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ON WEAKLY GRADED POSETS OF ORDER-PRESERVING MAPS UNDER THE NATURAL PARTIAL ORDER

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Abstract. In this paper, we simplify the natural partial ordering $\preccurlyeq$ on the semigroup $\mathcal{O}([n])$ under composition of all order-preserving maps on $[n] = \{1, \ldots, n\}$, and describe its maximal elements. Also, we show that the poset $(\mathcal{O}([n]), \preccurlyeq)$ is weakly graded and determine when $(\mathcal{O}([n]), \preccurlyeq)$ has a structure of $(\mathbf{i}+\mathbf{1})$-avoidance.

1. Introduction

Let X be a nonempty set and $T(X)$ the semigroup under composition of all transformations from X into itself. It is well-known that $T(X)$ is a regular semigroup, i.e., $\forall \alpha \in T(X)\ \exists \beta \in T(X),\ \alpha\beta\alpha = \alpha$. In 1980, the natural partial order on regular semigroups was independently studied by Hartwig [6] and Nambooripad [13]. Using terms of images and inverse images of transformations, in 1986, Kowol and Mitsch [9] described the natural partial order $\preccurlyeq$ on $T(X)$. Later, in 2003, Marques-Smith and Sullivan [12] extended some of previous work to the semigroup $P(X)$, consisting of all partial transformation from a subset of X into X. Additionally, $T(X)$ is a subsemigroup of $P(X)$. Since then, the natural partial order on the semigroup of transformations has been discovered for numerous subsemigroups of $P(X)$ (see [3, 14, 18, 19]) for X as a nonempty set, and also [4, 15] for X as a vector space.

For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. A map $\alpha \in T([n])$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in [n]$. We denote by $\mathcal{O}([n])$ the subsemigroups of $T([n])$ of all order-preserving maps. This type of semigroups has been extensively studied (see [7,8,10]), it remains open for partial ordering. It is well-known that the identity map is the maximum element on $(T([n]), \preccurlyeq)$ but not on $(\mathcal{O}([n]), \preccurlyeq)$. In this paper, we simplify the natural partial ordering $\preccurlyeq$ on $\mathcal{O}([n])$ and also describe the maximal elements.

The notion of $(\mathbf{i}+\mathbf{1})$-avoiding posets has been studied in many areas of combinatorics; for example, see [1, 11, 16]. Especially, the $(\mathbf{3}+\mathbf{1})$-avoiding posets

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play a role in the Stanley-Stembridge conjecture in [17]. For this reason, we direct our attention to study the structure of $(\mathbf{i}+\mathbf{1})$-avoidance in some sub-semigroups of $\mathcal{O}([n])$ with the natural partial order.

2. Preliminaries and notations

For a nonempty set X and $\alpha \in T(X)$, we denote $\alpha\alpha^{-1}$ as a subset of $X \times X$,

$$\alpha\alpha^{-1} = \{(x,y) \in X \times X : \exists z \in X,\ (x,z),(z,y) \in \alpha\}.$$

The following results describe the characterisations of the natural partial order $\preccurlyeq$ on $T(X)$ (see [9,12]).

Theorem 2.1. *For $\alpha, \beta \in T(X)$, we have that $\alpha \preccurlyeq \beta$ on $T(X)$ if and only if* $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ *and $\alpha = \beta\mu$ for some idempotent $\mu \in T(X)$.*

Theorem 2.2. *For $\alpha, \beta \in T(X)$, we have that $\alpha \preccurlyeq \beta$ on $T(X)$ if and only if the following statements hold.*
(1) $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$.
(2) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$.
(3) *For $x \in X$ with $x\beta \in \operatorname{ran}\alpha$, $x\alpha = x\beta$.*

Corollary 2.3. *If $\alpha, \beta \in T(X)$, then $\alpha \preccurlyeq \beta$ on $T(X)$ if and only if* $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ *and $(\alpha \cup \beta)\beta^{-1} \subseteq \alpha\alpha^{-1}$.*

We now recall some notations which will be useful later.

- For any $m, n \in \mathbb{N}$ with $m < n$, let
$$[m \to n] := \{m, m+1, \ldots, n\}.$$
- Given $\alpha = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \end{smallmatrix}\right) \in T([n])$, we may write $\alpha = \langle c_1, c_2, \ldots, c_n\rangle$.

Let $\alpha = \langle c_1, c_2, \ldots, c_n\rangle \in T([n])$. With non-commuting the product operator of $c_1c_2\cdots c_n$, denoted by $c_1^{t_1}c_{t_1+1}^{t_2}c_{t_1+t_2+1}^{t_3}\cdots$ when $c_1 \neq c_{t_1+1}$ and $c_{t_i+1} \neq c_{t_i}$ or $c_{t_i+t_{i+1}+1}$ for $i > 1$, so we denote

$$\pi_\alpha := (t_1, t_2, t_3, \ldots) \tag{2.1}$$

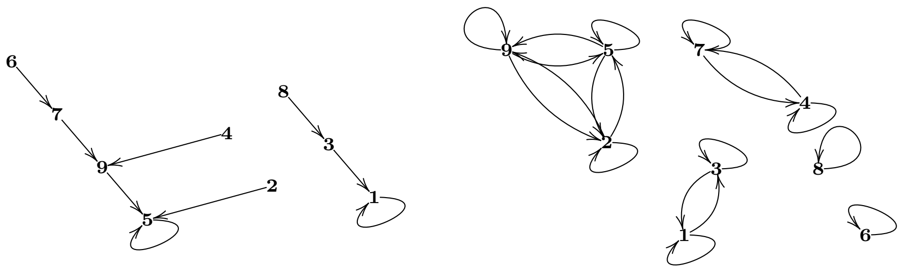
as a sequence of positive integers with respect to α. For example, consider $\alpha = \langle 1,1,1,3,1,1,4,3,4\rangle \in T([9])$. We have $\pi_\alpha = (3,1,2,1,1,1)$.

