

\mathcal{A} -GENERATORS FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN DEGREE $5(2^t - 1) + 6 \cdot 2^t$

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ABSTRACT. Let $P_s := \mathbb{F}_2[x_1, x_2, \dots, x_s] = \bigoplus_{n \geq 0} (P_s)_n$ be the polynomial algebra viewed as a graded left module over the mod 2 Steenrod algebra, \mathcal{A} . The grading is by the degree of the homogeneous terms $(P_s)_n$ of degree n in the variables x_1, x_2, \dots, x_s of grading 1. We are interested in the *hit problem*, set up by F. P. Peterson, of finding a minimal system of generators for \mathcal{A} -module P_s . Equivalently, we want to find a basis for the \mathbb{F}_2 -graded vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$.

In this paper, we study the hit problem in the case $s = 5$ and the degree $n = 5(2^t - 1) + 6 \cdot 2^t$ with t an arbitrary positive integer.

1. Introduction

Denote by $\mathbb{R}P^\infty$ the infinite real projective space and by \mathbb{F}_2 the prime field of two elements. We study the *hit problem* of determining a minimal set of generators for \mathbb{F}_2 -graded algebra

$$P_s := \mathbb{F}_2[x_1, x_2, \dots, x_s] = H^*((\mathbb{R}P^\infty)^{\times s}; \mathbb{F}_2)$$

as an unstable left module over the mod 2 Steenrod algebra, \mathcal{A} . Equivalently, we want to find a basis of the \mathbb{F}_2 -vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ in each degree n . Such a basis may be represented by a list of monomials of degree n . Here P_s is the polynomial algebra in s generators x_1, x_2, \dots, x_s , each of degree 1. The action of \mathcal{A} on P_s is determined by the elementary properties of the Steenrod squares Sq^i , in grading $i \geq 0$, and subject to the Cartan formula (see [14]).

The hit problem was first studied by Peterson [5, 6], Wood [23], Singer [12], and Priddy [10], who showed its relation to several classical problems in the homotopy theory. Several aspects of the hit problem were then investigated by many authors (see Crabb-Hubbuck [1], Kameko [2], Repka-Selick [11], Singer [13], Wood [24] and others).

Let $GL_s := GL(s, \mathbb{F}_2)$ be the general linear group of rank s over \mathbb{F}_2 . This group acts naturally on P_s by matrix substitution. Since the two actions of

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\mathcal{A} and GL_s upon P_s commute with each other, there is an inherited action of GL_s on $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$.

For a non-negative integer n , denote by $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n$ the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ consisting of all the classes represented by the homogeneous polynomials in $(P_s)_n$ and by $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n^{GL_s}$ the subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n$ consisting of all the GL_s -invariant classes of degree n . One of our main tools for studying the hit problem is the so-called Kameko's squaring operation

$$Sq^0 : \mathbb{F}_2 \otimes_{GL_s} PH_n((\mathbb{R}P^\infty)^{\times s}; \mathbb{F}_2) \longrightarrow \mathbb{F}_2 \otimes_{GL_s} PH_{2n+s}((\mathbb{R}P^\infty)^{\times s}; \mathbb{F}_2),$$

where $PH_n((\mathbb{R}P^\infty)^{\times s}; \mathbb{F}_2)$ denotes the primitive subspace consisting of all elements in the homology group $H_n((\mathbb{R}P^\infty)^{\times s}; \mathbb{F}_2)$ that are annihilated by every positive degree operation in \mathcal{A} . The dual of Sq^0 is the homomorphism $Sq_*^0 : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_{2n+s}^{GL_s} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n^{GL_s}$. This homomorphism is given by the $\mathbb{F}_2 GL_s$ -homomorphism $\widetilde{Sq_*^0} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_{2n+s} \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n$. The latter is given by the \mathbb{F}_2 -linear map $\psi : P_s \longrightarrow P_s$, determined by

$$\psi(x) = \begin{cases} u & \text{if } x = x_1 x_2 \cdots x_s u^2, \\ 0 & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_s$. The map ψ is not an \mathcal{A} -homomorphism. However, $\psi Sq^{2i} = Sq^i \psi$ and $\psi Sq^{2i+1} = 0$ for any $i \geq 0$.

The space $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ was explicitly determined for the cases $s \leq 4$ (see [2, 5, 16, 17]). From a result of Wood [23], the hit problem is reduced to the case of the degree d of the form

$$(1.1) \quad n = r(2^t - 1) + 2^t m,$$

with r, t, m non-negative integers such that $\mu(m) < r \leq s$ (see [17]). Here, $\mu(m)$ is the smallest number k for which it is possible to write $m = \sum_{1 \leq i \leq k} (2^{d_i} - 1)$, where $d_i > 0$.

For $r = s - 1$, the problem was partially studied by Crabb-Hubbuck [1], Nam [4], Repka-Selick [11], and Sum [15, 17] with $m > 0$, by Mothebe [3] and by us [8, 9] with $m = 0$. For $r = s = 5$, the problem was explicitly computed by Tín [20–22] with $m = 1, 2, 3$ and by Sum [18, 19] with $m = 5, 10$, unknown in general.

In this paper, we continue the investigation of the hit problem in the 5-variable case. More precisely, we explicitly determine the space $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ in the degree n of the form (1.1) for $r = s = 5$ and $m = 6$. The main result of the paper is the following.

Main Theorem. *Let $n = 5(2^t - 1) + 6 \cdot 2^t$ with t a positive integer. Then, $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_n = 566$ for $t = 1$, and $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_n = 2130$ for $t \geq 2$.*

Noting that the theorem has been proved in [7] for $t = 1$.

The rest of our work is as follows. In the next section, we recall some results related to the admissible monomials in P_s , Singer's criterion on the hit monomials and Kameko's homomorphism. Our main result will be proved in

Section 3. Finally, in Appendix, we list all the admissible monomials of degrees 18, 39 in P_4 and P_5 .

2. Some preliminaries on the hit problem

In this section, we review some needed information from Kameko [2], Singer [13], Phúc-Sum [8], and Sum [15, 17] which will be used in the next section.

Notation 2.1. Throughout the paper, we use the following notations.

$$\mathbb{N}_s = \{1, 2, \dots, s\},$$

$$X_{\mathbb{I}} = X_{\{i_1, i_2, \dots, i_r\}} = \prod_{i \in \mathbb{N}_s \setminus \mathbb{I}} x_i, \quad \mathbb{I} = \{i_1, i_2, \dots, i_r\} \subseteq \mathbb{N}_s.$$

In particular,

$$X_{\mathbb{N}_s} = 1,$$

$$X_{\emptyset} = x_1 x_2 \cdots x_s,$$

$$X_{\{i\}} = x_1 \cdots \hat{x}_i \cdots x_s, \quad i = 1, 2, \dots, s.$$

Let $\alpha_j(n)$ denote the j -th coefficients in dyadic expansion of a non-negative integer n . This means $n = \alpha_0(n)2^0 + \alpha_1(n)2^1 + \cdots + \alpha_j(n)2^j + \cdots$ for $\alpha_j(n) \in \{0, 1\}$ with $j \geq 0$.

Let $x = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \in P_s$. We set

$$\mathbb{I}_j(x) = \{i \in \mathbb{N}_s : \alpha_j(a_i) = 0\}$$

for $j \geq 0$. Then,

$$x = \prod_{j \geq 0} X_{\mathbb{I}_j(x)}^{2^j}.$$

Definition 2.2 (Kameko [2]). For a monomial $x = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \in P_s$, we define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \dots, \omega_j(x), \dots),$$

$$\sigma(x) = (a_1, a_2, \dots, a_s),$$

where $\omega_j(x) = \sum_{1 \leq i \leq s} \alpha_{j-1}(a_i) = \deg X_{\mathbb{I}_{j-1}(x)}$, $j \geq 1$. Noting that $\omega_j(x) \leq s$ for all j . The sequence $\omega(x)$ is called the *weight vector* of the monomial x and $\sigma(x)$ called the *exponent vector* of the monomial x .

Let $\omega = (\omega_1, \omega_2, \dots, \omega_j, \dots)$ be a sequence of non-negative integers. The sequence is called the weight vector if $\omega_i = 0$ for $i \gg 0$. The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω , we define $\deg \omega = \sum_{i \geq 1} 2^{i-1} \omega_i$. Denote by $P_s(\omega)$ the subspace of P_s spanned by all monomials $x \in P_s$ such that $\deg x = \deg \omega$, $\omega(x) \leq \omega$, and by $P_s^-(\omega)$ the subspace of P_s spanned by all monomials $x \in P_s$ such that $\omega(x) < \omega$.

Definition 2.3 (Kameko [2], Sum [17]). Let ω be a weight vector and f, g two homogeneous polynomials of the same degree in P_s .

- (i) $f \equiv g$ if and only if $(f + g) \in \mathcal{A}^+ \cdot P_s$. If $f \equiv 0$, then f is called *hit*.
- (ii) $f \equiv_\omega g$ if and only if $(f + g) \in \mathcal{A}^+ \cdot P_s + P_s^-(\omega)$. If $f \equiv_\omega 0$, then f is called ω -*hit*.

Here \mathcal{A}^+ denotes the augmentation ideal of \mathcal{A} .

Obviously, the relations \equiv and \equiv_ω are equivalence ones. Denote by $QP_s(\omega)$ the quotient of $P_s(\omega)$ by the equivalence relation \equiv_ω . Then we have

$$QP_s(\omega) = P_s(\omega) / ((\mathcal{A}^+ \cdot P_s \cap P_s(\omega)) + P_s^-(\omega)).$$

Furthermore, $QP_s(\omega)$ is an GL_s -module and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n \cong \bigoplus_{\deg \omega = n} QP_s(\omega)$ (see Sum [17]).

Definition 2.4 (Kameko [2]). Let x, y be monomials in P_s . We say that $x < y$ if and only if one of the following holds:

- (i) $\omega(x) < \omega(y)$;
- (ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5 (Kameko [2]). A monomial x is said to be inadmissible if there exist monomials y_1, y_2, \dots, y_m such that $y_t < x$ for $1 \leq t \leq m$ and $(x - \sum_{t=1}^m y_t) \in \mathcal{A}^+ \cdot P_s$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_s is a minimal set of \mathcal{A} -generators for P_s in degree n .

Definition 2.6 (Kameko [2]). A monomial x is said to be strictly inadmissible if and only if there exist monomials y_1, y_2, \dots, y_m such that $y_t < x$ for $1 \leq t \leq m$ and

$$x = \sum_{1 \leq t \leq m} y_t + \sum_{1 \leq j \leq 2^r - 1} Sq^j(h_j)$$

with $r = \max\{i : \omega_i(x) > 0\}$ and suitable polynomials $h_j \in P_s$.

It is easy to see that if x is strictly inadmissible, then it is inadmissible. In general, the opposite is not true. For instance, the monomial $x = x_1 x_2^2 x_3^2 x_4^2 x_5^2 x_6 \in P_6$ is inadmissible, but it is not strictly inadmissible.

Theorem 2.7 (Kameko [2], Sum [15]). Let x, y and u be monomials in P_s such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_t(u) \neq 0$ and $\omega_i(u) = 0$ for $i > t > 0$.

- (i) If u is inadmissible, then xu^{2^r} is also inadmissible.
- (ii) If u is strictly inadmissible, then uy^{2^t} is also strictly inadmissible.

For a positive integer n , we set

$$\mu(n) = \min\{k \in \mathbb{N} : n = \sum_{1 \leq i \leq k} (2^{d_i} - 1), d_i > 0\}.$$

Theorem 2.8 (Kameko [2]). *If $\mu(2n + s) = s$, then the homomorphism*

$$\widetilde{Sq}_*^0 = (\widetilde{Sq}_*^0)_{(s, 2n+s)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_{2n+s} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s)_n$$

is an isomorphism of the $\mathbb{F}_2 GL_s$ -modules.

We now recall a result of Singer [13] on the hit monomials in P_s .

Definition 2.9. A monomial $z = x_1^{b_1} x_2^{b_2} \cdots x_s^{b_s}$ is called a spike if $b_i = 2^{d_i} - 1$ for d_i a non-negative integer and $i = 1, 2, \dots, s$. If z is a spike with $d_1 > d_2 > \cdots > d_{r-1} \geq d_r > 0$ and $d_j = 0$ for $j > r$, then it is called a minimal spike.

Noting that if $\mu(n) \leq s$, then there exists uniquely a minimal spike of degree n in P_s (see Singer [13]).

Lemma 2.10 (see [8]). *All the spikes in P_s are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ is weakly decreasing and $\omega_1 \leq s$, then there is a spike z in P_s such that $\omega(z) = \omega$.*

The following is a criterion for the hit monomials in P_s .

