation of \( F \) given \( X \).

2. A simple formula for the generalized conditional expectation

In this section, we derive a simple evaluation formula for the generalized conditional expectations of functions on \( C[0,T] \) with an appropriate conditioning function.
Throughout the reminder of this paper, we assume that $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ is an arbitrary fixed partition of $[0, T]$ unless otherwise specified. To derive the desired simple evaluation formula for a generalized conditional expectation, we begin with letting for $t \in [0, T]$ \begin{equation}
abla \gamma_j(t) = \frac{\beta(t_j) - \beta(t)}{\beta(t_j) - \beta(t_{j-1})} \text{ and } \gamma_{2j}(t) = \frac{\beta(t) - \beta(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})}.
\end{equation}

For a function $f : [0, T] \to \mathbb{R}$, define the polygonal function $P_{\beta}^{n+1}(f)$ of $f$ by \begin{equation}
abla P_{\beta}^{n+1}(f)(t) = \sum_{j=1}^{n+1} \chi(t_{j-1}, t_j)[f(t_{j-1}) + \gamma_{2j}(t)[f(t_j) - f(t_{j-1})]] + \chi(0)(t)f(0)
\end{equation}
for $t \in [0, T]$, where $\chi$ denotes the characteristic function. Similarly, for $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$, the polygonal function $P_{\beta}^{n+1}(\vec{\eta}_{n+1})$ of $\vec{\eta}_{n+1}$ on $[0, T]$ is defined by (3) with replacing $f(t_j)$ by $\eta_j$. Then both $P_{\beta}^{n+1}(f)$ and $P_{\beta}^{n+1}(\vec{\eta}_{n+1})$ belong to $C[0, T]$, and $P_{\beta}^{n+1}(f)(t_j) = f(t_j)$, $P_{\beta}^{n+1}(\vec{\eta}_{n+1})(t_j) = \eta_j$ at each $t_j$. For $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, the polygonal function $P_{\beta}^{n}(\vec{\eta}_n)$ of $\vec{\eta}_n$ on $[0, t_n]$ is defined by (3) with replacing $f(t_j)$ by $\eta_j$ for $j = 0, 1, \ldots, n$.

For $s_1, s_2 \in [0, T]$, let \begin{equation}
abla \Gamma_j(s_1, s_2) = \gamma_{1j}(s_1)\gamma_{2j}(s_2)[\beta(t_j) - \beta(t_{j-1})].
\end{equation}

For $t \in [0, T]$, let \begin{equation}
abla \Gamma(t) = \sum_{j=1}^{n+1} \chi(t_{j-1}, t_j)[t_j][t_j] = \sum_{j=1}^{n+1} \chi(t_{j-1}, t_j)[t_j][t_j]
\end{equation}
and let $Z_t(f) = f(t) - P_{\beta}^{n+1}(f)(t)$ for a function $f : [0, T] \to \mathbb{R}$.

We now have the following two theorems from [5].

**Theorem 2.1.** Suppose that $Z_t$ is defined on $C[0, T]$ and let $X_{n+1} : C[0, T] \to \mathbb{R}^{n+2}$ be given by \begin{equation}
abla X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1})).
\end{equation}
Then the process $\{Z_t : 0 \leq t \leq T\}$ and $X_{n+1}$ are independent if $\varphi(\mathbb{R}) = 1$.

**Theorem 2.2.** Let $F : C[0, T] \to \mathbb{C}$ be integrable and $X_{n+1}$ be given by (6) of Theorem 2.1. Then we have for $m_{X_{n+1}}$, a.e. $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$,
\begin{equation}
abla GE[F|X_{n+1}]|_{\vec{\eta}_{n+1}} = \frac{1}{\varphi(\mathbb{R})} \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\vec{\eta}_{n+1}))dw_{n, \beta; \varphi}(x),
\end{equation}
where $m_{X_{n+1}}$ denotes the measure on $B(\mathbb{R}^{n+2})$ induced by $X_{n+1}$.

**Theorem 2.3.** Let $X_n : C[0, T] \to \mathbb{R}^{n+1}$ be given by \begin{equation}
abla X_n(x) = (x(t_0), x(t_1), \ldots, x(t_n))
\end{equation}
and let $F : C[0, T] \to \mathbb{C}$ be integrable. Let $m_{X_n}$ be the measure induced by $X_n$ on $B(\mathbb{R}^{n+1})$. Then we have for $m_{X_n}$ a.e. $\tilde{\eta}_n = (\eta_n, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

\begin{equation}
(7) \quad \text{GE}[F | X_n](\tilde{\eta}_n) = \frac{1}{\varphi(\mathbb{R})} \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1}) \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1})) dm_L(\eta_{n+1}),
\end{equation}

where $\tilde{\eta}_{n+1} = (\eta_n, \eta_1, \ldots, \eta_n, \eta_{n+1})$ and

\[ W_T(\eta_n, \eta_{n+1}) = \left[ \frac{1}{2\pi[\beta(T) - \beta(t_n)]} \right] \exp \left\{ -\frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2[\beta(T) - \beta(t_n)]} \right\}. \]

\textbf{Proof.} Let $m_{X_{n+1}}$ be the measure as in Theorem 2.2. Then for any Borel subset $B$ of $\mathbb{R}^{n+1}$, we have $X_{n-1}(B) = X_{n-1}(B \times \mathbb{R})$ so that we have by Theorem 2.2

\[ \int_{X_{n-1}(B)} F(x) dm_{\alpha, \beta}(x) = \int_{B \times \mathbb{R}} \text{GE}[F | X_n](\tilde{\eta}_{n+1}) dm_{X_n}(\tilde{\eta}_{n+1}) \]

\[ = \frac{1}{\varphi(\mathbb{R})} \int_{B \times \mathbb{R}} \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1})) dm_L(\eta_{n+1}) d\varphi(\eta_{n+1}) \]

By Theorem 1.1 and the Fubini’s theorem, we have for $\tilde{t}_n = (t_0, t_1, \ldots, t_n)$

\[ \int_{X_{n-1}(B)} F(x) dm_{\alpha, \beta}(x) \]

\[ = \frac{1}{\varphi(\mathbb{R})} \int_{\mathbb{R}^{n+2}} \chi_{B \times \mathbb{R}}(\tilde{\eta}_{n+1}) W_{n+1}(\tilde{t}_n, \tilde{t}_{n+1}), (\eta_1, \ldots, \eta_n, \eta_0) \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1})) dm_{\alpha, \beta, \varphi}(x) d\varphi(\eta_0) \]

\[ = \frac{1}{\varphi(\mathbb{R})} \int_{\mathbb{R}^n} \chi_{B}(\tilde{\eta}_n) W_n(\tilde{t}_n, \eta_0) \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1}) \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1})) dm_{\alpha, \beta, \varphi}(x) d\varphi(\eta_0) \]

\[ = \frac{1}{\varphi(\mathbb{R})} \int_{\mathbb{R}^n} \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1}) \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1})) dm_{\alpha, \beta, \varphi}(x) d\varphi(\eta_0) \]

where $W_{n+1}$ is given by (1) with replacing $n$ by $n+1$. Now, (7) follows by the definition of $\text{GE}[F | X_n]$. \hfill \Box