For $\alpha \in T([n])$, we have another way to view α and $\alpha\alpha^{-1}$ as two sets of *arcs* for two digraphs, namely $\Gamma_\alpha := ([n], \alpha)$ and $\Gamma_{\alpha\alpha^{-1}} := ([n], \alpha\alpha^{-1})$, respectively, where $[n]$ is the set all vertices. For example, let $\alpha = \langle 1,5,1,9,5,7,9,3,5\rangle \in T([9])$. The digraphs Γ_α and $\Gamma_{\alpha\alpha^{-1}}$ are in Fig. 1.

Let E be an $n \times n$ matrix of all ones. For two principal submatrices $E[X_1]$ and $E[X_2]$ of E, we say that $E[X_1]$ is *embedded* in $E[X_2]$ if $X_1 \subseteq X_2 \subseteq [n]$.

Let $\alpha \in T([n])$. If $\operatorname{ran}\alpha = \{a_1, \ldots, a_t\}$, we write $\mathcal{A}_\alpha$ for the symmetric matrix which is defined as follows

$$\mathcal{A}_\alpha := a_1E[a_1\alpha^{-1}] \oplus \cdots \oplus a_tE[a_t\alpha^{-1}], \tag{2.2}$$



(A) The digraph Γ_α

(B) The digraph $\Gamma_{\alpha\alpha^{-1}}$

FIGURE 1. Digraphs for α and $\alpha\alpha^{-1}$.

$\mathcal{A}_\alpha$ is called the *weighted adjacency matrix* of the digraph $\Gamma_{\alpha\alpha^{-1}}$.

Given distinct elements $y_1, \ldots, y_t$ of $[n]$ and a partition $\mathcal{Y}_1, \ldots, \mathcal{Y}_t$ of $[n]$, we let $\mathcal{A} = y_1 E[\mathcal{Y}_1] \oplus \cdots \oplus y_t E[\mathcal{Y}_t]$ be a weighted adjacency matrix on $[n]$ with respect to $\zeta_{\mathcal{A}} = \left(\begin{smallmatrix} \mathcal{Y}_1 & \mathcal{Y}_2 & \cdots & \mathcal{Y}_t \\ y_1 & y_2 & \cdots & y_t \end{smallmatrix}\right) \in T([n])$.

Example 2.4. Consider $\alpha = \langle 1,5,1,9,5,7,9,3,5 \rangle$, $\beta = \langle 1,1,3,5,5,5,7,9,9 \rangle \in T([9])$. We have

$$\begin{aligned}
\mathcal{A}_\alpha &= E[\{1,3\}] \oplus 3E[\{8\}] \oplus 5E[\{2,5,9\}] \oplus 7E[\{6\}] \oplus 9E[\{4,7\}] \\
&= \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 5 & 0 & 0 & 0 & 5 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 & 9 & 0 & 0 \\
0 & 5 & 0 & 0 & 5 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 5 & 0 & 0 & 5 & 0 & 0 & 0 & 5
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 5 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 5 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 5 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9
\end{pmatrix} \\
&= E[\{1,2\}] \oplus 3E[\{3\}] \oplus 5E[\{4,5,6\}] \oplus 7E[\{7\}] \oplus 9E[\{8,9\}] = \mathcal{A}_\beta.
\end{aligned}$$

Observe that if $\beta \in \mathcal{O}([n])$, then the weighted adjacency matrix with respect to β is a diagonal blocks matrix.

For convenience, given an order-preserving map on $[n]$,

$$\alpha = \langle \underbrace{c_1, \ldots, c_1}_{t_1}, \underbrace{c_2, \ldots, c_2}_{t_2}, \ldots, \underbrace{c_k, \ldots, c_k}_{t_k} \rangle$$

with $\pi_\alpha = (t_1, t_2, \ldots, t_k)$, we write $\operatorname{diag}(c_1^{t_1}, c_2^{t_2}, \ldots, c_k^{t_k})$ for the weighted adjacency matrix with respect to α,

$$\mathcal{A}_\alpha = c_1 E[[1 \to t_1]] \oplus c_2 E[[t_1 + 1 \to t_1 + t_2]] \oplus \cdots \oplus c_k E[[n - t_k + 1 \to n]].$$

3. Partial ordering through the weighted adjacency matrices

For $\alpha \in T([n])$ with $\operatorname{ran}\alpha = \{a_1, \ldots, a_t\}$, we denote a set of $N^\alpha_{a_1}, \ldots, N^\alpha_{a_t}$ forms a partition of $\alpha\alpha^{-1}$ where $N^\alpha_{a_i} = \{(x, y) \mid x, y \in a_i\alpha^{-1}\}$. By viewing $\Gamma_{N^\alpha_{a_i}} := (a_i\alpha^{-1}, N^\alpha_{a_i})$ as a subdigraph of $\Gamma_{\alpha\alpha^{-1}}$, we obtain that $E[a_i\alpha^{-1}]$ is the adjacency matrix for $\Gamma_{N^\alpha_{a_i}}$.