Theorem 2.11 (Singer [13]). *Suppose that $x \in P_s$ is a monomial of degree n , where $\mu(n) \leq s$. Let z be the minimal spike of degree n in P_s . If $\omega(x) < \omega(z)$, then x is hit.*

We set

$$\begin{aligned} P_s^0 &= \text{Span}\{x = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \in P_s \mid a_1 a_2 \cdots a_s = 0\}, \\ P_s^+ &= \text{Span}\{x = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \in P_s \mid a_1 a_2 \cdots a_s > 0\}. \end{aligned}$$

Then P_s^0 and P_s^+ are the \mathcal{A} -submodules of P_s . Furthermore, we have a direct summand decomposition of the \mathbb{F}_2 -vector spaces

$$\mathbb{F}_2 \otimes_{\mathcal{A}} P_s = (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s^0) \bigoplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s^+).$$

We end this section by recalling some notations and definitions in Sum [17].

We denote

$$\mathcal{N}_s := \{(i; I) \mid I = (i_1, i_2, \dots, i_r), 1 \leq i < i_1 < i_2 < \cdots < i_r \leq s, 0 \leq r < s\}.$$

Then, $|\mathcal{N}_s| = 2^s - 1$. Here, by convention, $I = \emptyset$, if $r = 0$. Denote by $r = \ell(I)$ the length of I .

Definition 2.12 (Sum [17]). Let $(i; I) \in \mathcal{N}_s$, $r = \ell(I)$, and let u be an integer with $1 \leq u \leq r$. A monomial $x = x_1^{a_1} x_2^{a_2} \cdots x_{s-1}^{a_{s-1}} \in P_{s-1}$ is said to be u -compatible with $(i; I)$ if all of the following hold:

- (i) $a_{i_1-1} = a_{i_2-1} = \cdots = a_{i_{(u-1)}-1} = 2^r - 1$,
- (ii) $a_{i_u-1} > 2^r - 1$,
- (iii) $\alpha_{r-t}(a_{i_u-1}) = 1$, $\forall t$, $1 \leq t \leq u$,
- (iv) $\alpha_{r-t}(a_{i_t-1}) = 1$, $\forall t$, $u < t \leq r$.

Clearly, a monomial x can be u -compatible with a given $(i; I) \in \mathcal{N}_s$ for at most one value of u . By convention, x is 1-compatible with $(i; \emptyset)$.

For $1 \leq i \leq s$, define the homomorphism $\tau_i : P_{s-1} \rightarrow P_s$ of algebras by substituting

$$\tau_i(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < i, \\ x_{j+1} & \text{if } i \leq j < s. \end{cases}$$

This map is also a monomorphism of \mathcal{A} -module.

Definition 2.13 (Sum [17]). Let $(i; I) \in \mathcal{N}_s, 0 < r < s$. Denote

$$x_{(I,u)} = x_{i_u}^{2^{r-1} + 2^{r-2} + \dots + 2^{r-u}} \prod_{u < t \leq r} x_{i_t}^{2^{r-t}}$$

for $1 \leq u \leq r$, $x_{(\emptyset,1)} = 1$. For a monomial $x \in P_{s-1}$, we define the monomial $\phi_{(i;I)}(x)$ in P_s by setting

$$\phi_{(i;I)}(x) = \begin{cases} \frac{x_i^{2^r-1} \tau_i(x)}{x_{(I,u)}} & \text{if there exists } u \text{ such that } x \text{ is } u\text{-compatible with } (i; I), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have an \mathbb{F}_2 -linear map $\phi_{(i;I)} : P_{s-1} \rightarrow P_s$. In particular, $\phi_{(i;\emptyset)} = \tau_i$. Noting that $\phi_{(i;I)}$ is not an \mathcal{A} -homomorphism.

For a subset $V \subset P_{s-1}$, we denote

$$\begin{aligned} \Phi^0(V) &= \bigcup_{1 \leq i \leq s} \phi_{(i;\emptyset)}(V) = \bigcup_{1 \leq i \leq s} \tau_i(V), \\ \Phi^+(V) &= \bigcup_{(i;I) \in \mathcal{N}_s, 1 \leq \ell(I) \leq s-1} \phi_{(i;I)}(V) \setminus P_s^0 \subset P_s^+. \end{aligned}$$

It is easy to see that if V is a minimal set of generators for an \mathcal{A} -module P_{s-1} in degree n , then $\Phi^0(V)$ is also a minimal set of generators for an \mathcal{A} -module P_s^0 in degree n .

Definition 2.14. For any $(i; I) \in \mathcal{N}_s$, we define the homomorphism $p_{(i;I)} : P_s \rightarrow P_{s-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < i, \\ \sum_{k \in I} x_{k-1} & \text{if } j = i, \\ x_{j-1} & \text{if } i < j \leq s. \end{cases}$$

Then $p_{(i;I)}$ is a homomorphism of \mathcal{A} -modules. In particular, we have $p_{(i;\emptyset)}(x_i) = 0$ for $1 \leq i \leq s$ and $p_{(i;I)}(\tau_i(u)) = u$ for any $u \in P_{s-1}$.

Lemma 2.15 (see [8]). *If x is a monomial in P_s , then $p_{(i;I)}(x) \in P_{s-1}(\omega(x))$.*

This lemma implies that if ω is a weight vector and $x \in P_s(\omega(x))$, then $p_{(i;I)}(x) \in P_{s-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to a homomorphism from $QP_s(\omega)$ to $QP_{s-1}(\omega)$.

Notation 2.16. For a polynomial $f \in P_s$, we denote by $[f]$ the classes in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_s$ represented by f . If ω is a weight vector and $f \in P_s(\omega)$, then denote by $[f]_\omega$ the classes in $QP_s(\omega)$ represented by f .

Let \mathcal{B} be a subset of P_s . Denote by $|\mathcal{B}|$ the cardinal of a set \mathcal{B} and $[\mathcal{B}] = \{[f] : f \in \mathcal{B}\}$. If $\mathcal{B} \subset P_s(\omega)$, then we set $[\mathcal{B}]_\omega = \{[f]_\omega : f \in \mathcal{B}\}$.

Denote by $\mathcal{B}_s(n)$ the set of all admissible monomials of degree n in P_s . We set

$$\mathcal{B}_s^0(n) := \mathcal{B}_s(n) \cap (P_s^0)_n, \quad \mathcal{B}_s^+(n) := \mathcal{B}_s(n) \cap (P_s^+)_n.$$

For a weight vector ω of degree n , we denote

$$\mathcal{B}_s(\omega) := \mathcal{B}_s(n) \cap P_s(\omega), \quad \mathcal{B}_s^+(\omega) := \mathcal{B}_s(n) \cap (P_s^+)_n.$$

Then, $[\mathcal{B}_s(\omega)]_\omega$ and $[\mathcal{B}_s^+(\omega)]_\omega$ are respectively the bases of the \mathbb{F}_2 -vector spaces $QP_s(\omega)$ and $QP_s^+(\omega) := QP_s(\omega) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_s^+)_n$.

3. Proof of Main Theorem

In this section, we prove the main result by explicitly determining all admissible monomials of degree $5(2^t - 1) + 6 \cdot 2^t$ in P_5 . It is easy to see that $\mu(5(2^t - 1) + 6 \cdot 2^t) = 5$ for any $t > 2$. So, Theorem 2.8 implies that

$$\widetilde{(Sq_*^0)}_{(5, 5(2^t - 1) + 6 \cdot 2^t)}^{t-2} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^t - 1) + 6 \cdot 2^t} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^2 - 1) + 6 \cdot 2^2}$$

is an isomorphism of GL_5 -modules for every $t \geq 2$. Therefore, we need only to compute $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^t - 1) + 6 \cdot 2^t}$ for $t = 1, 2$.

3.1. The case $t = 1$

For $t = 1$, $5(2^t - 1) + 6 \cdot 2^t = 17$. The space $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{17}$ has been computed in [7].

Theorem 3.1.1 (see [7]). *There exist exactly 566 admissible monomials of degree 17 in P_5 . Consequently, $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{17} = 566$.*

3.2. The admissible monomials of degree 18 in P_5

To compute $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^t - 1) + 6 \cdot 2^t}$ for $t = 2$, we need to determine the admissible monomials of degree 18 in P_5 .

Lemma 3.2.1. *If x is an admissible monomial of degree 18 in P_5 , then $\omega(x)$ is one of the following sequences:*

$$(2, 2, 1, 1), (2, 2, 3), (2, 4, 2), (4, 1, 1, 1), (4, 1, 3), (4, 3, 2).$$

Proof. Observe that $z = x_1^{15}x_2^3$ is the minimal spike of degree 18 in P_5 and $\omega(z) = (2, 2, 1, 1)$. Since $[x] \neq 0$, by Theorem 2.11, either $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If $\omega_1(x) = 2$, then $x = X_{\{i,j,\ell\}}y^2$ with y a monomial of degree 8 in P_5 and $1 \leq i < j < \ell \leq 5$. Since x is admissible, by Theorem 2.7, y is admissible. According to Tín [21], either $\omega(y) = (2, 1, 1)$ or $\omega(y) = (2, 3)$ or $\omega(y) = (4, 2)$.

If $\omega_1(x) = 4$, then $x = X_{\{i\}} u^2$ with u is an admissible monomial of degree 7 in P_5 and $1 \leq i \leq 5$. By Tín [21], $\omega(u)$ is one of the sequences $(1, 1, 1)$, $(1, 3)$, and $(3, 2)$. The lemma is proved. \square

Recall that $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{18} = (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{18} \oplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{18}$. According to Sum [17], $|\mathcal{B}_4(18)| = 126$ (see Subsection 4.1). Then, by a simple computation, we get

$$\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{18} = |\Phi^0(\mathcal{B}_4(18))| = 450.$$

Now, from Lemma 3.2.1, we have

$$\begin{aligned} (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{18} &= QP_5^+(2, 2, 1, 1) \bigoplus QP_5^+(2, 2, 3) \bigoplus QP_5^+(2, 4, 2) \\ &\quad \bigoplus QP_5^+(4, 1, 1, 1) \bigoplus QP_5^+(4, 1, 3) \bigoplus QP_5^+(4, 3, 2). \end{aligned}$$

Proposition 3.2.2. *We have the following:*

- (i) $\mathcal{B}_5^+(2, 2, 1, 1) = \Phi^+(\mathcal{B}_4(2, 2, 1, 1)) \cup \{x_1 x_2^2 x_3^4 x_4^3 x_5^8, x_1 x_2^2 x_3^2 x_4^4 x_5^3, x_1 x_2^2 x_3^4 x_4^9 x_5^2, x_1^3 x_2^4 x_3 x_4^2 x_5^8\}$.
- (ii) $\mathcal{B}_5^+(2, 2, 3)$ is the set of the following monomials:

$$\begin{aligned} &x_1 x_2^2 x_3^4 x_4^4 x_5^7, \quad x_1 x_2^2 x_3^4 x_4^7 x_5^4, \quad x_1 x_2^2 x_3^7 x_4^4 x_5^4, \\ &x_1 x_2^2 x_3^2 x_4^4 x_5^4, \quad x_1^7 x_2 x_3^2 x_4^4 x_5^4, \quad x_1 x_2^2 x_3^4 x_4^5 x_5^6, \\ &x_1 x_2^2 x_3^5 x_4^4 x_5^6, \quad x_1 x_2^2 x_3^5 x_4^6 x_5^4, \quad x_1 x_2^2 x_3^4 x_4^4 x_5^6, \\ &x_1 x_2^3 x_3^4 x_4^6 x_5^4, \quad x_1 x_2^3 x_3^6 x_4^4 x_5^4, \quad x_1^3 x_2 x_3^4 x_4^4 x_5^6, \\ &x_1^3 x_2 x_3^4 x_4^6 x_5^4, \quad x_1^3 x_2 x_3^6 x_4^4 x_5^4, \quad x_1^3 x_2 x_3^2 x_4^4 x_5^6. \end{aligned}$$
- (iii) $\mathcal{B}_5^+(2, 4, 2)$ is the set of the following monomials:

$$\begin{aligned} &x_1 x_2^2 x_3^3 x_4^6 x_5^6, \quad x_1 x_2^3 x_3^2 x_4^6 x_5^6, \quad x_1 x_2^3 x_3^6 x_4^2 x_5^6, \\ &x_1 x_2^3 x_3^6 x_4^6 x_5^2, \quad x_1^3 x_2 x_3^2 x_4^6 x_5^6, \quad x_1^3 x_2 x_3^6 x_4^2 x_5^6, \\ &x_1^3 x_2 x_3^6 x_4^6 x_5^2, \quad x_1^3 x_2^5 x_3^2 x_4^2 x_5^6, \quad x_1^3 x_2^5 x_3^2 x_4^6 x_5^2, \\ &x_1^3 x_2^5 x_3^6 x_4^2 x_5^2. \end{aligned}$$
- (iv) $\mathcal{B}_5^+(4, 1, 1, 1) = \Phi^+(\mathcal{B}_4(4, 1, 1, 1)) \cup \{x_1^3 x_2^5 x_3^8 x_4 x_5\}$.
- (v) $\mathcal{B}_5^+(4, 1, 3) = \Phi^+(\mathcal{B}_4(4, 1, 3))$.
- (vi) $\mathcal{B}_5^+(4, 3, 2) = \Phi^+(\mathcal{B}_4(4, 3, 2)) \cup \mathcal{D}$, where \mathcal{D} is the set of the following monomials:

$$\begin{aligned} &x_1^3 x_2^4 x_3 x_4^3 x_5^7, \quad x_1^3 x_2^4 x_3 x_4^7 x_5^3, \quad x_1^3 x_2^4 x_3^3 x_4 x_5^7, \\ &x_1^3 x_2^4 x_3^3 x_4^7 x_5, \quad x_1^3 x_2^4 x_3^7 x_4 x_5^3, \quad x_1^3 x_2^4 x_3^7 x_4^3 x_5, \\ &x_1^3 x_2^7 x_3^4 x_4 x_5^3, \quad x_1^3 x_2^7 x_3^4 x_4^3 x_5, \quad x_1^7 x_2^3 x_3^4 x_4 x_5^3, \\ &x_1^7 x_2^3 x_3^4 x_4^3 x_5, \quad x_1^3 x_2^4 x_3^3 x_4^5 x_5, \quad x_1^3 x_2^4 x_3^5 x_4^5 x_5^3. \end{aligned}$$

By a simple computation, one gets

$$|\mathcal{B}_5^+(2, 2, 1, 1)| = 25, \quad |\mathcal{B}_5^+(4, 1, 1, 1)| = 40, \quad |\mathcal{B}_5^+(4, 1, 3)| = 10, \quad |\mathcal{B}_5^+(4, 3, 2)| = 180.$$

These sets are explicitly determined as in Subsection 4.2. So, the following corollary is immediate.