\textbf{Remark 2.4.} (a) Let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})} \varphi$. Let $P_{X_n}$ be the probability distribution of $X_n$ on $\mathbb{R}^{n+1}$ and let $\text{GE}_{\varphi_0}[F | X_n]$ denote the conditional expectation of $F$ with respect to $w_{\alpha, \beta, \varphi_0}$. Since $B$ is a $P_{X_n}$ null-set if and only if $B$ is an $m_{X_n}$ null-set, (7) can be rewritten by

\[ \text{GE}[F | X_n](\tilde{\eta}_n) = \int_{\mathbb{R}} W_T(\eta_n, \eta_{n+1}) \int_{C[0, T]} F(x - P_{\beta}^{n+1}(x) \]


Theorem 3.1. Let \( s_1 \in [t_{j-1}, t_j] \) and \( s_2 \in [t_{k-1}, t_k] \) with \( 1 \leq j \leq n + 1 \) and \( 1 \leq k \leq n + 1 \). For \( x \in C[0,T] \), let \( G(x) = x(s_1)x(s_2) \). Suppose that \( \int_0^1 u^2d\varphi(u) < \infty \). Then \( G \) is \( w_{\alpha,\beta,\varphi} \)-integrable and we have the followings:

(a) If \( j \neq k \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_{n+1} \in \mathbb{R}^{n+2} \), we have

\[
GE[G|X_{n+1}]|\vec{\eta}_{n+1} = [Z_{s_1}(\alpha) + P_{\beta}^{n+1}(\vec{\eta}_{n+1})(s_1)]|Z_{s_2}(\alpha) + P_{\beta}^{n+1}(\vec{\eta}_{n+1})(s_2)].
\]

(b) If \( j = k \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_{n+1} \in \mathbb{R}^{n+2} \), we have

\[
GE[G|X_{n+1}]|\vec{\eta}_{n+1} = [Z_{s_1}(\alpha) + P_{\beta}^{n+1}(\vec{\eta}_{n+1})(s_1)]|Z_{s_2}(\alpha) + P_{\beta}^{n+1}(\vec{\eta}_{n+1})(s_2)] + \Gamma_j(s_1 \vee s_2, s_1 \land s_2),
\]

where \( s_1 \vee s_2 = \max\{s_1, s_2\} \), \( s_1 \land s_2 = \min\{s_1, s_2\} \) and \( \Gamma_j \) is given by (4), so that \( \text{Cov}(Z_{s_1}, Z_{s_2}) = \Gamma_j(s_1 \vee s_2, s_1 \land s_2) \) if \( \varphi(\mathbb{R}) = 1 \).

Theorem 3.2. Under the assumptions as in Theorem 3.1, we have the followings:

(a) If \( 1 \leq j \leq n \) and \( 1 \leq k \leq n \) with \( j \neq k \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_n \in \mathbb{R}^{n+1} \)

\[
GE[G|X_n]|\vec{\eta}_n = [Z_{s_1}(\alpha) + P_{\beta}^{n}(\vec{\eta}_n)(s_1)]|Z_{s_2}(\alpha) + P_{\beta}^{n}(\vec{\eta}_n)(s_2)].
\]

(b) If \( 1 \leq j \leq n \) and \( 1 \leq k \leq n \) with \( j = k \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_n \in \mathbb{R}^{n+1} \)

\[
GE[G|X_n]|\vec{\eta}_n = [Z_{s_1}(\alpha) + P_{\beta}^{n}(\vec{\eta}_n)(s_1)]|Z_{s_2}(\alpha) + P_{\beta}^{n}(\vec{\eta}_n)(s_2)] + \Gamma_j(s_1 \vee s_2, s_1 \land s_2).
\]

(c) If \( 1 \leq j \leq n \) and \( k = n + 1 \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1} \)

\[
GE[G|X_n]|\vec{\eta}_n = [Z_{s_1}(\alpha) + P_{\beta}^{n}(\vec{\eta}_n)(s_1)]|\alpha(s_2) - \alpha(t_n) + \eta_n].
\]

(d) If \( j = n + 1 \) and \( k = n + 1 \), then for \( m_{X_{n+1}} \) a.e. \( \vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1} \)

\[
GE[G|X_n]|\vec{\eta}_n = [\alpha(s_1) - \alpha(t_n) + \eta_n]|\alpha(s_2) - \alpha(t_n) + \eta_n]
\]

\[
+ 2\gamma_{2(n+1)}(s_1)|\beta(s_2) - \beta(t_n)| + \Gamma_j(s_1 \vee s_2, s_1 \land s_2),
\]

where \( \gamma_{2(n+1)} \) is given by (2) with \( j = n + 1 \).
Proof. If $s_1 \in [t_{j-1}, t_j]$ and $s_2 \in [t_{k-1}, t_k]$ with $1 \leq j \leq n$ and $1 \leq k \leq n$, then we have $P_{(n+1)}^{\alpha}(\bar{\eta}_{n+1})(s_1) = P_{(n+1)}^{\alpha}(\bar{\eta}_n)(s_1)$ for $l = 1, 2$, where $\bar{\eta}_{n+1} = (\bar{\eta}_n, \eta_{n+1})$, so that we have (a) and (b) in this theorem by (7), (a) and (b) of Theorem 3.1. If $s_2 \in [t_n, T]$, then we have

$$\int_{\mathbb{R}} P_{(n+1)}^{\alpha}(\bar{\eta}_{n+1})(s_2)W_T(\eta_n, \eta_{n+1})dm_L(\eta_{n+1})$$

$$= \left[ \frac{1}{2\pi[(\beta(T) - \beta(t_n))]} \right]^2 \int_{\mathbb{R}} \left[ \eta_n + \gamma_2(\eta_{n+1})(s_2)(\eta_{n+1} - \eta_n) \right]$$

$$\times \exp \left\{ - \frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2(\beta(T) - \beta(t_n))} \right\} dm_L(\eta_{n+1})$$

$$= \eta_n + \gamma_2(\eta_{n+1})(s_2)(\alpha(T) - \alpha(t_n)) = \eta_n - \alpha(t_n) + P_{(n+1)}^{\alpha}(\alpha)(s_2).$$

Since $Z_{x_2}(\alpha) + \eta_n - \alpha(t_n) + P_{(n+1)}^{\alpha}(\alpha)(s_2) = \alpha(s_2) - \alpha(t_n) + \eta_n$, we have (c) by Theorem 2.3 and (a) of Theorem 3.1. If $s_1, s_2 \in [t_n, T]$, then we have

$$\int_{\mathbb{R}} P_{(n+1)}^{\alpha}(\bar{\eta}_{n+1})(s_1)P_{(n+1)}^{\alpha}(\bar{\eta}_{n+1})(s_2)W_T(\eta_n, \eta_{n+1})dm_L(\eta_{n+1})$$

$$= \left[ \frac{1}{2\pi[(\beta(T) - \beta(t_n))]} \right]^2 \int_{\mathbb{R}} \left[ \eta_n + \gamma_2(\eta_{n+1})(s_1)(\eta_{n+1} - \eta_n) \right]$$

$$\times (\eta_{n+1} - \eta_n) \exp \left\{ - \frac{[\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)]^2}{2(\beta(T) - \beta(t_n))} \right\} dm_L(\eta_{n+1})$$

$$= \eta_n + \gamma_2(\eta_{n+1})(s_1)(\alpha(T) - \alpha(t_n))$$

$$\times \gamma_2(\eta_{n+1})(s_2)(\alpha(T) - \alpha(t_n))^2 + \beta(T) - \beta(t_n)$$

$$= \gamma_2(\eta_{n+1})(s_1)(\alpha(T) - \alpha(t_n))$$

$$\times \gamma_2(\eta_{n+1})(s_2)(\beta(s_2) - \beta(t_n))$$

$$+ \gamma_2(\eta_{n+1})(s_1)(\beta(s_2) - \beta(t_n))$$

$$= \eta_n - \alpha(t_n) + P_{(n+1)}^{\alpha}(\alpha)(s_1)$$

$$\times (\beta(s_2) - \beta(t_n))$$

so that we have (d) by Theorem 2.3 and (b) of Theorem 3.1. \qed

**Theorem 3.3.** For $x \in C[0, T]$, let $G_1(x) = [\int_0^T x(t) d\lambda(t)]^2$, where $\lambda$ is a finite complex measure on the Borel class of $[0, T]$. Suppose that

$$\int_0^T |\alpha(t)|^2 d|\lambda(t)| < \infty \quad \text{and} \quad \int_0^T u^2 d\varphi(u) < \infty.$$