Lemma 3.1. *Let $\alpha, \beta \in T([n])$ with*

$$\beta\beta^{-1} = \bigcup_{v \in \operatorname{ran}\beta} N^\beta_v \quad \text{and} \quad \alpha\alpha^{-1} = \bigcup_{u \in \operatorname{ran}\alpha} N^\alpha_u.$$

Suppose that $\alpha \preccurlyeq \beta$ on $T([n])$. Then the following statements hold.

(1) *If $N^\alpha_u \cap N^\beta_v \neq \emptyset$, then $N^\alpha_u \supseteq N^\beta_v$.*

(2) *There is a surjective map, denoted by $\varphi_{\beta\alpha}$, which sends N^β_v to N^α_u where $N^\beta_v \subseteq N^\alpha_u$. Moreover, for $v \in \operatorname{ran}\beta$, if $v \in \operatorname{ran}\alpha$, then $(N^\beta_v)\varphi_{\beta\alpha} = N^\alpha_v$.*

Proof. Suppose that $N^\alpha_u \cap N^\beta_v \neq \emptyset$. To show that $N^\alpha_u \supseteq N^\beta_v$, we let $(x, y) \in N^\alpha_u \cap N^\beta_v$ and assume on the contrary that there is $(a, b) \in N^\beta_v \backslash N^\alpha_u$. By applying Theorem 2.2(3), it forces that $v \notin \operatorname{ran}\alpha$. Using Theorem 2.2(2), $(a, b) \in N^\alpha_w$ for some $w \in \operatorname{ran}\alpha$, $w \neq u$. Since $(x, y), (a, b) \in N^\beta_v$, it implies that $x, y, a, b \in v\beta^{-1}$. That is, $(y, a) \in \beta\beta^{-1} \subseteq \alpha\alpha^{-1}$, so $(y, a) \in N^\alpha_z$ for some $z \in \operatorname{ran}\alpha$. From $(x, y) \in N^\alpha_u$, $(y, a) \in N^\alpha_z$ and $(a, b) \in N^\alpha_w$, we have $u = y\alpha = z = a\alpha = w$, a contradiction. Hence, (1) is proved.

For (2), let N^α_u be a class of $\alpha\alpha^{-1}$. By Theorem 2.2(1) and (3), we have $u \in \operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and let $x \in u\beta^{-1}$, that is, $x\beta = x\alpha$. Thus $(x, x) \in N^\beta_u \cap N^\alpha_u$. From (1), this finishes the proof. $\square$

As an immediate consequence of the above lemma, we get:

Theorem 3.2. *For $\alpha, \beta \in T([n])$ with*

$$\mathcal{A}_\alpha = a_1E[a_1\alpha^{-1}] \oplus \cdots \oplus a_tE[a_t\alpha^{-1}] \text{ and } \mathcal{A}_\beta = b_1E[b_1\beta^{-1}] \oplus \cdots \oplus b_kE[b_k\beta^{-1}]$$

where $\operatorname{ran}\alpha = \{a_1 < \cdots < a_t\}$ and $\operatorname{ran}\beta = \{b_1 < \cdots < b_k\}$, we have that $\alpha \preccurlyeq \beta$ on $T([n])$ if and only if the following statements hold.

(1) *For $i = 1, \ldots, t$, $a_i = b_i$ and $E[b_i\beta^{-1}]$ is embedded in $E[a_i\alpha^{-1}]$.*

(2) *For $i > t$, there is $j \in \{1, \ldots, t\}$ such that $E[b_i\beta^{-1}]$ and $E[b_j\beta^{-1}]$ can be embedded in $E[a_j\alpha^{-1}]$.*

Proof. From Corollary 2.3, we let $(x, y) \in (\alpha \cup \beta)\beta^{-1}$. Then there is $z \in [n]$ such that $(x, z) \in \alpha \cup \beta$ and $(y, z) \in \beta$. Suppose $z \in \operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, by (1). It follows that (x, y) is an arc in $\Gamma_{N^\beta_z}$. Since $E[z\beta^{-1}]$ is embedded in $E[z\alpha^{-1}]$, by (2), it implies that (x, y) is an arc in $\Gamma_{N^\alpha_z}$. Then $(x, y) \in \alpha\alpha^{-1}$. By the same process when $z \in \operatorname{ran}\beta \backslash \operatorname{ran}\alpha$, we have $(\alpha \cup \beta)\beta^{-1} \subseteq \alpha\alpha^{-1}$. Hence, $\alpha \preccurlyeq \beta$ on $T([n])$.

On the other hand, the proof can be done by applying Lemma 3.1. $\square$

Therefore, we directly obtain the following corollary.