Corollary 3.2.3. $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{18} = 730$.

To prove Proposition 3.2.2, we need some lemmas. By a simple computation, we obtain the following.

Lemma 3.2.4. *The following monomials are strictly inadmissible:*

- (i) $x_i^2 x_j x_\ell^3, x_i^2 x_j x_\ell^7, x_i^2 x_j^5 x_\ell^3, x_i^4 x_j^3 x_\ell^3, x_i^6 x_j x_\ell^3, x_i^2 x_j x_\ell^3 x_t^4, x_i^2 x_j x_\ell^5 x_t^2,$
 $x_i^2 x_j x_\ell^2 x_t^2 x_m^3, i < j,$
- (ii) $x_i^2 x_j^2 x_\ell x_t, x_i^2 x_j x_\ell x_t^3 x_m^3, i < j < \ell,$
 $x_i x_j^2 x_\ell^6 x_t, x_i x_j^6 x_\ell^2 x_t, x_i x_j^2 x_\ell^2 x_t^5, x_i x_j^2 x_\ell^2 x_t x_m^4, j < \ell < t,$
- (iii) $x_i^6 x_j x_\ell x_t^2, x_i^2 x_j x_\ell x_t^2 x_m^4, x_i^2 x_j x_\ell x_t x_m^5, i < j < \ell < t,$
- (iv) $x_i^2 x_j^2 x_\ell^3 x_t^3, x_i x_j^7 x_\ell^2 x_t^6 x_m^2, x_1^2 x_2 x_3 x_4 x_5, x_i^2 x_j^3 x_\ell^3 x_t^3 x_m^3, x_1^6 x_2 x_3 x_4 x_5.$

Here (i, j, ℓ, t, m) is a permutation of $(1, 2, 3, 4, 5)$.

Lemma 3.2.5. *If (j, ℓ, t, m) is a permutation of $(2, 3, 4, 5)$ such that $j < \ell$, then the monomial $x_1 x_j^6 x_\ell^3 x_t^6 x_m^2$ is strictly inadmissible.*

Proof. We have

$$\begin{aligned} x_1 x_j^6 x_\ell^3 x_t^6 x_m^2 &= Sq^1(x_1^2 x_j^3 x_\ell^3 x_t^5 x_m^4 + x_1^2 x_j^3 x_\ell^5 x_t^5 x_m^2 + x_1^2 x_j^5 x_\ell^3 x_t^5 x_m^2) \\ &\quad + Sq^2(x_1 x_j^5 x_\ell^3 x_t^5 x_m^2 + x_1 x_j^3 x_\ell^5 x_t^5 x_m^2 + x_1 x_j^3 x_\ell^3 x_t^5 x_m^4) \\ &\quad + x_1 x_j^3 x_\ell^6 x_t^6 x_m^2 \bmod(P_5^-(2, 4, 2)). \end{aligned}$$

This relation shows that $x_1 x_j^6 x_\ell^3 x_t^6 x_m^2$ is strictly inadmissible. The lemma is proved. \square

Lemma 3.2.6. *The following monomials are strictly inadmissible:*

$$\begin{aligned} x_1 x_2 x_3^6 x_4^8 x_5^2, \quad x_1 x_2^6 x_3 x_4^8 x_5^2, \quad x_1 x_2^6 x_3^8 x_4 x_5^2, \quad x_1 x_2^6 x_3^3 x_4^4 x_5^4, \\ x_1^3 x_2^3 x_3^4 x_4^4 x_5^4, \quad x_1^3 x_2^4 x_3^3 x_4^4 x_5^4, \quad x_1^3 x_2^4 x_3^4 x_4^3 x_5^4, \quad x_1^3 x_2^4 x_3^4 x_4^4 x_5^3, \\ x_1^3 x_2^4 x_3^2 x_4 x_5^9, \quad x_1^3 x_2^4 x_3 x_4^9 x_5, \quad x_1^3 x_2^4 x_3^9 x_4 x_5, \quad x_1^3 x_2^4 x_3 x_4^4 x_5^6, \\ x_1^3 x_2^4 x_3 x_4^6 x_5^4, \quad x_1^3 x_2^4 x_3^4 x_4 x_5^6, \quad x_1^3 x_2^4 x_3 x_4^5 x_5^5, \quad x_1^3 x_2^4 x_3 x_4 x_5^5, \\ x_1^3 x_2^4 x_3^5 x_4^5 x_5, \quad x_1^3 x_2^5 x_3^4 x_4 x_5^5, \quad x_1^3 x_2^5 x_3^4 x_4^5 x_5, \quad x_1^3 x_2^5 x_3^5 x_4^4 x_5, \\ x_1^3 x_2^4 x_3 x_4^8 x_5^2, \quad x_1^3 x_2^4 x_3^8 x_4 x_5^2, \quad x_1^3 x_2^4 x_3^4 x_4^5 x_5^2, \quad x_1^3 x_2^4 x_3 x_4^2 x_5^4, \\ x_1^3 x_2^4 x_3^5 x_4^4 x_5^2, \quad x_1^3 x_2^5 x_3^4 x_4^2 x_5^4, \quad x_1^3 x_2^5 x_3^4 x_4^4 x_5^2, \quad x_1^3 x_2^1 x_3 x_4 x_5^12. \end{aligned}$$

Proof. We prove the lemma for the monomials $u = x_1 x_2 x_3^6 x_4^8 x_5^2$ and $v = x_1^3 x_2^3 x_3^4 x_4^4 x_5^4$. The others can be proven by a similar computation.

We have $\omega(u) = (2, 2, 1, 1)$ and $\omega(v) = (2, 2, 3)$. By a direct computation using the Cartan formula, one gets

$$\begin{aligned} u &= x_1 x_2 x_3^2 x_4^6 x_5^8 + x_1 x_2 x_3^2 x_4^8 x_5^6 + x_1 x_2 x_3^4 x_4^2 x_5^{10} + x_1 x_2 x_3^4 x_4^{10} x_5^2 \\ &\quad + x_1 x_2 x_3^6 x_4^2 x_5^8 + Sq^1(h_1) + Sq^2(h_2) + Sq^4(h_4) \bmod(P_5^-(2, 2, 1, 1)), \end{aligned}$$

where

$$\begin{aligned} h_1 &= x_1 x_2^4 x_3^3 x_4^5 x_5 + x_1 x_2^4 x_3^3 x_4^5 x_5^4 + x_1 x_2^4 x_3^4 x_4^3 x_5^5 + x_1 x_2^4 x_3^4 x_4^5 x_5^3 \\ &\quad + x_1 x_2^4 x_3^5 x_4^3 x_5^4 + x_1 x_2^4 x_3^5 x_4^4 x_5^3 + x_1^2 x_2 x_3 x_4^5 x_5^8 + x_1^2 x_2 x_3 x_4^8 x_5^5 \\ &\quad + x_1^2 x_2 x_3^5 x_4^4 x_5^5 + x_1^2 x_2 x_3^5 x_4^5 x_5^4 + x_1^2 x_2^4 x_3 x_4^5 x_5^5 + x_1^4 x_2^4 x_3^3 x_4^3 x_5^3, \end{aligned}$$

$$\begin{aligned}
h_2 &= x_1 x_2 x_3^6 x_4^6 x_5^2 + x_1 x_2 x_3^5 x_4^5 x_5^4 + x_1 x_2 x_3 x_4^5 x_5^8 + x_1 x_2 x_3 x_4^8 x_5^5 \\
&\quad + x_1 x_2 x_3^5 x_4^4 x_5^5 + x_1 x_2 x_3^6 x_4^2 x_5^6 + x_1 x_2^2 x_3 x_4^6 x_5^6 + x_1 x_2^2 x_3^2 x_4^3 x_5^8 \\
&\quad + x_1 x_2^2 x_3^2 x_4^8 x_5^3 + x_1 x_2^2 x_3^3 x_4^4 x_5^6 + x_1 x_2^2 x_3^3 x_4^6 x_5^4 + x_1 x_2^2 x_3^4 x_4^3 x_5^4 \\
&\quad + x_1 x_2^2 x_3^6 x_4^4 x_5^3 + x_1 x_2^4 x_3 x_4^5 x_5^5 + x_1 x_2^4 x_3^2 x_4^3 x_5^6 + x_1 x_2^4 x_3^6 x_4^3 x_5^3 \\
&\quad + x_1^2 x_2^4 x_3^3 x_4^3 x_5^4 + x_1^2 x_2^4 x_3^3 x_4^4 x_5^3 + x_1^2 x_2^4 x_3^4 x_4^3 x_5^3,
\end{aligned}$$

$$h_4 = x_1 x_2 x_3^4 x_4^2 x_5^6 + x_1 x_2 x_3^4 x_4^6 x_5^2 + x_1 x_2^2 x_3^3 x_4^4 x_5^4 + x_1 x_2^2 x_3^4 x_4^3 x_5^4 + x_1 x_2^2 x_3^4 x_4^4 x_5^3.$$

By Definition 2.6, we see that u is strictly inadmissible. Similarly, we obtain

$$\begin{aligned}
v &= Sq^1(x_1^3 x_2^5 x_3 x_4^4 x_5^4 + x_1^5 x_2^3 x_3^4 x_4 x_5^4 + x_1^5 x_2^5 x_3 x_4^2 x_5^4 + x_1^6 x_2^5 x_3 x_4 x_5^4) \\
&\quad + Sq^2(x_1^3 x_2^3 x_3^4 x_4^2 x_5^4 + x_1^3 x_2^5 x_3^2 x_4^2 x_5^4 + x_1^3 x_2^6 x_3 x_4^2 x_5^4 \\
&\quad \quad + x_1^6 x_2^3 x_3^2 x_4 x_5^4 + x_1^6 x_2^6 x_3 x_4 x_5^2) \bmod(P_5^-(2, 2, 3)).
\end{aligned}$$

This equality implies that v is strictly inadmissible. The lemma follows. \square

Proof of Proposition 3.2.2. We prove the first part of the proposition. The others can be proved by a similar computation. Set

$$\mathcal{B} := \Phi^+(\mathcal{B}_4^+(2, 2, 1, 1)) \cup \{x_1 x_2^2 x_3^4 x_4^3 x_5^8, x_1 x_2^2 x_3^4 x_4^8 x_5^3, x_1 x_2^2 x_3^4 x_4^9 x_5^2, x_1^3 x_2^4 x_3 x_4^2 x_5^8\}.$$

Suppose that x is an admissible monomial of degree 18 in P_5 with $\omega(x) = (2, 2, 1, 1)$. Since $\omega_1(x) = 2$, $x = X_{\{i,j,\ell\}} y^2$ with y a monomial of degree 8 in P_5 and $1 \leq i < j < \ell \leq 5$. On the other hand, x is admissible; hence by Theorem 2.7, $y \in \mathcal{B}_5(8)$.