Then $G_1$ is $w_{\alpha, \beta, \omega}$-integrable and for $m_{X_{n+1}}$, a.e. $\bar{\eta}_{n+1} \in \mathbb{R}^{n+2}$, we have

$$GE[G_1|X_{n+1}](\bar{\eta}_{n+1}) = \left[ \int_0^T \left[ Z_{\beta}(\alpha) + P_{(n+1)}^{\alpha}(\bar{\eta}_{n+1})(t) \right] d\lambda(t) \right]$$

$$+ \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2),$$
where $\Lambda(s, t) = \sum_{j=1}^{n+1} \chi_{[t_{j-1}, t_j]}(s, t) \Gamma_j(s, t)$ for $(s, t) \in [0, T]^2$. In particular, if the support of $\lambda$ is contained in $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$, then for $m_{X_{n+1}}$ a.e. $\eta_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$, we have $GE[G_1|X_{n+1}](\eta_{n+1}) = \left(\sum_{j=0}^{n+1} \eta_j \lambda(t_j)\right)^2$.

Proof. Using the same method as used in the proof of [5, Theorem 4.3], we can prove the integrability of $G_1$ by (8). Now we evaluate $GE[G_1|X_{n+1}]$. For $m_{X_{n+1}}$ a.e. $\eta_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$, we have by Theorem 2.2

$$GE[G_1|X_{n+1}](\eta_{n+1})$$

$$= \int_{C[0,T]} \left[\int_0^T |Z_t(x) + P_{\beta}^{n+1}(\eta_{n+1})(t)|d\lambda(t)\right]^2 dw_{\alpha,\beta;\varphi_0}(x)$$

$$= \int_{C[0,T]} \left[\eta_0 \lambda}\{t_0}\right] + \sum_{j=1}^{n+1} \int_{(t_{j-1}, t_j]} \left[\int_0^T |Z_t(x) + P_{\beta}^{n+1}(\eta_{n+1})(t)|d\lambda(t)\right]^2 dw_{\alpha,\beta;\varphi_0}(x),$$

where $\varphi_0 = \frac{1}{\varphi(\theta)} \varphi$. Now we have by Theorem 3.1

$$GE[G_1|X_{n+1}](\eta_{n+1})$$

$$= |\eta_0 \lambda}\{t_0\}|^2 + 2\eta_0 \lambda\{t_0\} \sum_{j=1}^{n+1} \int_{(t_{j-1}, t_j]} \left[Z_t(\alpha) + P_{\beta}^{n+1}(\eta_{n+1})(\alpha)|Z_{s_1}(\alpha)\right]$$

$$+ 2 \sum_{1 \leq j < k \leq n+1} \int_{(t_{j-1}, t_j] \times (t_{k-1}, t_k]} \left[Z_{s_1}(\alpha) + P_{\beta}^{n+1}(\eta_{n+1})(s_1)|Z_{s_2}(\alpha)\right] + P_{\beta}^{n+1}(\eta_{n+1})(s_2)d\lambda^2(s_1, s_2) + \sum_{j=1}^{n+1} \int_{(t_{j-1}, t_j]^2} \left[Z_{s_1}(x)\right]$$

$$+ P_{\beta}^{n+1}(\eta_{n+1})(s_1)|Z_{s_2}(x) + P_{\beta}^{n+1}(\eta_{n+1})(s_2)|d\lambda^2(s_1, s_2)$$

$$= \int_0^T \int_0^T \left[Z_t(\alpha) + P_{\beta}^{n+1}(\eta_{n+1})(\alpha)|Z_{s_1}(\alpha) + P_{\beta}^{n+1}(\eta_{n+1})(\alpha)\right]$$

$$d\lambda^2(s_1, s_2) + \sum_{j=1}^{n+1} \int_{(t_{j-1}, t_j]^2} \Gamma_j(s_1 \vee s_2, s_1 \wedge s_2)d\lambda^2(s_1, s_2)$$

$$= \left[\int_0^T |Z_t(\alpha) + P_{\beta}^{n+1}(\eta_{n+1})(t)|d\lambda(t)\right]^2 + \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2)$$

$$d\lambda^2(s_1, s_2),$$

since $\Gamma_j(s_1, s_2) = 0$ if $s_1 = t_j$ or $s_2 = t_{j-1}$ for $l = 1, 2$. \hfill \Box

Remark 3.4. (1) The expressions for the results of [5, Theorem 4.3] and Theorem 3.3 are similar, but Theorem 3.3 is an extension of [5, Theorem 4.3] since the measure $\lambda$ in [5, Theorem 4.3] is a continuous complex measure and it
in Theorem 3.3 is an arbitrary complex measure. In addition, the proof of Theorem 3.3 is more complicated than that of [5, Theorem 4.3].

(2) Let $\varphi(\mathbb{R}) = 1$ and $Z(x) = \int_0^T Z_t(x) d\lambda(t)$ for $x \in C[0,T]$. Under the assumptions as in Theorem 3.3, we have $E[Z] = \int_0^T Z_t(\alpha) d\lambda(t)$ and $\text{Var}(Z) = \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2)$.

**Theorem 3.5.** Under the assumptions as in Theorem 3.3, we have for $m_{X_{n}}$ a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$,

$$GE[G_1|X_n](\vec{\eta}_n) = \left[ \int_0^{t_n} [Z_t(\alpha) + P^{n+1}_\beta(\vec{\eta}_n)(t)] d\lambda(t) + \int_{(t_n,T]} [\alpha(t) - \alpha(t_n) + \eta_n] d\lambda(t) \right]^2$$

$$+ \int_0^{t_n} \int_0^{t_n} \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2) + \int_{t_n}^{T} \int_{t_n}^{T} [\beta(s_1 \wedge s_2) - \beta(t_n)] d\lambda^2(s_1, s_2).$$

In particular, if the support of $\lambda$ is contained in $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$, then for $m_{X_{n}}$ a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, we have $GE[G_1|X_n](\vec{\eta}_n) = \left[ \sum_{j=0}^{n} \eta_j \lambda(\{t_j\}) + [\alpha(T) - \alpha(t_n) + \eta_n] \lambda(\{T\}) \right]^2 + [\beta(T) - \beta(t_n)] \lambda(\{T\})^2$.