Corollary 3.3. *Let $\mathcal{A} = \bigoplus_{i=1}^{t} y_i E[\mathcal{Y}_i]$ be a wighted adjacency matrix on $[n]$. For any wighted adjacency matrix on $[n]$, namely $\mathcal{B}$, we have that if $\zeta_{\mathcal{A}} \preccurlyeq \zeta_{\mathcal{B}}$ on $T([n])$, then $\mathcal{B}$ can be written in the form $\bigoplus_{i=1}^{t}(b_{i1}E[\mathcal{K}_{i1}] \oplus b_{i2}E[\mathcal{K}_{i2}] \oplus \cdots \oplus b_{ik_i}E[\mathcal{K}_{ik_i}])$ where*

(1) *each $\{\mathcal{K}_{i1}, \mathcal{K}_{i2}, \ldots, \mathcal{K}_{ik_i}\}$ forms a partition of $\mathcal{Y}_i$ and*
(2) *$y_1 = b_{11}, b_{12}, \ldots, b_{1k_1}, \ldots, y_t = b_{t1}, b_{t2}, \ldots, b_{tk_t}$ are all distinct in $[n]$.*

Example 3.4. Recall $\mathcal{A}_\alpha$ as in Example 2.4. We let $B_1 = 5E[\{2,5,8,9\}]$, $B_2 = 5E[\{2\}] \oplus 2E[\{5,9\}]$, $B_3 = 5E[\{2\}] \oplus 2E[\{5\}] \oplus 4E[\{9\}]$, and

$$\begin{aligned}
\mathcal{A}_1 &= E[\{1,3\}] \oplus B_1 \oplus 7E[\{6\}] \oplus 9E[\{4,7\}],\\
\mathcal{A}_2 &= E[\{1,3\}] \oplus 3E[\{8\}] \oplus B_2 \oplus 7E[\{6\}] \oplus 9E[\{4,7\}],\\
\mathcal{A}_3 &= E[\{1,3\}] \oplus 3E[\{8\}] \oplus B_3 \oplus 7E[\{6\}] \oplus 9E[\{4,7\}].
\end{aligned}$$

Then we have $\zeta_{\mathcal{A}_1} \preccurlyeq \alpha \preccurlyeq \zeta_{\mathcal{A}_2} \preccurlyeq \zeta_{\mathcal{A}_3}$.

To illustrate the embedding of any two principle submatrices which still satisfies the order-preserving property, we give a diagram (in Fig. 2) of all lower bounds of the map $\langle 1,1,3,4\rangle$ on $\mathcal{O}([4])$.

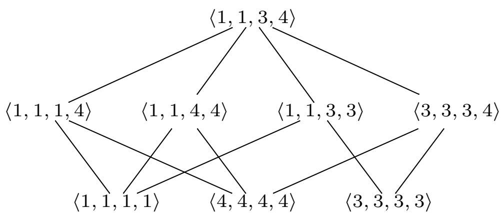


FIGURE 2. All lower bounds of $\langle 1,1,3,4\rangle$ on $\mathcal{O}([4])$.

From Corollary 3.3 and the observation in Fig. 2, the following theorem gives a characterization of the natural partial order on $\mathcal{O}([n])$.

Theorem 3.5. *Let $\alpha, \beta \in \mathcal{O}([n])$ with $\mathcal{A}_\alpha = \mathrm{diag}(a_1^{t_1}, a_2^{t_2}, \ldots, a_k^{t_k})$. Then $\alpha \preccurlyeq \beta$ on $\mathcal{O}([n])$ if and only if $\mathcal{A}_\beta = \mathrm{diag}(\mathcal{B}_1, \ldots, \mathcal{B}_k)$ where for each i,*

(i) $\mathcal{B}_i = \mathrm{diag}({b_{i1}}^{m_{i1}}, {b_{i2}}^{m_{i2}}, \ldots, {b_{it_i}}^{m_{it_i}})$,
(ii) $m_{i1} + m_{i2} + \cdots + m_{it_i} = t_i$,
(iii) $b_{11} < b_{12} < \cdots < b_{1t_1} < \cdots < b_{k1} < b_{k2} < \cdots < b_{kt_k} \in [n]$, *and*
(iv) $a_i \in \{b_{i1}, b_{i2}, \ldots, b_{it_i}\}$.

Remark 3.6. All constant maps are minimal on $(\mathcal{O}([n]), \preccurlyeq)$. The identity map is the maximum element on $(T([n]), \preccurlyeq)$ but not on $(\mathcal{O}([n]), \preccurlyeq)$.

To describe the maximal element of $\mathcal{O}([n])$, we prove the following proposition.

Proposition 3.7. *Let $\alpha \in \mathcal{O}([n])$ with $\mathcal{A}_\alpha = \mathrm{diag}(a_1^{t_1}, a_2^{t_2}, \ldots, a_k^{t_k})$. Then α is not maximal on $\mathcal{O}([n])$ if and only if one of the following statements holds.*

(1) $t_i > 1$ *and* $a_{i+1} - a_{i-1} > 2$ *for* $i \in \{2, \ldots, k-1\}$.
(2) $t_1 > 1$ *and* $a_2 - 1 > 1$.
(3) $t_k > 1$ *and* $n - a_{k-1} > 1$.

Proof. Suppose (1) holds. Let $1 < i < k$ be such that $t_i > 1$ and $a_{i+1} - a_{i-1} > 2$. Then there is $c_i \in [n]$, either $c_i \in [a_i \to a_{i+1}]$ or $c_i \in [a_{i-1} \to a_i]$. Without loss of generality, we assume $a_i < c_i < a_{i+1}$. Define

$$\mathcal{A} = \operatorname{diag}(a_1^{t_1}, a_2^{t_2}, \ldots, a_i^{t'_i}, c_i^{t''_i}, a_{i+1}^{t_{i+1}}, \ldots, a_k^{t_k}),$$

where $t'_i, t''_i > 0$ and $t'_i + t''_i = t_i$. By Theorem 3.5, we have $\alpha \preccurlyeq \zeta_{\mathcal{A}}$ on $\mathcal{O}([n])$. By using a similar technique, α is not maximal on $\mathcal{O}([n])$ if (2) or (3) holds.