By a direct computation we see that for all $u \in \mathcal{B}_5(8)$, $1 \leq i < j < \ell \leq 5$, such that $X_{\{i,j,\ell\}} u^2 \notin \mathcal{B}$, there is a monomial w which is given in one of Lemmas 3.2.4(i), (ii), (iii), and 3.2.5(ii) such that $X_{\{i,j,\ell\}} u^2 = w u_1^{2^r}$ with $u_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $X_{\{i,j,\ell\}} u^2$ is inadmissible. Since $x = X_{\{i,j,\ell\}} y^2$ admissible and $y \in \mathcal{B}_5(8)$, one gets $x \in \mathcal{B}$. This implies $\mathcal{B}_5^+(2, 2, 1, 1) \subset \mathcal{B}$.

We now prove the set $[\mathcal{B}]$ is linearly independent in $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{18}$. We denote the classes in $[\mathcal{B}]$ represented by a_k , $1 \leq k \leq 25$ (see Subsection 4.2). Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq k \leq 25} \gamma_k a_k \equiv 0,$$

where $\gamma_k \in \mathbb{F}_2$, $\forall k$, $1 \leq k \leq 25$. We express $p_{(i,I)}(\mathcal{S})$, $(i; I) \in \mathcal{N}_5$ in terms of the admissible monomials in $(P_4^+)_18$. Computing directly from the relations $p_{(1;2)}(\mathcal{S}) \equiv 0$, $p_{(1;3)}(\mathcal{S}) \equiv 0$, $p_{(2;3)}(\mathcal{S}) \equiv 0$, and $p_{(3;4)}(\mathcal{S}) \equiv 0$, we obtain $\gamma_k = 0$, $k = 1, 2, \dots, 25$. This finishes the proof. \square

3.3. The case $t = 2$

For $t = 2$, we have $5(2^t - 1) + 6 \cdot 2^t = 39$. Since Kameko's homomorphism

$$(\widetilde{Sq_*^0})_{(5, 39)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{39} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{17}$$

is an epimorphism, $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{39} = \text{Ker}((\widetilde{Sq}_*)_{(5,39)}) \bigoplus (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{17}$. So, we need to compute $\text{Ker}((\widetilde{Sq}_*)_{(5,39)})$.

Set

$$\begin{aligned}\omega_{(1)} &= (3, 2, 2, 1, 1), \quad \omega_{(2)} = (3, 2, 2, 3), \quad \omega_{(3)} = (3, 2, 4, 2), \\ \omega_{(4)} &= (3, 4, 1, 1, 1), \quad \omega_{(5)} = (3, 4, 1, 3), \quad \omega_{(6)} = (3, 4, 3, 2).\end{aligned}$$

Then we obtain the following.

Lemma 3.3.1. *If x is an admissible monomial of degree 39 in P_5 and $[x] \in \text{Ker}((\widetilde{Sq}_*)_{(5,39)})$, then $\omega(x)$ is one of the sequences $\omega_{(k)}$, $1 \leq k \leq 6$.*

Proof. It is easy to see that $z = x_1^{31}x_2^7x_3$ is the minimal spike of degree 39 in P_5 and $\omega(z) = \omega_{(1)}$. Since $[z] \neq 0$, by Theorem 2.11, either $\omega_1(x) = 3$ or $\omega_1(x) = 5$. If $\omega_1(x) = 5$, then $x = X_\emptyset y^2$ with y a monomial of degree 17 in P_5 . Since x is admissible, by Theorem 2.7, $y \in \mathcal{B}_5(17)$. Hence, $(\widetilde{Sq}_*)_{(5,39)}([x]) = [y] \neq [0]$. This contradicts the fact that $[x] \in \text{Ker}((\widetilde{Sq}_*)_{(5,39)})$, so $\omega_1(x) = 3$. Then, we have $x = X_{\{i,j\}}y_1^2$ with $1 \leq i < j \leq 5$ and y_1 an admissible monomial of degree 18 in P_5 . Now, the lemma follows from Lemma 3.2.1. \square

Combining this lemma and a result of Sum [17], we get

$$\begin{aligned}\text{Ker}((\widetilde{Sq}_*)_{(5,39)}) &= (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{39} \bigoplus \left(\text{Ker}((\widetilde{Sq}_*)_{(5,39)}) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{39} \right), \\ \text{Ker}((\widetilde{Sq}_*)_{(5,39)}) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{39} &= \bigoplus_{1 \leq k \leq 6} QP_5^+(\omega_{(k)}), \\ \dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_{39} &= |\Phi^0(\mathcal{B}_4(39))| = 915.\end{aligned}$$

Noting that $|\mathcal{B}_4(39)| = 225$ (see Subsection 4.3).

Proposition 3.3.2. *The set $\{[d_k] : 1 \leq k \leq 649\}$ is a basis of the \mathbb{F}_2 -vector space $\text{Ker}(\widetilde{Sq}_*) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_{39}$. Here the monomials $d_k = d_{39,k}$, with $1 \leq k \leq 649$, are determined as in Subsection 4.4.*

We prove this proposition by proving some lemmas.

Lemma 3.3.3. *The set $QP_5^+(\omega_{(1)})$ is spanned by the set $\{[d_k]_{\omega_{(1)}} : 1 \leq k \leq 485\}$.*

The following lemma is proved by a direct computation.

Lemma 3.3.4. *If (i, j, ℓ, t, m) is a permutation of $(1, 2, 3, 4, 5)$, then the following monomials are strictly inadmissible:*

- (i) $x_i^2 x_j x_\ell x_t^3, x_i x_j^2 x_\ell^6 x_t^7 x_m^7, x_i x_j^6 x_\ell^2 x_t^7 x_m^7, x_i^2 x_j x_\ell^2 x_t^3 x_m^3, x_i^2 x_j^2 x_\ell x_t^3 x_m^3, i < j < \ell$.
- (ii) $x_i^2 x_j x_\ell x_t x_m^2, x_1 x_2^2 x_3^2 x_4 x_5, x_i^3 x_j^{12} x_\ell x_t^7, x_i^3 x_j^{12} x_\ell^7 x_t, x_i^3 x_j^{12} x_\ell^3 x_t^5, x_i^3 x_j^4 x_\ell^9 x_t^7, x_i^3 x_j^5 x_\ell^9 x_t^6, x_i^3 x_j^5 x_\ell^8 x_t^7, x_1^7 x_2^8 x_3^3 x_4^5, x_i x_j^6 x_\ell^3 x_t^6 x_m^7, x_i x_j^6 x_\ell^6 x_t^3 x_m^7, i < j < \ell < t$.
- (iii) $x_i^2 x_j^3 x_\ell^3 x_t^3$.

Lemma 3.3.5. *The following monomials are strictly inadmissible:*

$$\begin{aligned}
& x_1x_2^2x_3^6x_4^3x_5^3, \quad x_1x_2^6x_3^2x_4^3x_5^3, \quad x_1x_2^6x_3^3x_4^2x_5^3, \quad x_1x_2^6x_3^3x_4^3x_5^2, \\
& x_1^3x_2^4x_3x_4^8x_5^7, \quad x_1^3x_2^4x_3^8x_4x_5^7, \quad x_1^3x_2^4x_3^8x_4^7x_5, \quad x_1^7x_2^8x_3^3x_4^4x_5, \\
& x_1^3x_2^4x_3x_4^9x_5^6, \quad x_1^3x_2^4x_3^9x_4x_5^6, \quad x_1^3x_2^4x_3^9x_4^6x_5, \quad x_1^3x_2^4x_3^4x_4^9x_5^9, \\
& x_1^3x_2^4x_3^3x_4^{12}x_5, \quad x_1^3x_2^{12}x_3^3x_4^4x_5, \quad x_1^3x_2^4x_3^7x_4^4x_5, \quad x_1^3x_2^4x_3^7x_4^5x_5^4, \\
& x_1^3x_2^5x_3^7x_4^4x_5^4, \quad x_1^3x_2^7x_3^4x_4^4x_5^5, \quad x_1^3x_2^7x_3^4x_4^5x_5^4, \quad x_1^3x_2^7x_3^5x_4^4x_5^4, \\
& x_1^7x_2^3x_3^4x_4^4x_5^5, \quad x_1^7x_2^3x_3^4x_4^5x_5^4, \quad x_1^7x_2^3x_3^5x_4^4x_5^4, \quad x_1^3x_2^4x_3^8x_4^3x_5^5, \\
& x_1^3x_2^4x_3^{11}x_4^4x_5^5, \quad x_1^3x_2^5x_3^3x_4^8x_5^6, \quad x_1^3x_2^5x_3^8x_4x_5^6, \quad x_1^3x_2^5x_3^8x_4^6x_5, \\
& x_1^3x_2^5x_3^6x_4^4x_5^5, \quad x_1^3x_2^5x_3^6x_4^5x_5^4, \quad x_1^3x_2^{12}x_3x_4x_5^6, \quad x_1^3x_2^{12}x_3x_4x_5^6.
\end{aligned}$$

The proof of this lemma is straightforward.

Lemma 3.3.6. *The following monomials are strictly inadmissible:*

$$\begin{aligned}
& x_1x_2^6x_3^3x_4^{12}x_5^{17}, \quad x_1x_2^6x_3^3x_4^{13}x_5^{16}, \quad x_1x_2^6x_3^3x_4^{24}x_5^5, \quad x_1x_2^6x_3^{11}x_4^4x_5^{17}, \\
& x_1x_2^6x_3^{11}x_4^{17}x_5^4, \quad x_1x_2^6x_3^{11}x_4^{5}x_5^{16}, \quad x_1x_2^6x_3^{11}x_4^{16}x_5^5, \quad x_1x_2^6x_3^{11}x_4^{20}x_5, \\
& x_1x_2^7x_3^{10}x_4^{16}x_5^5, \quad x_1^7x_2x_3^{10}x_4^{16}x_5^5, \quad x_1x_2^7x_3^{10}x_4^{17}x_5^4, \quad x_1^7x_2x_3^{10}x_4^{17}x_5^4, \\
& x_1x_2^7x_3^{11}x_4^{16}x_5^4, \quad x_1^7x_2x_3^{11}x_4^{16}x_5^4, \quad x_1^7x_2^1x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \\
& x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \\
& x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \quad x_1^7x_2^{11}x_3x_4^{16}x_5^4, \\
& x_1^3x_2^4x_3^{11}x_4^{17}x_5^4, \quad x_1^3x_2^5x_3^9x_4^{18}x_5^4, \quad x_1^3x_2^5x_3^{24}x_4^2x_5^5, \quad x_1^3x_2^5x_3^{24}x_4^3x_5^4, \\
& x_1^3x_2^5x_3^{25}x_4^2x_5^4, \quad x_1^3x_2^7x_3^8x_4^4x_5^{17}, \quad x_1^3x_2^7x_3^8x_4^{17}x_5^4, \quad x_1^7x_2^3x_3^8x_4^4x_5^{17}, \\
& x_1^7x_2^3x_3^8x_4^{17}x_5^4, \quad x_1^3x_2^7x_3^8x_4^5x_5^{16}, \quad x_1^3x_2^7x_3^8x_4^{16}x_5^5, \quad x_1^7x_2^3x_3^8x_4^5x_5^{16}, \\
& x_1^7x_2^3x_3^8x_4^{16}x_5^5, \quad x_1^3x_2^7x_3^8x_4^5x_5^{20}, \quad x_1^7x_2^3x_3^8x_4^{20}x_5, \quad x_1^3x_2^7x_3^{12}x_4x_5^{16}, \\
& x_1^3x_2^7x_3^{12}x_4^{16}x_5, \quad x_1^7x_2^3x_3^{12}x_4x_5^{16}, \quad x_1^7x_2^3x_3^{12}x_4^{16}x_5, \quad x_1^3x_2^7x_3^{24}x_4x_5^4, \\
& x_1^3x_2^7x_3^{24}x_4^4x_5^5, \quad x_1^7x_2^3x_3^{24}x_4x_5^4, \quad x_1^7x_2^3x_3^{24}x_4^4x_5^5, \quad x_1^3x_2^{12}x_3x_4^2x_5^{21}, \\
& x_1^3x_2^{12}x_3x_4^3x_5^{20}, \quad x_1^3x_2^{12}x_3x_4x_5^{20}, \quad x_1^3x_2^{28}x_3x_4^2x_5^5, \quad x_1^3x_2^{28}x_3x_4^3x_5^4, \\
& x_1^3x_2^{28}x_3x_4x_5^4.
\end{aligned}$$