**Proof.** By Theorems 2.3 and 3.3, we have for $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$GE[G_1|X_n](\vec{\eta}_n) = \int \mathcal{W}_T(\eta_n, \eta_{n+1}) \left[ \int_0^{T} [Z_t(\alpha) + P^{n+1}_\beta(\vec{\eta}_{n+1})(t)] d\lambda(t) \right]^2 - \lambda_{L}(\eta_{n+1})$$

$$+ \int_0^{T} \int_0^{T} \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2),$$

where $\vec{\eta}_{n+1} = (\vec{\eta}_n, \eta_{n+1})$. Since $\mathcal{W}_T$ is Gaussian, we have by Theorem 3.2

$$\int \mathcal{W}_T(\eta_n, \eta_{n+1}) \left[ \int_0^{T} [Z_t(\alpha) + P^{n+1}_\beta(\vec{\eta}_{n+1})(t)] d\lambda(t) \right]^2 - \lambda_{L}(\eta_{n+1})$$

$$= \int \mathcal{W}_T(\eta_n, \eta_{n+1}) \left[ \int_0^{t_n} [Z_t(\alpha) + P^{n}_\beta(\vec{\eta}_n)(t)] d\lambda(t) + \int_{(t_n,T]} [Z_t(\alpha) + P^{n+1}_\beta(\vec{\eta}_{n+1})(t)] d\lambda(t) \right]^2$$

$$+ \left[ \int_0^{t_n} [Z_t(\alpha) + P^{n}_\beta(\vec{\eta}_n)(t)] d\lambda(t) + \int_{(t_n,T]} [\alpha(t) - \alpha(t_n) + \eta_n] d\lambda(t) \right]^2$$

$$+ [\beta(T) - \beta(t_n)] \left[ \int_{t_n}^{T} \gamma^{(n+1)}_2(t) d\lambda(t) \right]^2.$$
We also have
\[
\begin{align*}
\Gamma (T, t_n) &= \int_{t_n}^{T} \gamma_{n+1}(t) d\lambda(t) + \int_{[t_n, T]^2} \Gamma_{n+1}(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2) \\
&= \frac{1}{\beta(T) - \beta(t_n)} \left[ \int_{\Delta_1} [\beta(s_1) - \beta(t_n)][\beta(s_2) - \beta(t_n) + \beta(T) - \beta(s_2)] d\lambda^2(s_1, s_2) \right] + \int_{[t_n, T]^2} [\beta(s_2) - \beta(t_n)][\beta(s_1) - \beta(t_n) + \beta(T) - \beta(s_1)] d\lambda^2(s_1, s_2)
\end{align*}
\]
where \( \Delta_1 = \{(s_1, s_2) : t_n \leq s_1 \leq s_2 \leq T_n \} \) and \( \Delta_2 = \{(s_1, s_2) : t_n \leq s_2 < s_1 \leq T_n \} \). Now we have

\[
\begin{align*}
GE[\alpha_{\hat{t}_{n+1}}(X_{n+1})] &= \left[ \int_0^{\alpha_{\hat{t}_{n}}} [Z_{\alpha}(\alpha) + P_{\beta}^{\alpha}(\hat{\eta}_n)](t) d\lambda(t) + \int_{[t_n, T]} [\alpha(t) - \alpha(t_n) + \eta_n] d\lambda(t) \right]^2 \\
&+ \int_0^{\alpha_{\hat{t}_{n}}} \int_0^{\alpha_{\hat{t}_{n}}} \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1, s_2) + \left[ \int_{[t_n, T]} [\beta(s_2) - \beta(t_n)] d\lambda^2(s_1, s_2) \right]
\end{align*}
\]

which completes the proof.

The following theorem is useful for the remainder results of this section [5].

**Theorem 3.6.** For \( m \in \mathbb{N} \) and \( t \in [0, T] \), let \( F_t(x) = [x(t)]^m \) for \( x \in C[0, T] \) and suppose that \( \int_0^T |u|^m d\varphi(u) < \infty \). Then \( F_t \) is \( w_{0, B, \varphi} \)-integrable and for \( m_{X_{n+1}} \) a.e. \( \tilde{\eta}_{n+1} \in \mathbb{R}^{n+2}, GE[F_{t_{m_{X_{n+1}}}}](\tilde{\eta}_{n+1}) \) is given by

\[
(9) \quad GE[F_t|X_{n+1}](\tilde{\eta}_{n+1}) = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{m!}{2^k k! (m - 2k)!} [P_{\beta}^{n+1}(\tilde{\eta}_{n+1})](t) + Z_t(\alpha)]^{m-2k} \Gamma(t)^k,
\]

where \( \Gamma(t) \) is given by (5) and \( \left\lfloor \frac{m}{2} \right\rfloor \) denotes the greatest integer less than or equal to \( \frac{m}{2} \). In particular, if \( t = t_j \) for some \( j \in \{0, 1, \ldots, n, n+1 \} \), then we
have $GE[F_i|X_{n+1}](\vec{\eta}_{n+1}) = \eta_0^m$ for $m_{X_{n+1}}$ a.e. $\vec{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+2}$.

**Theorem 3.7.** Under the assumptions as in Theorem 3.6, we have the following:

(a) If $t \in [0, t_n]$, then for $m_{X_n}$ a.e. $\vec{\eta}_n \in \mathbb{R}^{n+1}$,

$$GE[F_i|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{m!}{2^l!(m - 2l)!}[P^m_{\beta}(\vec{\eta}_n)(t) + Z_i(\alpha)]^{m-2k}[\Gamma(t)]^k.$$ 

In particular, if $t = t_j$ for some $j \in \{0, 1, \ldots, n\}$, then $GE[F_i|X_n](\vec{\eta}_n) = \eta_j^m$ for $m_{X_n}$ a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$.

(b) If $t \in (t_n, T)$, then for $m_{X_n}$ a.e. $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$,

$$GE[F_i|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{m!}{2^l!(m - 2l)!}[\eta_n + \alpha(t)]^{m-2k}[(\Gamma(t))^{m-2k}]^{\beta(t) - \beta(t_n)}.$$ 

In particular, if $t = T$, then $GE[F|X_n](\vec{\eta}_n)$ is reduced to

$$GE[F|X_n](\vec{\eta}_n) = \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l!(m - 2l)!}[\eta_n + \alpha(T) - \alpha(t_n)]^{m-2l}[(\Gamma(T))^{m-2l}]^{\beta(T) - \beta(t_n)}.$$ 

**Proof.** (a) follows from (9) of Theorem 3.6 since $W_T$ is a probability density. To prove (b), let $t \in (t_n, T)$. Then we have by the change of variable theorem

$$\int_{\mathbb{R}} \left[\frac{1}{2\pi |\beta(T) - \beta(t_n)|} \right]^\frac{1}{2} \int_{\mathbb{R}} \left[\frac{1}{2\pi |\gamma_2(n+1)(t)\eta_{n+1} - \eta_n + \alpha(t) - \alpha(t_n)|} \right]^\frac{1}{2} \frac{d\eta_n}{\mathbb{R}}$$

$$\times \exp\left\{ -\frac{u^2}{2\gamma_2(n+1)(t)|\beta(t) - \beta(t_n)|} \right\} dm_L(u).$$

Using the same process as used in the proof of [2, Theorem 3.1], we have by Theorem 3.6

$$GE[F_i|X_n](\vec{\eta}_n) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{m!}{2^l!(m - 2l)!}[\eta_n + \alpha(t)]^{m-2k}[(\Gamma(t))^{m-2k}]^{\beta(t) - \beta(t_n)}$$

$$- \alpha(t_n)]^{m-2l}[\gamma_2(n+1)(t)|\beta(t) - \beta(t_n)|]^{\beta(t) - \beta(t_n)}.$$
which proves (b). □

By Theorem 3.7 and [5, Theorem 4.5], we can prove the following theorem.