Conversely, assume that α is not maximal on $\mathcal{O}([n])$. We directly obtain the result from the conditions of Theorem 3.5. □

Observe that the map $\langle 1, 1, 3, 4 \rangle$ satisfies the condition 2 in Proposition 3.7. Then $\langle 1, 1, 3, 4 \rangle$ is not maximal on $\mathcal{O}([4])$.

Remark 3.8. Let $\mathfrak{S}$ be the set of all finite sequences of positive integers. We define a partial order $\trianglelefteq$ on $\mathfrak{S}$ as follows:

For any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{S}$ with $\mathfrak{a} = (a_1, \ldots, a_m)$ and $\mathfrak{b} = (b_1, b_2, \ldots, b_n)$,

$$\mathfrak{a} \trianglelefteq \mathfrak{b} \quad \Leftrightarrow \quad \begin{array}{l} \exists \text{ a strictly increasing function } \theta : [m] \to [n] \text{ such that} \\ a_1 = b_1 + \cdots + b_{(1)\theta} \quad \text{and} \\ a_i = b_{(i-1)\theta+1} + \cdots + b_{(i)\theta} \ \text{ for all } i \geq 2, \end{array}$$

then we call $\trianglelefteq$ the *block partitions order* on $\mathfrak{S}$.

Using Theorem 3.5 and the poset $(\mathfrak{S}, \trianglelefteq)$, it gives us some necessary conditions when two transformations are being compared.

Corollary 3.9. *If* $\alpha \preccurlyeq \beta$ *on* $\mathcal{O}([n])$, *then* $\operatorname{ran} \alpha \subseteq \operatorname{ran} \beta$ *and* $\pi_\alpha \trianglelefteq \pi_\beta$ *on* $\mathfrak{S}$.

The converse does not hold. For example, $\alpha = \langle 4, 4, 4, 5, 5 \rangle$ and $\beta = \langle 1, 1, 1, 4, 5 \rangle$ are incomparable under $\preccurlyeq$ on $\mathcal{O}([4])$ but $\pi_\alpha = (3, 2) \trianglelefteq (3, 1, 1) = \pi_\beta$ on $\mathfrak{S}$.

In next section, the poset $(\mathfrak{S}, \trianglelefteq)$ is a useful tool to understand the structure of the poset $(\mathcal{O}([n]), \preccurlyeq)$.

4. Graded posets

Given any poset $(P, \leq)$ and for $x, y \in P$, x is said to be *covered* by y if $x < y$ and there is no $z \in P$ such that $x < z < y$ and we let $C_P(y)$ stands for the set of all elements in P which is covered by y.

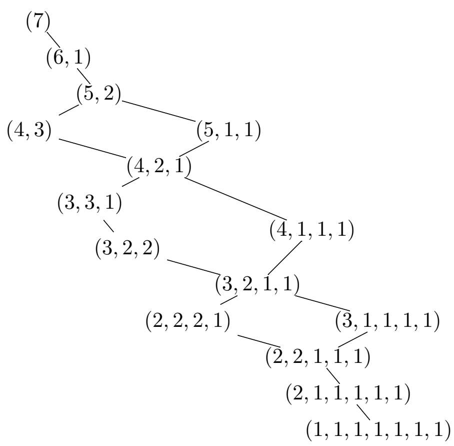
The *length* of a finite chain C, denoted $l(C)$, is $|C| - 1$. The *rank* of a finite poset P is defined to be the maximum length of chains of P. The poset P is *weakly graded* if there exists a *rank function* $\mathrm{rk} : P \to \mathbb{N}_0$ such that (i) $(x)\mathrm{rk} = 0$ if x is minimal in P, and (ii) $(x)\mathrm{rk} = (y)\mathrm{rk} - 1$ if $x \in C_P(y)$. A weakly poset P is called *strongly graded* if every maximal chain of P has the same length.

A subposet Q of a poset P is called a *copy of* $(\mathbf{i}+\mathbf{j})$ if Q is isomorphic to the disjoint union of two chains of length $i-1$ and $j-1$. If P contains no copy of $(\mathbf{i}+\mathbf{j})$, we say that P is $(\mathbf{i}+\mathbf{j})$-*avoiding*.

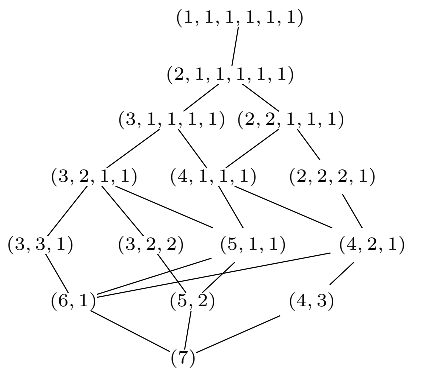
Remark 4.1. For $n \in \mathbb{N}$, the partitions of n, denoted by P_n, is the subset of $\mathfrak{S}$ which contains all nonincreasing sequence. Under the dominance order $\ll$ (or majorization order) on P_n in the sense that

$$a_1 + a_2 + \cdots + a_k \leq b_1 + b_2 + \cdots + b_k \text{ for all } k,$$

we observe that $(2,2,1) \ll (3,1,1)$ on $(P_5, \ll)$ whereas $(2,2,1)$ and $(3,1,1)$ are incomparable on $(\mathfrak{S}, \trianglelefteq)$. It was shown in [2,5], that $(P_n, \ll)$ is not weakly graded when $n \geq 7$.