Proof. We prove the lemma for the monomials $u = x_1^3x_2^4x_3^{11}x_4^{17}x_5^4$, and $v = x_1^7x_2^{11}x_3^{16}x_4x_5^4$. The others can be proved by a similar computation. By a direct computation, we have

$$\begin{aligned}
u &= x_1^3x_2^2x_3^{13}x_4^{17}x_5^4 + x_1^3x_2x_3^{13}x_4^{18}x_5^4 + x_1^3x_2x_3^{11}x_4^{20}x_5^4 + x_1^3x_2^4x_3^9x_4^{21}x_5^2 \\
&\quad + x_1^3x_2^2x_3^9x_4^{21}x_5^4 + x_1^3x_2x_3^{12}x_4^{21}x_5^2 + x_1^3x_2x_3^{10}x_4^{21}x_5^4 + x_1^3x_2x_3^9x_4^{22}x_5^4 \\
&\quad + x_1^2x_2x_3^{13}x_4^{21}x_5^2 + x_1^2x_2x_3^{11}x_4^{21}x_5^4 + x_1^2x_2x_3^7x_4^{25}x_5^4 + x_1^3x_2^7x_3x_4^{17}x_5^8 \\
&\quad + x_1^3x_2x_3^7x_4^{20}x_5^8 + x_1^3x_2^4x_3^{11}x_4^{16}x_5^5 + x_1^3x_2^4x_3^8x_4^{21}x_5^3 + x_1^2x_2^4x_3^{13}x_4^{17}x_5^3 \\
&\quad + x_1^2x_2^4x_3^{11}x_4^{17}x_5^5 + x_1x_2^4x_3^{18}x_4^{13}x_5^3 + x_1x_2^4x_3^{14}x_4^{17}x_5^3 + x_1x_2^2x_3^{17}x_4^{13}x_5^6 \\
&\quad + x_1x_2^2x_3^7x_4^{17}x_5^{12} + x_1x_2^2x_3^{13}x_4^{17}x_5^6 + x_1x_2^2x_3^7x_4^{13}x_5^{16} + x_1^3x_2x_3^{18}x_4^{13}x_5^4 \\
&\quad + x_1^2x_2x_3^{13}x_4^{18}x_5^5 + x_1^2x_2x_3^{11}x_4^{20}x_5^5 + x_1x_2^2x_3^{17}x_4^{14}x_5^5 + x_1^3x_2x_3^{14}x_4^{17}x_5^4 \\
&\quad + x_1x_2^2x_3^{13}x_4^{18}x_5^5 + x_1x_2x_3^{18}x_4^{14}x_5^5 + x_1x_2x_3^{14}x_4^{18}x_5^5 + x_1x_2x_3^{17}x_4^{14}x_5^6 \\
&\quad + x_1x_2x_3^{13}x_4^{18}x_5^6 + x_1^3x_2x_3^{16}x_4^{14}x_5^5 + x_1^3x_2x_3^{13}x_4^{16}x_5^6 + x_1^3x_2x_3^{13}x_4^{18}x_5^4 \\
&\quad + x_1^3x_2x_3^{17}x_4^{14}x_5^4 + Sq^1(u_1) + Sq^2(u_2) + Sq^4(u_4) + Sq^8(u_8) \bmod(P_5^-(\omega_{(1)})),
\end{aligned}$$

where

$$\begin{aligned}
u_1 &= x_1^5 x_2 x_3^{11} x_4^{17} x_5^4 + x_1^5 x_2 x_3^{11} x_4^{13} x_5^8 + x_1^5 x_2 x_3^7 x_4^{17} x_5^8 + x_1^5 x_2 x_3^9 x_4^{21} x_5^2 \\
&\quad + x_1^5 x_2 x_3^{13} x_4^{13} x_5^3 + x_1^3 x_2 x_3^{13} x_4^{13} x_5^8 + x_1^5 x_2 x_3^{11} x_4^{13} x_5^5 + x_1^3 x_2 x_3^{13} x_4^{13} x_5^5 \\
&\quad + x_1 x_2 x_3^{13} x_4^{13} x_5^3 + x_1 x_2 x_3^{11} x_4^{13} x_5^5 + x_1^3 x_2 x_3^{11} x_4^{18} x_5^5 + x_1^3 x_2 x_3^{11} x_4^{14} x_5^9 \\
&\quad + x_1 x_2 x_3^{13} x_4^{14} x_5^9, \\
u_2 &= x_1^3 x_2^2 x_3^{11} x_4^{17} x_5^4 + x_1^3 x_2 x_3^{11} x_4^{18} x_5^4 + x_1^6 x_2 x_3^7 x_4^{21} x_5^2 + x_1^3 x_2^2 x_3^9 x_4^{21} x_5^2 \\
&\quad + x_1^3 x_2 x_3^{10} x_4^{21} x_5^2 + x_1^3 x_2 x_3^9 x_4^{22} x_5^2 + x_1^2 x_2 x_3^{11} x_4^{21} x_5^2 + x_1^2 x_2 x_3^7 x_4^{25} x_5^2 \\
&\quad + x_1^3 x_2 x_3^{11} x_4^{13} x_5^8 + x_1^3 x_2 x_3^{11} x_4^{14} x_5^8 + x_1^3 x_2 x_3^7 x_4^{17} x_5^8 + x_1^3 x_2 x_3^7 x_4^{18} x_5^8 \\
&\quad + x_1^3 x_2 x_3^{11} x_4^{13} x_5^6 + x_1^6 x_2 x_3^{11} x_4^{13} x_5^3 + x_1^3 x_2 x_3^{11} x_4^{14} x_5^5 + x_1^3 x_2 x_3^{13} x_4^{14} x_5^3 \\
&\quad + x_1^3 x_2 x_3^{14} x_4^{13} x_5^3 + x_1^2 x_2 x_3^{11} x_4^{13} x_5^3 + x_1^2 x_2 x_3^{11} x_4^{17} x_5^3 + x_1 x_2 x_3^{11} x_4^{14} x_5^3 \\
&\quad + x_1 x_2 x_3^{11} x_4^{13} x_5^{10} + x_1^2 x_2 x_3^{11} x_4^{18} x_5^5 + x_1^2 x_2 x_3^{11} x_4^{14} x_5^9 + x_1^5 x_2 x_3^{11} x_4^{14} x_5^6, \\
u_4 &= x_1^{10} x_2 x_3^7 x_4^{13} x_5^4 + x_1^4 x_2 x_3^7 x_4^{21} x_5^2 + x_1^3 x_2 x_3^{11} x_4^{14} x_5^5 + x_1^4 x_2 x_3^{11} x_4^{13} x_5^3 \\
&\quad + x_1^3 x_2 x_3^{14} x_4^{13} x_5^4 + x_1^3 x_2 x_3^{11} x_4^{16} x_5^3 + x_1 x_2 x_3^{14} x_4^{13} x_5^3 + x_1 x_2 x_3^{11} x_4^{13} x_5^6 \\
&\quad + x_1 x_2 x_3^{13} x_4^{13} x_5^6 + x_1 x_2 x_3^7 x_4^{13} x_5^{12} + x_1 x_2 x_3^{13} x_4^{14} x_5^5 + x_1 x_2 x_3^{14} x_4^{14} x_5^5 \\
&\quad + x_1 x_2 x_3^{13} x_4^{14} x_5^6 + x_1^3 x_2 x_3^{12} x_4^{14} x_5^5 + x_1^3 x_2 x_3^{11} x_4^{14} x_5^6 + x_1^3 x_2 x_3^{13} x_4^{14} x_5^4, \\
u_8 &= x_1^3 x_2^4 x_3^8 x_4^{13} x_5^3 + x_1^6 x_2 x_3^7 x_4^{13} x_5^4.
\end{aligned}$$

Hence, u is strictly inadmissible. By a similar computation, we obtain

$$\begin{aligned}
v &= x_1^3 x_2^7 x_3 x_4^{20} x_5^8 + x_1^3 x_2 x_3^4 x_4^{17} x_5^8 + x_1^3 x_2 x_3^5 x_4^{16} x_5^8 + x_1^3 x_2 x_3^8 x_4^{17} x_5^4 \\
&\quad + x_1^3 x_2 x_3^{16} x_4^5 x_5^8 + x_1^3 x_2 x_3^{17} x_4^4 x_5^8 + x_1^3 x_2 x_3^2 x_4^{21} x_5^4 + x_1^3 x_2 x_3^4 x_4^{19} x_5^4 \\
&\quad + x_1^3 x_2 x_3^{18} x_4^5 x_5^4 + x_1^3 x_2 x_3^{20} x_4^3 x_5^4 + x_1^3 x_2 x_3^{11} x_4^{20} x_5^4 + x_1^3 x_2 x_3^{16} x_4^5 x_5^4 \\
&\quad + x_1^3 x_2 x_3^{11} x_4^{17} x_5^4 + x_1^3 x_2 x_3^{12} x_4^{19} x_5^4 + x_1^3 x_2 x_3^{12} x_4^{17} x_5^3 + x_1^3 x_2 x_3^{14} x_4^{17} x_5^4 \\
&\quad + x_1^3 x_2 x_3^{14} x_4^{17} x_5^4 + x_1^3 x_2^{17} x_3^4 x_4^7 x_5^8 + x_1^3 x_2^{17} x_3^6 x_4^5 x_5^8 + x_1^3 x_2^{17} x_3^8 x_4^7 x_5^4 \\
&\quad + x_1^3 x_2^{17} x_3^9 x_4^6 x_5^4 + x_1^3 x_2^{18} x_3 x_4^{13} x_5^4 + x_1^3 x_2^{18} x_3^5 x_4^5 x_5^8 + x_1^3 x_2^{18} x_3^5 x_4^9 x_5^4 \\
&\quad + x_1^3 x_2^{18} x_3^{13} x_4 x_5^4 + x_1^3 x_2^{19} x_3 x_4^{12} x_5^4 + x_1^3 x_2^{19} x_3^4 x_4^5 x_5^8 + x_1^3 x_2^{19} x_3^5 x_4^8 x_5^4 \\
&\quad + x_1^3 x_2^{20} x_3 x_4^7 x_5^8 + x_1^3 x_2^{20} x_3 x_4^{11} x_5^4 + x_1^3 x_2^{20} x_3^3 x_4^5 x_5^8 + x_1^3 x_2^{20} x_3^3 x_4^9 x_5^4 \\
&\quad + x_1^3 x_2^{21} x_3^2 x_4^9 x_5^4 + x_1^3 x_2^{21} x_3^3 x_4^4 x_5^8 + x_1^3 x_2^{21} x_3^3 x_4^8 x_5^4 + x_1^3 x_2^{21} x_3^4 x_4^3 x_5^8 \\
&\quad + x_1^3 x_2^{24} x_3^3 x_4^5 x_5^4 + x_1^3 x_2^{24} x_3^5 x_4^3 x_5^4 + x_1^3 x_2^{25} x_3^3 x_4^4 x_5^4 + x_1^3 x_2^{25} x_3^4 x_4^3 x_5^4 \\
&\quad + x_1^4 x_2^7 x_3 x_4^{19} x_5^8 + x_1^4 x_2^7 x_3^{17} x_4^3 x_5^8 + x_1^4 x_2^{11} x_3 x_4^{19} x_5^4 + x_1^4 x_2^{11} x_3^{17} x_4^3 x_5^4 \\
&\quad + x_1^4 x_2^{19} x_3 x_4^7 x_5^8 + x_1^4 x_2^{19} x_3 x_4^{11} x_5^4 + x_1^4 x_2^{19} x_3^3 x_4^5 x_5^8 + x_1^4 x_2^{19} x_3^3 x_4^9 x_5^4 \\
&\quad + x_1^4 x_2^{21} x_3^3 x_4^8 x_5^8 + x_1^4 x_2^{25} x_3^3 x_4^3 x_5^4 + x_1^5 x_2 x_3^2 x_4^{21} x_5^8 + x_1^5 x_2 x_3^2 x_4^{25} x_5^4 \\
&\quad + x_1^5 x_2 x_3^3 x_4^{20} x_5^8 + x_1^5 x_2 x_3^3 x_4^{24} x_5^4 + x_1^5 x_2 x_3^4 x_4^{11} x_5^{16} + x_1^5 x_2 x_3^5 x_4^{10} x_5^{16}
\end{aligned}$$