**Theorem 3.8.** For $m \in \mathbb{N}$, let $F(x) = \int_0^T x(t)^m \, d\lambda(t)$ for $x \in C[0, T]$, where $\lambda$ is a finite complex measure on the Borel class of $[0, T]$, and suppose that $\int_0^T |x(t)|^m \, d\varphi(u) < \infty$ and $\int_0^T |\alpha(t)|^m \, d\lambda(t) < \infty$. Then for $mX_n \ a.e. \ \eta_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, $GE[F|X_n](\eta_n)$ is given by

$$GE[F|X_n](\eta_n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{m!}{2^k k!(m-2k)!} \int_0^T \left[ P_n^0(\eta_n)(t) + Z_l(\alpha)^{m-2k} \Gamma(t) \right]^k \, d\lambda(t)$$

$$+ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor-k} \frac{m!}{2^{k+l} k! l! (m-2k-2l)!} \int_{(t_n, T]} [\eta_n + \alpha(t)]^k \, d\lambda(t),$$

$$- \alpha(t_n)[m-2k](\beta(t) - \beta(t_n))^l.$$}

In particular, if the support of $\lambda$ is contained in $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$, then $GE[F|X_n](\eta_n)$ is reduced to

$$GE[F|X_n](\eta_n) = \sum_{j=0}^{n} \lambda(\{t_j\}) \eta_j^m + \lambda(\{T\}) \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{m!}{2^l l! (m-2l)!} [\eta_n + \alpha(T)$$

$$- \alpha(t_n)[m-2k](\beta(T) - \beta(t_n))^l].$$

Applying the integration by parts formula to the result of Theorem 3.8 repeatedly, we have the following corollary [3].

**Corollary 3.9.** Suppose that $\alpha(t) = P_n^0(\alpha(t))$ for $t \in [0, t_n]$ and $\alpha(t) = \alpha(t_n)$ for $t \in (t_n, T]$. For $m \in \mathbb{N}$, let $F(x) = \int_0^T x(t)^m \, d\beta(t)$ for $x \in C[0, T]$ and suppose that $\int_0^T |x(t)|^m \, d\varphi(u) < \infty$. Then, for $mX_n \ a.e. \ \eta_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$, $GE[F|X_n](\eta_n)$ is given by

$$GE[F|X_n](\eta_n) = \sum_{j=1}^{n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - k} \frac{m!(l+k)!}{2^l l!(m-2l-k)!} [\beta(t_j) - \beta(t_{j-1})]^l \eta_j^{m-2k-2l}$$

$$+ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor-k} \frac{m!(2l+k)!}{2^{k+l} k! l!} \eta_j^{m-2k-2l} [\beta(T) - \beta(t_n)]^l.$$}

**Example 3.10.** For $l = 1, 2, 3$, let $F(x) = \int_0^T x(t)^l \, d\beta(t)$ for $x \in C[0, T]$. By [5, Example 4.6], we have the followings:
(a) If $\int_{\mathbb{R}} |u|d\varphi(u) < \infty$ and $\int_{0}^{T} |\alpha(t)|d\beta(t) < \infty$, then for $m_{X_n}$ a.e. $\tilde{\eta}_n = \left(\eta_0, \eta_1, \ldots, \eta_n\right) \in \mathbb{R}^{n+1}$

$$GE[F_1|X_n](\tilde{\eta}_n) = \int_{0}^{T} \alpha(t)d\beta(t) + \frac{1}{2} \sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\eta_j - \alpha(t_j) + \eta_{j-1} - \alpha(t_{j-1})] + [\beta(T) - \beta(t_n)][\eta_n - \alpha(t_n)].$$

(b) If $\int_{\mathbb{R}} u^2d\varphi(u) < \infty$ and $\int_{0}^{T} |\alpha(t)|^2d\beta(t) < \infty$, then for $m_{X_n}$ a.e. $\tilde{\eta}_n = \left(\eta_0, \eta_1, \ldots, \eta_n\right) \in \mathbb{R}^{n+1}$

$$GE[F_2|X_n](\tilde{\eta}_n)$$

$$= \int_{0}^{T} \alpha(t) \left[\int_{\mathbb{R}} [\alpha(t) + 2P_{\beta}^{n+1}(\tilde{\eta}_{n+1} - \alpha)(t)]W_T(\eta_n, \eta_{n+1})dm_{L}(\eta_{n+1})d\beta(t)\right]$$

$$+ \frac{1}{6} \sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1})] + 2[\eta_j - \alpha(t_j)]^2 + [\eta_j - \alpha(t_j)]$$

$$\times [\eta_j - \alpha(t_j - 1)] + [\eta_j - \alpha(t_j - 1)]^2] + \frac{1}{2} [\beta(T) - \beta(t_n)][\beta(T) - \beta(t_n)]$$

$$+ 2[\eta_n - \alpha(t_n)]^2]$$

$$= \int_{0}^{T} \alpha(t) \left[\int_{\mathbb{R}} [\alpha(t) + 2P_{\beta}^{n+1}(\tilde{\eta}_{n+1} - \alpha)(t)]W_T(\eta_n, \eta_{n+1})dm_{L}(\eta_{n+1})d\beta(t)\right]$$

$$+ \frac{1}{6} \sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1})] + 2[\eta_j - \alpha(t_j)]^2 + [\eta_j - \alpha(t_j)]$$

$$- \alpha(t_j)[\eta_j - \alpha(t_j - 1)] + [\eta_j - \alpha(t_j - 1)]^2] + \frac{1}{2} [\beta(T) - \beta(t_n)][\beta(T) - \beta(t_n)]$$

$$- \beta(t_n) + 2[\eta_n - \alpha(t_n)]^2]$$

Since $\int_{\mathbb{R}} P_{\beta}^{n+1}(\tilde{\eta}_{n+1} - \alpha)(t)W_T(\eta_n, \eta_{n+1})dm_{L}(\eta_{n+1}) = \eta_n - \alpha(t_n)$, where $\tilde{\eta}_{n+1} = (\tilde{\eta}_n, \eta_{n+1})$.