(A) On $(P_7, \ll)$



(B) On $(P_7, \trianglelefteq)$

FIGURE 3. $(P_7, \ll)$ is not weakly graded and $(P_7, \trianglelefteq)$ is strongly graded.

For $n \in \mathbb{N}$, we denote

$$\mathfrak{P}_n = \{\mathfrak{a} = (a_1, \ldots) \in \mathfrak{S} \, : \, \sum \mathfrak{a} = a_1 + \cdots = n\}.$$

Note that each $\mathfrak{a} \in \mathfrak{P}_n$ can be viewed as an equivalence class of $\alpha \in \mathcal{O}([n])$, given by $\pi_\alpha = \mathfrak{a}$. To study the poset $(\mathcal{O}([n]), \preccurlyeq)$, we first show that $(\mathfrak{P}_n, \trianglelefteq)$ is strongly graded. The following lemma is needed.

Lemma 4.2. *For* $\mathfrak{a} = (a_1, a_2, \ldots, a_m) \in (\mathfrak{S}, \trianglelefteq)$*, suppose that* $\mathfrak{a}$ *is not minimal. Then for any* $\mathfrak{b} \in C_{\mathfrak{S}}(\mathfrak{a})$*, there exists* $j \in \mathbb{N}$ *such that* $a_j, a_{j+1} \neq 0$ *and* $\mathfrak{b} = (a_1, \ldots, a_{j-1}, a_j + a_{j+1}, a_{j+2}, \ldots)$.

Proof. Let $\mathfrak{a} = (a_1, a_2, \ldots, a_m), \mathfrak{b} = (b_1, b_2, \ldots, b_n) \in \mathfrak{S}$. Suppose $\mathfrak{a}$ is not minimal and $\mathfrak{b} \trianglelefteq \mathfrak{a}$. Then there is $\theta : [n] \to [m]$ a strictly increasing function such that $b_1 = a_1 + \cdots + a_{(1)\theta}$ and $b_i = a_{(i-1)\theta+1} + \cdots + a_{(i)\theta}$ for $i > 1$. Assume that $\mathfrak{b} \neq \mathfrak{a}$. We have θ is not the identity function. Suppose there are $1 < j_1 < j_2 \in \mathbb{N}$ such that $(j_1)\theta - (j_1 - 1)\theta \geq 2$ and $(j_2)\theta - (j_2 - 1)\theta \geq 2$. It follows that $b_{j_1} = a_{(j_1-1)\theta+1} + \cdots + a_{(j_1)\theta}$ and $b_{j_2} = a_{(j_2-1)\theta+1} + \cdots + a_{(j_2)\theta}$. Let $\mathfrak{c} = (b_1, \ldots, b_{j_1-1}, b_{j_1} - a_{(j_1)\theta}, a_{(j_1)\theta}, b_{j_1+1}, \ldots, b_{j_2}, \ldots) \neq \mathfrak{b}$. It is clear that $\mathfrak{b} \trianglelefteq \mathfrak{c} \trianglelefteq \mathfrak{a}$ with $\mathfrak{c} \neq \mathfrak{a}$. These imply that $\mathfrak{b} \notin C_{\mathfrak{S}}(\mathfrak{a})$. Next, if $\mathfrak{b} = (a_1, \ldots, a_{j-1}, a_j + a_{j+1} + a_{j+2} + \cdots + a_{j+k}, a_{j+k+1}, \ldots)$ for $k \geq 2$, then $\mathfrak{b} \trianglelefteq (a_1, \ldots, a_{j-1}, a_j + a_{j+1}, a_{j+2}, a_{j+3}, \ldots) \trianglelefteq \mathfrak{a}$, that is, $\mathfrak{b} \notin C_{\mathfrak{S}}(\mathfrak{a})$. Hence, the lemma is proved. □

Theorem 4.3. *For $n \in \mathbb{N}$, the posets $(\mathfrak{P}_n, \trianglelefteq)$ and $(P_n, \trianglelefteq)$ are strongly graded of rank $n - 1$.*

Proof. Define $\mathrm{rk} : \mathfrak{P}_n \to \mathbb{N}_0$ by $(\mathfrak{a})\mathrm{rk} = k - 1$ where $\mathfrak{a} = (a_1, a_2, \ldots, a_k) \in \mathfrak{P}_n$. By applying Lemma 4.2, rk is the rank function. □

The following proposition is directly obtained from Theorem 3.5 and Lemma 4.2.

Proposition 4.4. *For $\beta \in \mathcal{O}([n])$ with $\mathcal{A}_\beta = \mathrm{diag}(b_1^{t_1}, b_2^{t_2}, \ldots, b_k^{t_k})$, if $\alpha \in \mathcal{O}([n])$ and α is covered by β, then $\exists j \in \{1, \ldots, k-1\}$ such that*

$$\mathcal{A}_\alpha = \mathrm{diag}(b_1^{t_1}, b_2^{t_2}, \ldots, (b_j * b_{j+1})^{t_j + t_{j+1}}, b_{j+2}^{t_{j+2}}, \ldots, b_k^{t_k}),$$

where $b_j * b_{j+1}$ is either b_j or b_{j+1}.