$$\begin{aligned}
& + x_1^5 x_2^3 x_3^8 x_4^7 x_5^{16} + x_1^5 x_2^3 x_3^9 x_4^6 x_5^{16} + x_1^5 x_2^7 x_3 x_4^{18} x_5^8 + x_1^5 x_2^7 x_3^2 x_4^9 x_5^{16} \\
& + x_1^5 x_2^7 x_3^3 x_4^8 x_5^{16} + x_1^5 x_2^7 x_3^{18} x_4 x_5^8 + x_1^5 x_2^{11} x_3 x_4^{18} x_5^4 + x_1^5 x_2^{11} x_3^2 x_4^5 x_5^{16} \\
& + x_1^5 x_2^{11} x_3^3 x_4^4 x_5^{16} + x_1^5 x_2^{11} x_3^{18} x_4 x_5^4 + x_1^5 x_2^{18} x_3 x_4^7 x_5^8 + x_1^5 x_2^{18} x_3^3 x_4^5 x_5^8 \\
& + x_1^5 x_2^{18} x_3^7 x_4 x_5^8 + x_1^7 x_2^3 x_3^4 x_4^9 x_5^{16} + x_1^7 x_2^3 x_3^4 x_4^{17} x_5^8 + x_1^7 x_2^3 x_3^5 x_4^8 x_5^{16} \\
& + x_1^7 x_2^3 x_3^5 x_4^{16} x_5^8 + x_1^7 x_2^3 x_3^8 x_4^{17} x_5^4 + x_1^7 x_2^3 x_3^9 x_4^{16} x_5^4 + x_1^7 x_2^5 x_3^2 x_4^{17} x_5^8 \\
& + x_1^7 x_2^5 x_3^3 x_4^{16} x_5^8 + x_1^7 x_2^7 x_3 x_4^{16} x_5^8 + x_1^7 x_2^7 x_3^{16} x_4 x_5^8 + x_1^7 x_2^8 x_3 x_4^7 x_5^{16} \\
& + x_1^7 x_2^8 x_3^3 x_4^5 x_5^{16} + x_1^7 x_2^8 x_3^7 x_4 x_5^{16} + x_1^7 x_2^9 x_3 x_4^{18} x_5^4 + x_1^7 x_2^9 x_3^2 x_4^5 x_5^{16} \\
& + x_1^7 x_2^9 x_3^2 x_4^{17} x_5^4 + x_1^7 x_2^9 x_3^3 x_4^4 x_5^{16} + x_1^7 x_2^9 x_3^3 x_4^{16} x_5^4 + x_1^7 x_2^9 x_3^{18} x_4 x_5^4 \\
& + x_1^7 x_2^{11} x_3 x_4^{16} x_5^4 + S q^1(v_1) + S q^2(v_2) + S q^4(v_4) + S q^8(v_8) \bmod(P_5^-(\omega_{(1)})),
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= x_1^3 x_2^7 x_3 x_4^{19} x_5^8 + x_1^3 x_2^7 x_3^{17} x_4^3 x_5^8 + x_1^3 x_2^{11} x_3 x_4^{19} x_5^4 + x_1^3 x_2^{11} x_3^{17} x_4^3 x_5^4 \\
&\quad + x_1^3 x_2^{19} x_3 x_4^7 x_5^8 + x_1^3 x_2^{19} x_3 x_4^{11} x_5^4 + x_1^3 x_2^{19} x_3^3 x_4^5 x_5^8 \\
&\quad + x_1^3 x_2^{19} x_3^3 x_4^9 x_5^4 + x_1^3 x_2^{21} x_3^3 x_4^3 x_5^8 + x_1^3 x_2^{25} x_3^3 x_4^3 x_5^4 \\
&\quad + x_1^7 x_2^{13} x_3 x_4^9 x_5^8 + x_1^7 x_2^{13} x_3^5 x_4^5 x_5^8 + x_1^7 x_2^{13} x_3^9 x_4 x_5^8, \\
v_2 &= x_1^3 x_2^7 x_3^8 x_4^{11} x_5^8 + x_1^3 x_2^7 x_3^9 x_4^{10} x_5^8 + x_1^3 x_2^{11} x_3^5 x_4^{10} x_5^8 + x_1^3 x_2^{11} x_3^6 x_4^9 x_5^8 \\
&\quad + x_1^3 x_2^{11} x_3^8 x_4^7 x_5^8 + x_1^3 x_2^{11} x_3^{10} x_4^5 x_5^8 + x_1^3 x_2^{14} x_3 x_4^{11} x_5^8 + x_1^3 x_2^{14} x_3^3 x_4^9 x_5^8 \\
&\quad + x_1^3 x_2^{14} x_3^{11} x_4 x_5^8 + x_1^5 x_2^7 x_3^2 x_4^{19} x_5^4 + x_1^5 x_2^7 x_3^{18} x_4^3 x_5^4 + x_1^5 x_2^{19} x_3^2 x_4^7 x_5^4 \\
&\quad + x_1^5 x_2^{19} x_3^3 x_4^6 x_5^4 + x_1^5 x_2^{22} x_3^3 x_4^3 x_5^8 + x_1^7 x_2^3 x_3^2 x_4^{21} x_5^4 + x_1^7 x_2^3 x_3^3 x_4^{20} x_5^4 \\
&\quad + x_1^7 x_2^3 x_3^4 x_4^7 x_5^{16} + x_1^7 x_2^3 x_3^5 x_4^6 x_5^{16} + x_1^7 x_2^3 x_3^8 x_4^{11} x_5^8 + x_1^7 x_2^3 x_3^9 x_4^{10} x_5^8 \\
&\quad + x_1^7 x_2^7 x_3 x_4^{18} x_5^4 + x_1^7 x_2^7 x_3^2 x_4^5 x_5^{16} + x_1^7 x_2^7 x_3^4 x_4^{16} + x_1^7 x_2^7 x_3^8 x_4^7 x_5^8 \\
&\quad + x_1^7 x_2^7 x_3^9 x_4^6 x_5^8 + x_1^7 x_2^7 x_3^{18} x_4 x_5^4 + x_1^7 x_2^9 x_3^2 x_4^{11} x_5^8 + x_1^7 x_2^9 x_3^3 x_4^{10} x_5^8 \\
&\quad + x_1^7 x_2^{11} x_3^6 x_4^5 x_5^8 + x_1^7 x_2^{11} x_3^9 x_4^2 x_5^8 + x_1^7 x_2^{11} x_3^{10} x_4 x_5^8 + x_1^7 x_2^{14} x_3 x_4^7 x_5^8 \\
&\quad + x_1^7 x_2^{14} x_3^3 x_4^5 x_5^8 + x_1^7 x_2^{14} x_3^7 x_4 x_5^8, \\
v_4 &= x_1^3 x_2^7 x_3^2 x_4^{19} x_5^4 + x_1^3 x_2^7 x_3^5 x_4^{12} x_5^8 + x_1^3 x_2^7 x_3^{12} x_4^9 x_5^4 + x_1^3 x_2^7 x_3^{18} x_4^3 x_5^4 \\
&\quad + x_1^3 x_2^{11} x_3^4 x_4^{13} x_5^4 + x_1^3 x_2^{13} x_3^4 x_4^{11} x_5^4 + x_1^3 x_2^{13} x_3^5 x_4^6 x_5^8 + x_1^3 x_2^{13} x_3^5 x_4^{10} x_5^4 \\
&\quad + x_1^3 x_2^{13} x_3^6 x_4^5 x_5^8 + x_1^3 x_2^{14} x_3 x_4^{13} x_5^4 + x_1^3 x_2^{14} x_3^9 x_4^5 x_5^4 + x_1^3 x_2^{14} x_3^{13} x_4 x_5^4 \\
&\quad + x_1^3 x_2^{19} x_3^2 x_4^7 x_5^4 + x_1^3 x_2^{19} x_3^3 x_4^6 x_5^4 + x_1^3 x_2^{22} x_3^3 x_4^3 x_5^4 + x_1^5 x_2^3 x_3^2 x_4^{21} x_5^4 \\
&\quad + x_1^5 x_2^3 x_3^3 x_4^{20} x_5^4 + x_1^5 x_2^3 x_3^4 x_4^7 x_5^{16} + x_1^5 x_2^3 x_3^5 x_4^6 x_5^{16} + x_1^5 x_2^7 x_3 x_4^{18} x_5^4 \\
&\quad + x_1^5 x_2^7 x_3^2 x_4^5 x_5^{16} + x_1^5 x_2^7 x_3^3 x_4^4 x_5^{16} + x_1^5 x_2^7 x_3^8 x_4^7 x_5^8 + x_1^5 x_2^7 x_3^9 x_4^6 x_5^8 \\
&\quad + x_1^5 x_2^7 x_3^{18} x_4 x_5^4 + x_1^5 x_2^{11} x_3^5 x_4^6 x_5^8 + x_1^5 x_2^{11} x_3^6 x_4^5 x_5^8 + x_1^5 x_2^{14} x_3 x_4^7 x_5^8 \\
&\quad + x_1^5 x_2^{14} x_3^3 x_4^5 x_5^8 + x_1^5 x_2^{14} x_3^7 x_4 x_5^8 + x_1^{11} x_2^3 x_3^4 x_4^{13} x_5^4 + x_1^{11} x_2^3 x_3^5 x_4^{12} x_5^4
\end{aligned}$$

$$\begin{aligned}
& + x_1^{11}x_2^5x_3^2x_4^{13}x_5^4 + x_1^{11}x_2^5x_3^3x_4^{12}x_5^4 + x_1^{11}x_2^5x_3^4x_4^7x_5^8 + x_1^{11}x_2^5x_3^5x_4^6x_5^8 \\
& + x_1^{11}x_2^7x_3x_4^{12}x_5^4 + x_1^{11}x_2^7x_3^4x_4^5x_5^8 + x_1^{11}x_2^7x_3^5x_4^4x_5^8 + x_1^{11}x_2^7x_3^{12}x_4x_5^4, \\
v_8 = & x_1^3x_2^7x_3^4x_4^{13}x_5^4 + x_1^3x_2^7x_3^{12}x_4^5x_5^4 + x_1^3x_2^{13}x_3^4x_4^7x_5^4 + x_1^3x_2^{13}x_3^5x_4^6x_5^4 \\
& + x_1^3x_2^{14}x_3^5x_4^5x_5^4 + x_1^7x_2^3x_3^4x_4^{13}x_5^4 + x_1^7x_2^3x_3^5x_4^{12}x_5^4 + x_1^7x_2^5x_3^2x_4^{13}x_5^4 \\
& + x_1^7x_2^5x_3^3x_4^{12}x_5^4 + x_1^7x_2^5x_3^4x_4^7x_5^8 + x_1^7x_2^5x_3^5x_4^6x_5^8 + x_1^7x_2^7x_3x_4^{12}x_5^4 \\
& + x_1^7x_2^7x_3^4x_4^5x_5^8 + x_1^7x_2^7x_3^5x_4^4x_5^8 + x_1^7x_2^7x_3^{12}x_4x_5^4 + x_1^7x_2^8x_3x_4^7x_5^8 \\
& + x_1^7x_2^8x_3^3x_4^5x_5^8 + x_1^7x_2^8x_3^5x_4^7x_5^8.
\end{aligned}$$

Hence, v is strictly inadmissible. The lemma is proved. \square

Proof of Lemma 3.3.3. Let x be an admissible monomial of degree 39 in P_5^+ such that $\omega(x) = \omega_{(1)}$. Then $x = X_{\{i,j\}}u^2$ with $1 \leq i < j \leq 5$ and $u \in \mathcal{B}_5(2, 2, 1, 1)$.

Let $y \in \mathcal{B}_5(2, 2, 1, 1)$ such that $X_{\{i,j\}}y^2 \in P_5^+$. By a direct computation using the results in Subsection 3.2, we see that if $X_{\{i,j\}}y^2 \neq d_{39,k}$, $\forall k$, $1 \leq k \leq 485$, then there is a monomial w which is given in one of Lemmas 3.3.4, 3.3.5, and 3.3.6 such that $X_{\{i,j\}}y^2 = wy_1^{2\ell}$ with $y_1 \in P_5$, and $\ell = \max\{r \in \mathbb{Z} : \omega_r(w) > 0\}$. By Theorem 2.7, $X_{\{i,j\}}y^2$ is inadmissible. Since $x = X_{\{i,j\}}u^2$ with $u \in \mathcal{B}_5(2, 2, 1, 1)$, and x is admissible, one can see that $x = d_k$ for some k , $1 \leq k \leq 485$. This implies $\mathcal{B}_5^+(\omega_{(1)}) \subset \{d_k : 1 \leq k \leq 485\}$. The proposition follows. \square

Lemma 3.3.7. $\mathcal{B}_5(\omega_{(2)}) = \mathcal{B}_5^+(\omega_{(2)}) = \emptyset$. This means $QP_5^+(\omega_{(2)}) = 0$.