(c) If $\int_{\mathbb{R}} u^3d\varphi(u) < \infty$ and $\int_{0}^{T} |\alpha(t)|^3d\beta(t) < \infty$, then for $m_{X_n}$ a.e. $\tilde{\eta}_n = \left(\eta_0, \eta_1, \ldots, \eta_n\right) \in \mathbb{R}^{n+1}$

$$GE[F_3|X_n](\tilde{\eta}_n)$$

$$= \int_{0}^{T} \int_{\mathbb{R}} [\alpha(t)[\alpha(t)]^2 + 3[\Gamma(t) + P_{\beta}^{n+1}(\tilde{\eta}_{n+1} - \alpha)(t)]P_{\beta}^{n+1}(\tilde{\eta}_{n+1} - \alpha)(t)$$

$$+ \alpha(t)][W_T(\eta_n, \eta_{n+1})dm_{L}(\eta_{n+1})d\beta(t) + \frac{1}{4} \sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1})] + 2[\eta_j - \alpha(t_j)]^2 + [\eta_j - \alpha(t_j)]^2]$$

$$\times [\eta_j - \alpha(t_j - 1)] + [\eta_j - \alpha(t_j - 1)]^2 + [\eta_j - \alpha(t_j)]^2]$$

$$\times [\eta_j - \alpha(t_j - 1)] + [\eta_j - \alpha(t_j - 1)]^2 + [\eta_j - \alpha(t_j)]^2]$$
For $t \in (t_n, T)$, we have by the change of variable theorem

$$
\int_{\mathbb{R}} \left[ P_{\beta}^{n+1} (\tilde{\eta}_{n+1} - \alpha)(t) \right]^2 W_T(\eta_n, \eta_{n+1}) \, dm_L(\eta_{n+1})
= \left[ \frac{1}{2\pi[\beta(T) - \beta(t_n)]} \right]^\frac{1}{2} \int_{\mathbb{R}} \left[ \gamma_2(n+1)(t) |\eta_{n+1} - \eta_n - \alpha(T) + \alpha(t_n)| \right] + |\eta_n - \alpha(t_n)|^2 \, dm_L(\eta_{n+1})
= \gamma_2(n+1)(t)[\beta(t) - \beta(t_n)] + |\eta_n - \alpha(t_n)|^2
$$

and $T(t) + \gamma_2(n+1)(t)[\beta(t) - \beta(t_n)] = \beta(t) - \beta(t_n)$, so that

$$GE[F_3|X_n](\tilde{\eta}_n)
= \int_0^{t_n} \alpha(t)[|\alpha(t)|^2 + 3|\Gamma(t) + P_{\beta}(\tilde{\eta}_n - \alpha)(t)[P_{\beta}^{n}(\tilde{\eta}_n - \alpha)(t) + \alpha(t)]]|d\beta(t)
+ \int_{t_n}^T \alpha(t)[|\alpha(t)|^2 + 3[\beta(t) - \beta(t_n) + |\eta_n - \alpha(t_n)|]|\eta_n + \alpha(t) - \alpha(t_n)|] \, d\beta(t)
\left[ \frac{1}{4} \sum_{j=1}^n [\beta(t_j) - \beta(t_{j-1})][|\beta(t_j) - \beta(t_{j-1})|][\beta_j - \beta(t_j)] + |\gamma_j - \alpha(t_j)| + |\gamma_j - \alpha(t_j)|
+ [\gamma_j - \alpha(t_j)]^2 + [\gamma_j - \alpha(t_j)][\gamma_j - \alpha(t_j)] + |\gamma_j - \alpha(t_j)|
\times [\gamma_j - \alpha(t_j)] + [\gamma_j - \alpha(t_j)]^2 + [\gamma_j - \alpha(t_j)][\gamma_j - \alpha(t_j)] + |\gamma_j - \alpha(t_j)|
\times |3[\beta(T) - \beta(t_n)]| |\eta_n - \alpha(t_n)|
\times |3[\beta(T) - \beta(t_n)] + 2|\eta_n - \alpha(t_n)||]
\right].
$$

**Theorem 3.11.** Let $F(x) = \exp\{\int_0^T x(t) \, d\beta(t)\}$ for $x \in C[0, T]$. Suppose that $\lim_{T \to \infty} \alpha(t) = \alpha(T)$. Let $\tau: 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ be any partition of $[0, T]$ and let $X_{\tau}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$ for $x \in C[0, T]$. Then, for $w_{\alpha, \beta, \gamma} \ a.e. \ y \in C[0, T]$, we have

$$\lim_{||\tau|| \to 0} GE[F|X_{\tau}](X_{\tau}(y)) = F(y).$$

**Proof.** For $w_{\alpha, \beta, \gamma} \ a.e. \ y \in C[0, T]$, we have

$$GE[F|X_{\tau}](X_{\tau}(y))
= \exp\left\{ \frac{1}{2} \sum_{j=1}^n [\beta(t_j) - \beta(t_{j-1})][y(t_{j-1}) + y(t_j)] \right\} \int_{C[0, T]} \exp\left\{ \int_0^T Z_t(x) \right\} \, d\beta(t) \, dw_{\alpha, \beta, \gamma}(x) \int_{\mathbb{R}} W_T(y(t_n), \eta_{n+1}) \exp\left\{ \frac{1}{2}[\beta(T) - \beta(t_n)]|y(t_n) + \eta_{n+1}| \right\} \, dm_L(\eta_{n+1})
$$
\[
\exp \left\{ \frac{1}{2} \sum_{j=1}^{n} [\beta(t_j) - \beta(t_{j-1})][y(t_{j-1}) + y(t_j)] + [\beta(T) - \beta(t_n)][2y(t_n) + \alpha(T) - \alpha(t_n)] + \frac{1}{8}[\beta(T) - \beta(t_n)]^3 \right\} \int_{C[0,T]} \exp \left\{ \int_{0}^{T} Z_t(x)d\beta(t) \right\} \]

d\omega_{\alpha,\beta,\varphi_0}(x)

by Theorem 2.3 and the same process used in Example 3.10, where \( \varphi_0 = \frac{1}{\varphi(B)} \varphi \).

Letting \( \|\tau\| \to 0 \), we have

\[
\lim_{\|\tau\| \to 0} GE[F|X_\tau](X_\tau(y)) = F(y)
\]
because \( \lim_{\|\tau\| \to 0} Z_t(x) = 0 \) for \( x \in C[0,T] \) and both \( \alpha \) and \( \beta \) are left-continuous at \( T \).

\section{Translation theorems for the generalized conditional expectation}

In this section we derive translation theorems for the Radon-Nikodym derivatives of the functions on \( C[0,T] \) with the conditioning functions \( X_n \) and \( X_{n+1} \). To do this, we need the following translation theorem.

\textbf{Theorem 4.1.} Let \( h \) be continuous and of bounded variation on \([0,T]\). Suppose that \( \alpha \) is of bounded variation or continuous. Let \( a \in \mathbb{R} \) and define \( x_0 \) by \( x_0(t) = \int_{0}^{t} h(u)d\beta(u) + a \) for \( t \in [0,T] \). Let \( \varphi_a \) be the measure defined by \( \varphi_a(B) = \varphi(B + a) \) for \( B \in \mathcal{B}(\mathbb{R}) \). If \( F : C[0,T] \to \mathbb{C} \) is \( w_{\alpha,\beta,\varphi_a} \)-integrable, then \( F(\cdot + x_0) \) is \( w_{\alpha,\beta,\varphi_a \cdot \varphi}(x) \)-integrable and

\[
\int_{C[0,T]} F(x)dw_{\alpha,\beta,\varphi_a}(x) = J_1(h) \int_{C[0,T]} F(x + x_0)J_2(h,x)dw_{\alpha,\beta,\varphi}(x),
\]

where \( J_1(h) = \exp\{ -\frac{1}{2} \int_{0}^{T} [h(t)]^2d\beta(t) + \int_{0}^{T} h(t)d\alpha(t) \} \) and \( J_2(h,x) = \exp\{ -\int_{0}^{T} h(t)dx(t) \} \) for \( x \in C[0,T] \).

\textbf{Proof.} Suppose that \( F \) is bounded and continuous, and vanishes on the set \( \{ x \in C[0,T] : \|x\|_{\infty} > M \} \) for some real number \( M > 0 \). For nonnegative integer \( n \), let \( s_n+1 = (s_0, s_1, \ldots, s_n, s_{n+1}) = (0, \frac{T}{n+1}, \frac{2T}{n+1}, \ldots, \frac{nT}{n+1}, \frac{(n+1)T}{n+1}) \).