Proposition 4.5. *For $0 < k < n$, the poset $(\mathcal{O}([n]), \preccurlyeq)$ has a maximal chain C with $l(C) = k$.*

Proof. Suppose that $n > 2$. For $i \in \{1, \ldots, n-2\}$, we let $\mathfrak{a}_i = (n-i, \underbrace{1, \ldots, 1}_{i \text{ copies}}) \in \mathfrak{P}_n$. Define an order-preserving map ω_i on $[n]$ which $\pi_{\omega_i} = \mathfrak{a}_i$ by

$$\omega_i = \langle \underbrace{1, \ldots, 1}_{n-i \text{ copies}}, 2, 2+1, \ldots, 2+(i-1) \rangle.$$

We observe that $\mathcal{A}_{\omega_i} = \mathrm{diag}(1^{n-i}, 2, 2+1, \ldots, 2+(i-1))$. By using Proposition 3.7, it follows that ω_i is a maximal element on $(\mathcal{O}([n]), \preccurlyeq)$. Applying Proposition 4.4, the length of all maximal chains having ω_i as the maximum element is i. □

As a consequence of Proposition 4.5, we have the following result.

Theorem 4.6. *For $n \in \mathbb{N}$ with $n > 2$, the poset $(\mathcal{O}([n]), \preccurlyeq)$ is weakly graded but not strongly graded.*

Remark 4.7. It was shown in [8] that the cardinality of $\mathcal{E}(\mathcal{O}([n]))$, the set of all idempotents of $\mathcal{O}([n])$, is F_{2n} (the alternate Fibonacci number given by $F_1 = F_2 = 1$). In Fig. 4, we list all elements of $\mathcal{E}(\mathcal{O}([5]))$ and exhibit some maximal chains of length 4 in $\mathcal{E}(\mathcal{O}([5]))$ such as $\langle 1,1,1,1,1\rangle \preccurlyeq \langle 1,3,3,3,3\rangle \preccurlyeq \langle 1,3,3,3,5\rangle \preccurlyeq \langle 1,3,3,4,5\rangle \preccurlyeq \langle 1,2,3,4,5\rangle$.

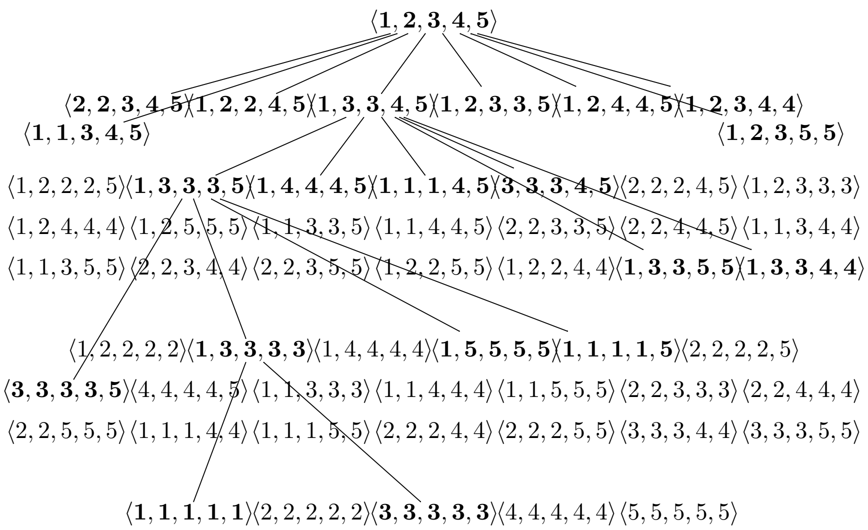


FIGURE 4. All idempotent elements of $\mathcal{O}([5])$.

Proposition 4.8. *For $n \in \mathbb{N}$, the following statements hold.*
(1) *The poset $\mathcal{E}(\mathcal{O}([n]))$ is strongly graded of rank $n-1$.*
(2) *The number of all maximal chains of $\mathcal{E}(\mathcal{O}([n]))$ is $2^{n-1}(n-1)!$.*