Proof. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(2)}$. Then $x = X_{\{i,j\}}y^2$ with $1 \leq i < j \leq 5$, and $y \in \mathcal{B}_5(2, 2, 3)$. By a direct computation using the results Proposition 3.2.2, Theorem 2.7, and Lemma 3.3.4(i), (ii), we see that x is a permutation of one of the monomials:

$$\begin{aligned}
& x_1^3x_2^4x_3^8x_4^{11}x_5^{13}, \quad x_1^3x_2^4x_3^9x_4^{9}x_5^{14}, \quad x_1^3x_2^4x_3^{10}x_4^{13}, \quad x_1^3x_2^4x_3^9x_4^{11}x_5^{12}, \\
& x_1^3x_2^5x_3^8x_4^9x_5^{14}, \quad x_1^3x_2^5x_3^8x_4^{10}x_5^{13}, \quad x_1^3x_2^5x_3^8x_4^{11}x_5^{12}, \quad x_1^3x_2^5x_3^9x_4^{10}x_5^{12}, \\
& x_1^7x_2^3x_3^8x_4^8x_5^{13}, \quad x_1^7x_2^3x_3^8x_4^9x_5^{12}, \quad x_1^7x_2^3x_3^8x_4^8x_5^9, \quad x_1^7x_2^8x_3^{11}x_4^4x_5^9, \\
& x_1^7x_2^8x_3^{11}x_4^5x_5^8, \quad x_1^7x_2^9x_3^{10}x_4^4x_5^9, \quad x_1^7x_2^9x_3^{10}x_4^5x_5^8, \quad x_1^{15}x_2^3x_3^4x_4^8x_5^9, \\
& x_1^{15}x_2^3x_3^5x_4^8x_5^8.
\end{aligned}$$

A simple computation shows that

$$\begin{aligned}
x_1^3x_2^5x_3^8x_4^9x_5^{14} = & Sq^1(x_1^5x_2^5x_3x_4^5x_5^{22}) \\
& + Sq^2(x_1^3x_2^5x_3x_4^6x_5^{22} + x_1^3x_2^5x_3^2x_4^5x_5^{22} + x_1^3x_2^6x_3x_4^5x_5^{22}) \\
& + Sq^4(x_1^3x_2^5x_3^4x_4^9x_5^{14} + x_1^3x_2^5x_3^8x_4^5x_5^{14}) \\
& + Sq^8(x_1^3x_2^5x_3^4x_4^5x_5^{14}) \text{ mod}(P_5^-(\omega_{(2)})).
\end{aligned}$$

Hence $[x_1^3x_2^5x_3^8x_4^9x_5^{14}]_{\omega_{(2)}} = [0]_{\omega_{(2)}}$. The others can be proved by a similar computation. The lemma is proved. \square

The following lemma is an immediate consequence of Lemmas 3.3.4(i), (ii), and 3.3.5.

Lemma 3.3.8. *We have $\mathcal{B}_5(\omega_{(3)}) = \mathcal{B}_5^+(\omega_{(3)}) = \emptyset$. Consequently,*

$$QP_5^+(\omega_{(3)}) = 0.$$

Lemma 3.3.9. *The space $QP_5(\omega_{(4)})$ is spanned by the set $\{[d_k]_{\omega_{(4)}} : 486 \leq k \leq 604\}$, where the monomials d_k are determined as in Subsection 4.4.*

Lemma 3.3.10. *The following monomials are strictly inadmissible:*

$$\begin{aligned} & x_1x_2^2x_3^7x_4^{10}x_5^{19}, \quad x_1x_2^7x_3^2x_4^{10}x_5^{19}, \quad x_1x_2^7x_3^{10}x_4^2x_5^{19}, \quad x_1x_2^7x_3^{10}x_4^{19}x_5^2, \\ & x_1^7x_2x_3^2x_4^{10}x_5^{19}, \quad x_1^7x_2x_3^{10}x_4^2x_5^{19}, \quad x_1^7x_2x_3^{10}x_4^{19}x_5^2, \quad x_1x_2^2x_3^7x_4^{11}x_5^{18}, \\ & x_1x_2^7x_3^2x_4^{11}x_5^{18}, \quad x_1x_2^7x_3^{11}x_4^2x_5^{18}, \quad x_1x_2^7x_3^{11}x_4^{18}x_5^2, \quad x_1^7x_2x_3^2x_4^{11}x_5^{18}, \\ & x_1^7x_2x_3^{11}x_4^2x_5^{18}, \quad x_1^7x_2x_3^{11}x_4^{18}x_5^2, \quad x_1^7x_2^2x_3^{11}x_4^{18}x_5^2, \quad x_1^7x_2x_3^2x_4^{18}x_5^2, \\ & x_1x_2^2x_3^7x_4^2x_5^3, \quad x_1x_2^7x_3^2x_4^2x_5^3, \quad x_1x_2^7x_3^2x_4^2x_5^3, \quad x_1x_2^7x_3^2x_4^3x_5^2, \\ & x_1^7x_2x_3^2x_4^2x_5^3, \quad x_1^7x_2x_3^2x_4^2x_5^3, \quad x_1^7x_2x_3^2x_4^3x_5^2, \quad x_1x_2^7x_3^{10}x_4^3x_5^8, \\ & x_1x_2^7x_3^{10}x_4^{18}x_5^3, \quad x_1^7x_2x_3^{10}x_4^3x_5^{18}, \quad x_1^7x_2x_3^{10}x_4^{18}x_5^3, \quad x_1^7x_2^{11}x_3^{17}x_4^2x_5^2. \end{aligned}$$

Proof. We prove the lemma for $x = x_1x_2^2x_3^7x_4^{10}x_5^{19}$. The others are proved by a similar computation. By a direct computation using Cartan formula, we have

$$\begin{aligned} x = & Sq^1(x_1x_2^2x_3^7x_4^5x_5^{23} + x_1x_2^2x_3^9x_4^3x_5^{23} + x_1x_2^2x_3^7x_4^3x_5^{25} + x_1x_2^4x_3^7x_4^3x_5^{23} \\ & + x_1^4x_2x_3^7x_4^3x_5^{23}) + Sq^2(x_1^2x_2^2x_3^7x_4^3x_5^{23}) + Sq^4(x_1x_2^2x_3^{11}x_4^6x_5^{15} + x_1x_2^2x_3^6x_4^3x_5^{23}) \\ & + Sq^8(x_1x_2^2x_3^7x_4^6x_5^{15}) + x_1x_2^2x_3^6x_4^3x_5^{27} + x_1x_2^2x_3^7x_4^3x_5^{26} \text{ mod}(P_5^-(\omega_{(4)})). \end{aligned}$$

This equality implies that x is strictly inadmissible. The lemma follows. \square

Proof of Lemma 3.3.9. Let x be an admissible monomial such that $\omega(x) = \omega_{(4)}$. Then, $x = X_{\{i,j\}}u^2$ with $u \in \mathcal{B}_5(4, 1, 1)$.

Let $y \in \mathcal{B}_5(4, 1, 1)$ such that $X_{\{i,j\}}y^2 \in P_5^+$. By a direct computation using Proposition 3.2.2, we see that if $X_{\{i,j\}}y^2 \neq d_k$, $486 \leq k \leq 604$, then there is a monomial w which is given in one of Lemmas 3.3.5(i), (iii), 3.3.6, and 3.3.10 such that $X_{\{i,j\}}y^2 = wy_1^{2^\ell}$ with suitable monomial $y_1 \in P_5$, and $\ell = \max\{r \in \mathbb{Z} : \omega_r(w) > 0\}$. By Theorem 2.7, $X_{\{i,j\}}y^2$ is inadmissible. Since $x = X_{\{i,j\}}u^2$ with $u \in \mathcal{B}_5(4, 1, 1)$ and x admissible, one gets $x = d_k$, for some k , $486 \leq k \leq 604$. This concludes the proof. \square

Lemma 3.3.11. *The space $QP_5(\omega_{(5)})$ is spanned by the set $\{[d_k]_{\omega_{(5)}} : 605 \leq k \leq 609\}$, where the monomials d_k are determined as in Subsection 4.4.*

By a direct computation, one gets the following.

Lemma 3.3.12. *All permutations of the following monomials are strictly inadmissible:*

$$x_1x_2^6x_3^{10}x_4^{11}x_5^{11}, \quad x_1x_2^7x_3^{10}x_4^{10}x_5^{11}.$$

Proof of Lemma 3.3.11. Let x be an admissible monomial such that $\omega(x) = \omega_{(5)}$. Then $x = x_i x_j x_\ell y^2$ with $1 \leq i < j < \ell \leq 5$ and $y \in \mathcal{B}_5(4, 1, 3)$.

Let $z \in \mathcal{B}_5(4, 1, 3)$ such that $x_i x_j x_\ell z^2 \in P_5^+$. By a direct computation using the results in Subsection 3.2, we see that if $x_i x_j x_\ell z^2 \neq d_k$, $605 \leq k \leq 609$, then there is a monomial w which is given in one of Lemmas 3.3.5(i), (iii), 3.3.6, and 3.3.12 such that $x_i x_j x_\ell z^2 = w z_1^{2r}$ with suitable monomials $z_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $x_i x_j x_\ell z^2$ is inadmissible. Since $x = x_i x_j x_\ell y^2$ and x is admissible, one gets $x = d_k$ for some k , $605 \leq k \leq 609$. This implies $\mathcal{B}_5^+(\omega_{(5)}) \subset \{d_k : 605 \leq k \leq 609\}$. \square

Lemma 3.3.13. *The space $QP_5(\omega_{(6)})$ is spanned by the set $\{[d_k]_{\omega_{(6)}} : 610 \leq k \leq 649\}$, where the monomials d_k are determined as in Subsection 4.4.*

Lemma 3.3.14. *All permutations of the following monomials are strictly inadmissible:*

$$x_1 x_2^3 x_3^6 x_4^{14} x_5^{15}, \quad x_1^3 x_2^5 x_3^2 x_4^{14} x_5^{15}, \quad x_1^3 x_2^{13} x_3^2 x_4^6 x_5^{15}.$$

Proof. We prove the lemma for $x = x_1 x_2^3 x_3^6 x_4^{14} x_5^{15}$. The others are proved by a similar computation. A direct computation shows that

$$\begin{aligned} x_1 x_2^3 x_3^6 x_4^{14} x_5^{15} &= Sq^1(x_1 x_2^3 x_3^6 x_4^{13} x_5^{15} + x_1 x_2^3 x_3^{12} x_4^7 x_5^{15} + x_1^4 x_2^{12} x_3^3 x_4^7 x_5^{15} \\ &\quad + x_1^2 x_2^9 x_3^5 x_4^7 x_5^{15} + x_1^4 x_2^3 x_3^5 x_4^{11} x_5^{15} + x_1^4 x_2^3 x_3^9 x_4^7 x_5^{15} + x_1^4 x_2^9 x_3^3 x_4^{15} x_5^{15}) \\ &\quad + Sq^2(x_1^2 x_2^3 x_3^6 x_4^{11} x_5^{15} + x_1^2 x_2^3 x_3^{10} x_4^7 x_5^{15}) \\ &\quad + Sq^4(x_1^2 x_2^5 x_3^6 x_4^7 x_5^{15}) \text{ mod}(P_5^-(\omega_{(6)})). \end{aligned}$$