Let \( P_{\beta}^{n+1} \) be the polygonal function given by (3) with replacing \( t_j \) by \( s_j \). Then we have by Theorem 1.1

\[
\int_{C[0,T]} F(P_{\beta}^{n+1}(x))dw_{\alpha,\beta,\varphi}(x)
\]

\[= \int_{\mathbb{R}^{n+2}} F(P_{\beta}^{n+1}(\vec{u}_{n+2}))W_{n+1}(\vec{s}_{n+1}, \vec{u}_{n+1}, u_0)dm_{\beta}^{n+1}(\vec{u}_{n+1})d\varphi(u_0)
\]

\[= \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n+1} \frac{[x_0(s_j) - x_0(s_{j-1})]^2}{\beta(s_j) - \beta(s_{j-1})} \right\} \int_{\mathbb{R}^{n+2}} F(P_{\beta}^{n+1}(\vec{u}_{n+2}))W_{n+1}(\vec{s}_{n+1}, \vec{u}_{n+1}, u_0)
\]
where $\vec{u}_{n+1} = (u_1, \ldots, u_n, u_{n+1})$, $\vec{u}_{n+2} = (u_0, \vec{u}_{n+1})$ and $\vec{x}_1 = (x_0(s_1), x_0(s_2), \ldots, x_0(s_n), x_0(s_{n+1}))$. Let $\vec{x}_0 = (x_0(s_0), x_0(s_1), \ldots, x_0(s_n), x_0(s_{n+1}))$. By the change of variable theorem and Theorem 1.1 again, we have

$$
\int_{C[0,T]} F(P_{\alpha,\beta}^{n+1}(x)) d\varphi(x)
= \exp\left\{-\frac{1}{2} \sum_{j=1}^{n+1} \frac{[x_0(s_j) - x_0(s_{j-1})]^2}{\beta(s_j) - \beta(s_{j-1})}\right\} \int_{\mathbb{R}^{n+2}} F(P_{\alpha,\beta}^{n+1}(\vec{u}_{n+2} + \vec{x}_0)) \times W_{n+1}(\vec{u}_{n+1}, \vec{u}_{n+2}, u_0) \exp\left\{-\sum_{j=1}^{n+1} \frac{x_0(s_j) - x_0(s_{j-1})}{\beta(s_j) - \beta(s_{j-1})} [u_j - u_{j-1}]
- \alpha(s_j) + \alpha(s_{j-1})\right\} dm_{\alpha,\beta}^{n+1}(\vec{u}_{n+1}) d\varphi(u_0)
= \exp\left\{-\frac{1}{2} \sum_{j=1}^{n+1} \frac{[x_0(s_j) - x_0(s_{j-1})]^2}{\beta(s_j) - \beta(s_{j-1})}\right\} \int_{C[0,T]} F(P_{\alpha,\beta}^{n+1}(x + x_0)) \exp\left\{-\sum_{j=1}^{n+1} \frac{x_0(s_j) - x_0(s_{j-1})}{\beta(s_j) - \beta(s_{j-1})} [x(s_j) - x(s_{j-1}) - \alpha(s_j) + \alpha(s_{j-1})]\right\} d\varphi(x).
$$

Since $h$ is continuous on $[0, T]$, we have for $j = 1, \ldots, n, n + 1$

$$
x_0(s_j) - x_0(s_{j-1}) = \int_{s_{j-1}}^{s_j} h(t) d\beta(t) = h(\xi_j) [\beta(s_j) - \beta(s_{j-1})]
$$

for some $\xi_j \in [s_{j-1}, s_j]$ by the mean value theorem for integral. Now, we have

$$
\int_{C[0,T]} F(P_{\alpha,\beta}^{n+1}(x)) d\varphi(x)
= \exp\left\{-\frac{1}{2} \sum_{j=1}^{n+1} [h(\xi_j)]^2 [\beta(s_j) - \beta(s_{j-1})]\right\} \int_{C[0,T]} F(P_{\alpha,\beta}^{n+1}(x + x_0)) \times \exp\left\{-\sum_{j=1}^{n+1} h(\xi_j) [x(s_j) - x(s_{j-1}) - \alpha(s_j) + \alpha(s_{j-1})]\right\} d\varphi(x).
$$

Note that $(P_{\alpha,\beta}^{n+1})_{n=0}^\infty$ converges uniformly to the identity function on $C[0,T]$. Letting $n \to \infty$, we have (10) by using the same process used in the proof of [6, Theorem 3.1]. For the other cases of $F$, (10) can be proved by using the same process used in the proof of [10, Theorem 2.1]. \qed
Remark 4.2. Let \( \beta \) be continuously differentiable with \( \beta' > 0 \). Let \( h \) be in \( C[0,T] \) and let \( \frac{h}{\beta'} \) be of bounded variation. Replacing \( h \) in Theorem 4.1 by \( \frac{h}{\beta'} \), we can obtain [10, Theorem 2.1]. In Theorem 4.1, the conditions those \( \beta' > 0 \) and \( \beta' \) is continuous are removed so that Theorem 4.1 generalizes [10, Theorem 2.1]. Note that, to prove [10, Theorem 2.1], Ryu used the mean value theorem for differentiation, but the mean value theorem for integral is used in the proof of Theorem 4.1.

We now derive translation theorems for the generalized conditional expectations of functions on \( C[0,T] \) with the conditioning functions \( X_n \) and \( X_{n+1} \).

**Theorem 4.3.** Under the assumptions as in Theorem 4.1, we have for \( m_{X_{n+1}} \) a.e. \( \bar{\eta}_{n+1} \in \mathbb{R}^{n+2} \)

\[
GE[F|X_{n+1}]|\bar{\eta}_{n+1})
= \int_{C[0,T]} F(x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0)) J_1(h, x)\, \text{d}w_{\alpha, \beta; \varphi_a}(x)
\]

where \( \bar{\eta}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n), x_0(t_{n+1})) \), \( GE_{w_{\varphi_a}} \) denotes the generalized conditional expectation with respect to \( w_{\alpha, \beta; \varphi_a} \), and for \( m \in \mathbb{N} \), \( \bar{\eta}_m = (\eta_0, \eta_1, \ldots, \eta_m) \in \mathbb{R}^{m+1} \)

\[
J_3(h, \bar{\eta}_m) = \exp \left\{ \sum_{j=1}^{m} x_0(t_j) - x_0(t_{j-1}) \cdot \left[ \eta_j - \eta_{j-1} - \alpha(t_j) \right.ight.
+ \left. \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})] \right\}.
\]