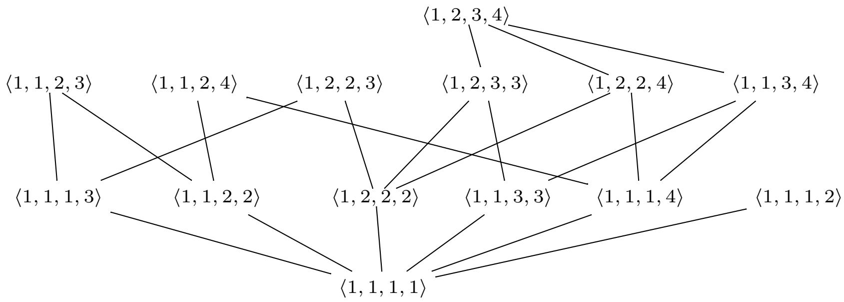
Proof. For (1), it follows from Proposition 4.4. Next, we let $\beta \in \mathcal{E}(\mathcal{O}([n]))$ with $\mathcal{A}_\beta = \operatorname{diag}(b_1^{t_1}, b_2^{t_2}, \ldots, b_k^{t_k})$. Since $\beta = \beta^2$, we have that $b_1 \in [1 \to t_1]$, $b_k \in [n - t_k + 1 \to n]$ and each $0 < l < k$, $b_l \in [\sum_{i=1}^{l} t_i + 1 \to \sum_{i=1}^{l} t_i + t_{l+1}]$. Using Proposition 4.4 and applying the diagram in Fig. 4, we obtain that $|C_{\mathcal{E}(\mathcal{O}([n]))}(\beta)| = 2(k-1)$. As the identity map is the maximum element on $\mathcal{E}(\mathcal{O}([n]))$, it follows that there are $2(n-1)\cdot 2(n-2)\cdot 2(n-3)\cdots 2(n-(n-1))$ chains of $\mathcal{E}(\mathcal{O}([n]))$ having length $n-1$. $\square$

Proposition 4.9. *For $i \geq 3$, the poset $(\mathcal{O}([n]), \preccurlyeq)$ is weakly graded $(\mathbf{i}+\mathbf{1})$-avoiding if $n < i$.*

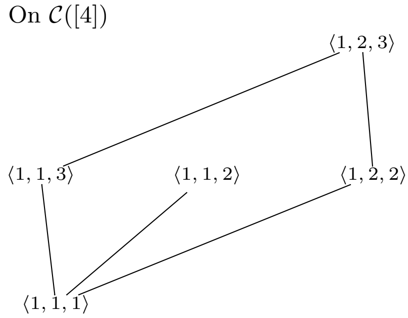
Proof. As in the proof of Proposition 4.5, we consider a maximal element $\omega_1 = \langle \underbrace{1, \ldots, 1}_{n-1 \text{ copies}}, 2\rangle$. Next, we choose a chain C of $\mathcal{E}(\mathcal{O}([n]))$ with $l(C) = n-1$ and $\min(C) = \langle 3, \ldots, 3\rangle$. Since the set of all lower bounds of ω_1 is

$\{\langle 1,\dots,1\rangle,\langle 2,\dots,2\rangle\}$, it follows that $(\mathcal{O}([n]),\preccurlyeq)$ is not an $(\mathbf{n}+\mathbf{1})$-avoiding poset. □

Remark 4.10. A map $\alpha \in T([n])$ is called *regressive* if $x\alpha \le x$ for all $x \in [n]$. We denote $\mathcal{R}([n])$ the subsemigroup of $T([n])$ of all regressive maps. In [10], Laradji and Umar proved that the cardinality of the semigroup $\mathcal{O}([n]) \cap \mathcal{R}([n])$, denoted by $\mathcal{C}([n])$, is $\frac{1}{n+1}\binom{2n}{n}$.



(A) On $\mathcal{C}([4])$



(B) On $\mathcal{R}([3])$

Figure 5. The poset $(\mathcal{C}([4]),\preccurlyeq)$ is $(\mathbf{4}+\mathbf{1})$-avoiding and the poset $(\mathcal{R}([3]),\preccurlyeq)$ is $(\mathbf{3}+\mathbf{1})$-avoiding.

Under the natural partial ordering $\preccurlyeq$, the constant map $\langle 1,\dots,1\rangle$ is the minimum element on $\mathcal{C}([n])$ and $\mathcal{R}([n])$. Then the following proposition is clear.

Proposition 4.11. *The following statements hold.*
(1) *The poset* $(\mathcal{C}([n]),\preccurlyeq)$ *is* $(\mathbf{n}+\mathbf{1})$*-avoiding.*
(2) *The poset* $(\mathcal{R}([n]),\preccurlyeq)$ *is* $(\mathbf{n}+\mathbf{1})$*-avoiding.*

Remark 4.12. For $\alpha \in \mathcal{O}([n])$, let $\mathcal{O}_n(\mathbf{i}+\alpha)$ be the collections of copies $(\mathbf{i}+\mathbf{1})$ in $\mathcal{O}([n])$ which α is incomparable to all members of a chain of length $i-1$. Note that we define $\mathcal{R}_n(\mathbf{i}+\alpha)$ and $\mathcal{C}_n(\mathbf{i}+\alpha)$ in a similar way. We observe that

1. $|\mathcal{O}_3(\mathbf{3}+\langle 2,3,3\rangle)| = 3,\ |\mathcal{O}_3(\mathbf{2}+\langle 2,3,3\rangle)| = 9.$

2. $|\mathcal{R}_3(\mathbf{3} + \langle 1,1,2\rangle)| = 0,\ |\mathcal{R}_3(\mathbf{2} + \langle 1,1,2\rangle)| = 2.$
3. $|\mathcal{C}_4(\mathbf{4}+\langle 1,1,1,2\rangle)| = 0,\ |\mathcal{C}_4(\mathbf{3}+\langle 1,1,1,2\rangle)| = 6,\ |\mathcal{C}_4(\mathbf{2}+\langle 1,1,1,2\rangle)| = 18.$

Question. What is a formula for $|\mathcal{S}_n(\mathbf{i} + \alpha)|$ when α is maximal in $\mathcal{S}([n])$, where $\mathcal{S}$ is $\mathcal{O}$, $\mathcal{R}$, or $\mathcal{C}$?

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