**Proof.** Let \( \varphi_1 = \frac{1}{\varphi_a(\mathbb{R})} \varphi_a \). By Theorems 2.2 and 4.1, we have for \( m_{X_{n+1}} \) a.e. \( \bar{\eta}_{n+1} \in \mathbb{R}^{n+2} \)

\[
GE[F|X_{n+1}]|\bar{\eta}_{n+1})
= \frac{1}{\varphi(\mathbb{R})} \int_{C[0,T]} F(x - p_{\beta}^{n+1}(x) + \frac{p_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0))} {J_1(h)} \, \text{d}w_{a, \beta; \varphi_a}(x)
\]

\[
= \int_{C[0,T]} F(x_0 + x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0)) J_2(h, x)\, \text{d}w_{a, \beta; \varphi_a}(x)
\]

\[
= \int_{C[0,T]} F(x_0 + x - P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0)) J_2(h, x)
- P_{\beta}^{n+1}(x) + P_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0)) J_2(h, x)\, \text{d}w_{a, \beta; \varphi_a}(x)
\]

since \( \varphi_a(\mathbb{R}) = \varphi(\mathbb{R} + a) = \varphi(\mathbb{R}) \). Since \( \varphi_1 \) is a probability measure, \( x - P_{\beta}^{n+1}(x) \) and \( P_{\beta}^{n+1}(x) \) are independent with respect to \( w_{a, \beta; \varphi_1} \) by Theorem 2.1. Hence we have

\[
GE[F|X_{n+1}]|\bar{\eta}_{n+1}) = J_1(h)GE_{w_{\varphi_a}} [F(x_0 + x) J_2(h, x)|X_{n+1}]|\bar{\eta}_{n+1} - \bar{x}_0) J_2(h, x)
\]
- \( P_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0) \) \( \int_{C[0,T]} J_2(h, P_{\beta}^{n+1}(x)) \, dw_{\alpha,\beta,\varphi_1}(x) \)

since \( P_{\beta}^{n+1}(x_0) = P_{\beta}^{n+1}(\bar{x}_0) \). Note that for \( \bar{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \)

\[
\int_0^T h(t) \, dP_{\beta}^{n+1}(\bar{\eta}_{n+1} - x_0)(t) = \sum_{j=1}^{n+1} \frac{x_0(t_j) - x_0(t_{j-1})}{\beta(t_j) - \beta(t_{j-1})} [\eta_j - \eta_{j-1} - x_0(t_j) + x_0(t_{j-1})] 
\]

and by Theorem 1.1, we have

\[
\int_{C[0,T]} J_2(h, P_{\beta}^{n+1}(x)) \, dw_{\alpha,\beta,\varphi_1}(x) 
\]

\[
= \int_{C[0,T]} \exp \left\{ \sum_{j=1}^{n+1} \frac{1}{\beta(t_j) - \beta(t_{j-1})} \left( x_0(t_j) - x_0(t_{j-1}) \right) \right\} \, dw_{\alpha,\beta,\varphi_1}(x) 
\]

\[
= \exp \left\{ \sum_{j=1}^{n+1} \frac{\left( x_0(t_j) - x_0(t_{j-1}) \right)^2}{2(\beta(t_j) - \beta(t_{j-1}))} \right\} 
\]

Now, we have the desired result by a simple calculation.

**Theorem 4.4.** Under the assumptions as in Theorem 4.1, we have for \( m_{X_n} \) a.e. \( \bar{\eta}_n \in \mathbb{R}^{n+1} \)

\[ GE[F|X_n](\bar{\eta}_n) = J_1(h)J_2(h, \bar{\eta}_n)GE_{w_n}[F(\cdot + x_0)J_2(h, \cdot)|X_n](\bar{\eta}_n - \bar{x}_0) \]

where \( \bar{x}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n)) \) and \( J_3 \) is as given in Theorem 4.3.

**Proof.** Let \( \bar{\gamma}_0 = (x_0(t_0), x_0(t_1), \ldots, x_0(t_n), x_0(t_{n+1})) \). By Theorems 2.3 and 4.3, we have for \( m_{X_n} \) a.e. \( \bar{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1} \)

\[ GE[F|X_n](\bar{\eta}_n) = J_1(h) \int_{\mathbb{R}} J_3(h, \bar{\eta}_{n+1})GE_{w_n}[F(\cdot + x_0)J_2(h, \cdot)|X_{n+1}](\bar{\eta}_{n+1} - \bar{\gamma}_0) W_{\alpha}(\eta_n, \eta_{n+1}) \, dm_{L}(\eta_{n+1}) \]

where \( \bar{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \). By the change of variable theorem with a simple calculation, we have

\[ GE[F|X_n](\bar{\eta}_n) = J_1(h) \int_{\mathbb{R}} J_3(h, \bar{\eta}_n)GE_{w_n}[F(\cdot + x_0)J_2(h, \cdot)|X_{n+1}](\bar{\eta}_n - \bar{x}_0, \eta_{n+1}) \]

\[ - x_0(t_{n+1}) W_{\alpha}(\eta_n - x_0(t_n), \eta_{n+1} - x_0(t_{n+1})) \, dm_{L}(\eta_{n+1}) \]

\[ = J_1(h)J_3(h, \bar{\eta}_n) \int_{\mathbb{R}} GE_{w_n}[F(\cdot + x_0)J_2(h, \cdot)|X_{n+1}](\bar{\eta}_n - \bar{x}_0, \eta_{n+1}) \]

\[ \times W_{\alpha}(\eta_n - x_0(t_n), \eta_{n+1} - x_0(t_{n+1})) \, dm_{L}(\eta_{n+1}). \]

Now, the theorem follows from Theorem 2.3 as desired.
Finally, we have the next example which is useful in the Feynman integration theory.

**Example 4.5.** Let the assumptions be as in Theorem 4.1. Let $A(\lambda, x) = \exp\{\lambda \int_{0}^{T} h(t) dt\} \varphi(x)$ for $\lambda \in \mathbb{R}$ and for $x \in C[0, T]$. Letting $\alpha = 0$ and $F \equiv 1$, we have the followings:

(a) By Theorem 4.1, we have

$$
\int_{C[0, T]} A(\lambda, x) dw_{\alpha, \beta, \varphi}(x) = \varphi(\mathbb{R}) \exp\left\{\frac{\lambda^2}{2} \int_{0}^{T} [h(t)]^2 \beta(t) + \lambda \int_{0}^{T} h(t) dt\right\}
$$

which can also be obtained by applications of [4, Theorem 3.4] and [4, Corollary 3.7].

(b) By Theorem 4.3, we have for $m_{X_{n+1}}$ a.e. $\bar{\eta}_{n+1} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \in \mathbb{R}^{n+1}$

$$
GE[A(1, \cdot)|X_{n+1}](\bar{\eta}_{n+1})
$$

$$
= \exp\left\{\frac{1}{2} \int_{0}^{T} [h(t)]^2 dt + \int_{0}^{T} h(t) dt + \sum_{j=1}^{n+1} x_0(t_j) - x_0(t_{j-1}) \beta(t_j) - \beta(t_{j-1}) \right\}
$$

$$
\times \left[\eta_j - \eta_{j-1} - \alpha(t_j) + \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})]\right]\}
$$

(c) By Theorem 4.4, we have for $m_{X_{n}}$ a.e. $\bar{\eta}_{n} = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+1}$

$$
GE[A(1, \cdot)|X_{n}](\bar{\eta}_{n})
$$

$$
= \exp\left\{\frac{1}{2} \int_{0}^{T} [h(t)]^2 dt + \int_{0}^{T} h(t) dt + \sum_{j=1}^{n} x_0(t_j) - x_0(t_{j-1}) \beta(t_j) - \beta(t_{j-1}) \right\}
$$

$$
\times \left[\eta_j - \eta_{j-1} - \alpha(t_j) + \alpha(t_{j-1}) - \frac{1}{2}[x_0(t_j) - x_0(t_{j-1})]\right]\}
$$

**References**


DONG HYUN CHO
DEPARTMENT OF MATHEMATICS
KYONGGI UNIVERSITY
SUWON 16227, KOREA
Email address: j94385@kyonggi.ac.kr