Hence x is strictly inadmissible. The lemma follows. \square

Lemma 3.3.15. *The following monomials are strictly inadmissible:*

$$\begin{aligned} &x_1 x_2^3 x_3^{14} x_4^7 x_5^{14}, \quad x_1 x_2^3 x_3^{14} x_4^7 x_5^7, \quad x_1^3 x_2 x_3^{14} x_4^7 x_5^{14}, \quad x_1^3 x_2 x_3^{14} x_4^{14} x_5^7, \\ &x_1^3 x_2^5 x_3^2 x_4^{14} x_5^{15}, \quad x_1^3 x_2^5 x_3^2 x_4^{15} x_5^{14}, \quad x_1^3 x_2^5 x_3^{14} x_4^2 x_5^{15}, \quad x_1^3 x_2^5 x_3^{14} x_4^{15} x_5^2, \\ &x_1^3 x_2^5 x_3^{15} x_4^2 x_5^{14}, \quad x_1^3 x_2^5 x_3^5 x_4^{14} x_5^2, \quad x_1^3 x_2^5 x_3^5 x_4^2 x_5^{14}, \quad x_1^3 x_2^5 x_3^5 x_4^4 x_5^2, \\ &x_1^{15} x_2^3 x_3^5 x_4^2 x_5^{14}, \quad x_1^{15} x_2^3 x_3^5 x_4^{14} x_5^2, \quad x_1^3 x_2^5 x_3^6 x_4^{10} x_5^{15}, \quad x_1^3 x_2^5 x_3^6 x_4^{15} x_5^{10}, \\ &x_1^3 x_2^5 x_3^6 x_4^6 x_5^{15}, \quad x_1^3 x_2^5 x_3^6 x_4^{15} x_5^6, \quad x_1^3 x_2^5 x_3^{15} x_4^6 x_5^{10}, \quad x_1^3 x_2^5 x_3^{15} x_4^{10} x_5^6, \\ &x_1^3 x_2^{15} x_3^5 x_4^6 x_5^{10}, \quad x_1^3 x_2^{15} x_3^5 x_4^6 x_5^{10}, \quad x_1^5 x_2^3 x_3^5 x_4^6 x_5^{10}, \quad x_1^{15} x_2^3 x_3^5 x_4^{10} x_5^6, \\ &x_1^3 x_2^5 x_3^6 x_4^{11} x_5^{14}, \quad x_1^3 x_2^5 x_3^6 x_4^{14} x_5^{11}, \quad x_1^3 x_2^5 x_3^6 x_4^6 x_5^{11}, \quad x_1^3 x_2^5 x_3^{14} x_4^{11} x_5^6, \\ &x_1^3 x_2^5 x_3^7 x_4^{10} x_5^{14}, \quad x_1^3 x_2^5 x_3^7 x_4^{14} x_5^{10}, \quad x_1^3 x_2^5 x_3^{10} x_4^7 x_5^{14}, \quad x_1^3 x_2^5 x_3^{10} x_4^{14} x_5^7, \\ &x_1^3 x_2^5 x_3^{14} x_4^7 x_5^{10}, \quad x_1^3 x_2^5 x_3^{14} x_4^{10} x_5^7, \quad x_1^3 x_2^5 x_3^{14} x_4^3 x_5^{14}, \quad x_1^3 x_2^5 x_3^{14} x_4^{14} x_5^3, \\ &x_1^3 x_2^{13} x_3^3 x_4^6 x_5^{14}, \quad x_1^3 x_2^{13} x_3^3 x_4^{14} x_5^6, \quad x_1^3 x_2^{13} x_3^6 x_4^3 x_5^{14}, \quad x_1^3 x_2^{13} x_3^6 x_4^{14} x_5^3, \\ &x_1^3 x_2^{13} x_3^{14} x_4^3 x_5^6, \quad x_1^3 x_2^{13} x_3^{14} x_4^6 x_5^3, \quad x_1^3 x_2^{13} x_3^2 x_4^6 x_5^{15}, \quad x_1^3 x_2^{13} x_3^2 x_4^{15} x_5^6, \\ &x_1^3 x_2^{13} x_3^6 x_4^2 x_5^{15}, \quad x_1^3 x_2^{13} x_3^6 x_4^2 x_5^2, \quad x_1^3 x_2^{13} x_3^{15} x_4^2 x_5^6, \quad x_1^3 x_2^{13} x_3^{15} x_4^6 x_5^2, \\ &x_1^3 x_2^{15} x_3^{13} x_4^2 x_5^6, \quad x_1^3 x_2^{15} x_3^{13} x_4^6 x_5^2, \quad x_1^5 x_2^3 x_3^{13} x_4^2 x_5^6, \quad x_1^{15} x_2^3 x_3^{13} x_4^6 x_5^2, \\ &x_1^3 x_2^{13} x_3^2 x_4^7 x_5^{14}, \quad x_1^3 x_2^{13} x_3^2 x_4^7 x_5^5, \quad x_1^3 x_2^{13} x_3^7 x_4^2 x_5^{14}, \quad x_1^3 x_2^{13} x_3^7 x_4^{14} x_5^2, \\ &x_1^3 x_2^{13} x_3^{14} x_4^2 x_5^7, \quad x_1^3 x_2^{13} x_3^{14} x_4^7 x_5^2, \quad x_1^3 x_2^{13} x_3^6 x_4^6 x_5^{11}, \quad x_1^3 x_2^{13} x_3^6 x_4^{11} x_5^6, \\ &x_1^3 x_2^{13} x_3^6 x_4^7 x_5^{10}, \quad x_1^3 x_2^{13} x_3^6 x_4^{10} x_5^7, \quad x_1^3 x_2^{13} x_3^7 x_4^6 x_5^{10}, \quad x_1^3 x_2^{13} x_3^7 x_4^{10} x_5^6. \end{aligned}$$

Proof. We prove the lemma for the monomials $x = x_1^3x_2^5x_3^6x_4^{11}x_5^{14}$, and $y = x_1^7x_2^9x_3^3x_4^6x_5^{14}$. The others can be proved by a similar computation. By a direct computation, we have

$$\begin{aligned} x &= Sq^1(x_1^3x_2^6x_3^5x_4^{11}x_5^{13} + x_1^3x_2^5x_3^3x_4^{13}x_5^{14} + x_1^3x_2^5x_3^6x_4^{11}x_5^{13} + x_1^3x_2^6x_3^9x_4^7x_5^{13} \\ &\quad + x_1^3x_2^{10}x_3^5x_4^7x_5^{13} + x_1^6x_2^3x_3^5x_4^{11}x_5^{13} + x_1^6x_2^5x_3^3x_4^{11}x_5^{13}) \\ &+ Sq^2(x_1^3x_2^5x_3^3x_4^{13}x_5^{13} + x_1^3x_2^6x_3^3x_4^{11}x_5^{14} + x_1^3x_2^3x_3^5x_4^{13}x_5^{13} + x_1^3x_2^3x_3^6x_4^{11}x_5^{14} \\ &\quad + x_1^5x_2^3x_3^7x_4^{17} + x_1^5x_2^3x_3^9x_4^7x_5^{13} + x_1^5x_2^3x_3^9x_4^9x_5^{11} + x_1^5x_2^5x_3^3x_4^7x_5^{17} \\ &\quad + x_1^5x_2^9x_3^3x_4^7x_5^{13} + x_1^5x_2^9x_3^3x_4^9x_5^{11}) \\ &+ Sq^4(x_1^3x_2^5x_3^3x_4^{11}x_5^{13} + x_1^3x_2^9x_3^3x_4^7x_5^{13} + x_1^3x_2^5x_3^3x_4^7x_5^{17} + x_1^3x_2^3x_3^5x_4^{11}x_5^{13} \\ &\quad + x_1^3x_2^5x_3^6x_4^7x_5^{14} + x_1^3x_2^3x_3^9x_4^7x_5^{13} + x_1^3x_2^3x_3^5x_4^7x_5^{17} + x_1^3x_2^3x_3^9x_4^9x_5^{11} \\ &\quad + x_1^3x_2^9x_3^3x_4^9x_5^{11}) + x_1^3x_2^5x_3^3x_4^{14}x_5^{14} + x_1^3x_2^3x_3^6x_4^{13}x_5^{14} \text{ mod}(P_5^-(\omega_{(6)})). \end{aligned}$$

This relation shows that x is strictly inadmissible.

By a similar computation, we obtain

$$\begin{aligned} y &= Sq^1(x_1^7x_2^7x_3^3x_4^8x_5^{13} + x_1^7x_2^7x_3^5x_4^6x_5^{13} + x_1^7x_2^7x_3^8x_4^5x_5^{11} + x_1^7x_2^7x_3^5x_4^8x_5^{11} \\ &\quad + x_1^3x_2^7x_3^9x_4^6x_5^{13} + x_1^3x_2^7x_3^5x_4^{10}x_5^{13} + x_1^7x_2^9x_3^{10}x_4^5x_5^7 + x_1^7x_2^9x_3^5x_4^{10}x_5^7 \\ &\quad + x_1^7x_2^{12}x_3^9x_4^3x_5^7 + x_1^7x_2^9x_3^{12}x_4^3x_5^7 + x_1^7x_2^{12}x_3^3x_4^9x_5^7 + x_1^7x_2^9x_3^3x_4^{12}x_5^7 \\ &\quad + x_1^7x_2^3x_3^{12}x_4^9x_5^7 + x_1^7x_2^3x_3^9x_4^{12}x_5^7 + x_1^7x_2^3x_3^{12}x_4^5x_5^{11} + x_1^7x_2^3x_3^9x_4^6x_5^{13} \\ &\quad + x_1^7x_2^3x_3^5x_4^{12}x_5^{11} + x_1^7x_2^3x_3^6x_4^9x_5^{13}) \\ &+ Sq^2(x_1^7x_2^3x_3^6x_4^7x_5^{14} + x_1^3x_2^{11}x_3^3x_4^6x_5^{14} + x_1^3x_2^7x_3^3x_4^{10}x_5^{14} + x_1^7x_2^7x_3^6x_4^6x_5^{11} \\ &\quad + x_1^3x_2^{11}x_3^6x_4^6x_5^{11} + x_1^7x_2^3x_3^{10}x_4^6x_5^7 + x_1^7x_2^3x_3^{10}x_4^6x_5^{11} + x_1^7x_2^3x_3^6x_4^{10}x_5^{11} \\ &\quad + x_1^7x_2^{10}x_3^{10}x_4^3x_5^7 + x_1^7x_2^7x_3^3x_4^6x_5^{14} + x_1^7x_2^{10}x_3^6x_4^{10}x_5^7) \\ &+ Sq^4(x_1^5x_2^7x_3^3x_4^5x_5^{14} + x_1^5x_2^7x_3^6x_4^6x_5^{11} + x_1^3x_2^7x_3^6x_4^6x_5^{13} + x_1^{11}x_2^5x_3^6x_4^6x_5^7) \\ &+ Sq^8(x_1^7x_2^5x_3^6x_4^6x_5^7) + x_1^7x_2^3x_3^9x_4^6x_5^{14} + x_1^7x_2^3x_3^6x_4^9x_5^{14} + x_1^3x_2^{13}x_3^3x_4^6x_5^{14} \\ &+ x_1^3x_2^{13}x_3^6x_4^6x_5^{11} \text{ mod}(P_5^-(\omega_{(6)})). \end{aligned}$$

Hence, y is strictly inadmissible. The lemma is proved. \square

Proof of Lemma 3.3.13. Let x be an admissible monomial such that $\omega(x) = \omega_{(6)}$. Then $x = x_i x_j x_\ell y^2$ with $1 \leq i < j < \ell \leq 5$ and $y \in \mathcal{B}_5(4, 3, 2)$.

Let $u \in \mathcal{B}_5(4, 3, 2)$ such that $x_i x_j x_\ell u^2 \in P_5^+$. By a direct computation using Proposition 3.2.2, we see that if $x_i x_j x_\ell u^2 \neq d_k$, $610 \leq t \leq 649$. then there is a monomial w which is given in one of Lemmas 3.3.5, 3.3.6, 3.3.14, and 3.3.15 such that $x_i x_j x_\ell u^2 = w u_1^{2r}$ with suitable monomial $u_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $x_i x_j x_\ell u^2$ is inadmissible. Since $x = x_i x_j x_\ell y^2$, and x is admissible, one gets $x = d_k$ for some k , $610 \leq k \leq 649$. This implies $\mathcal{B}_5^+(\omega_{(6)}) \subset \{d_k : 610 \leq k \leq 649\}$. \square

Proof of Proposition 3.3.2. From Lemmas 3.3.3, 3.3.7, 3.3.8, 3.3.9, 3.3.11, and 3.3.13, we see that the space $\text{Ker}((\widetilde{Sq}_*)_{(5,39)}) \cap (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^+)_3$ is spanned by the set $\{[d_k] : 1 \leq k \leq 649\}$. Furthermore, the set $\{[d_k] : 1 \leq k \leq 649\}$ is linearly independent in $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_3$. Indeed, suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq k \leq 649} \gamma_k d_k \equiv 0,$$

where $\gamma_k \in \mathbb{F}_2$, for $1 \leq k \leq 649$. We explicitly compute $p_{(i,I)}(\mathcal{S})$ in terms of the admissible monomials in $(P_4^+)_3$. Computing directly from the relations

$$p_{(i,I)}(\mathcal{S}) \equiv 0, \quad \forall (i; I) \in \mathcal{N}_5, \quad \ell(I) > 0,$$

we obtain $\gamma_k = 0$ for all k . The proposition follows. \square

Recall that $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5^0)_3 = 915$. Hence, from Theorem 3.1.1 and Proposition 3.3.2, we get $\dim(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_3 = 2130$. The main result is completely proved.

4. Appendix

In this section, we list all admissible monomials of degrees 18, 39 in P_4 and P_5 . We order a set of some monomials in P_s by using the order as in Definition 2.4.

4.1. The admissible monomials of degree 18 in P_4

$\mathcal{B}_4(18)$ is the set of 126 monomials:

1. $x_2x_3^2x_4^{15}$	2. $x_2x_3^{15}x_4^2$	3. $x_2x_3^3x_4^{14}$	4. $x_2x_3^{14}x_4^3$
5. $x_2^3x_3x_4^{14}$	6. $x_2^3x_4^{15}$	7. $x_2^{15}x_3^3$	8. $x_2^3x_4^{15}$
9. $x_2^3x_3^{15}$	10. $x_2^{15}x_4^3$	11. $x_2^{15}x_3^3$	12. $x_2x_3^6x_4^{11}$
13. $x_2x_3^7x_4^{10}$	14. $x_2^7x_3x_4^{10}$	15. $x_2^7x_4^{11}$	16. $x_2^7x_4^{11}$
17. $x_2^7x_3^{11}$	18. $x_2^3x_3^{13}x_4^2$	19. $x_2^3x_3^5x_4^{10}$	20. $x_2^3x_3^3x_4^{12}$
21. $x_2^{15}x_3x_4^2$	22. $x_1x_3^2x_4^{15}$	23. $x_1x_3^{15}x_4^2$	24. $x_1x_2^2x_4^{15}$
25. $x_1x_2^2x_3^{15}$	26. $x_1x_2^{15}x_4^2$	27. $x_1x_2^{15}x_3^2$	28. $x_1x_3^3x_4^{14}$
29. $x_1x_3^{14}x_4^3$	30. $x_1x_2^3x_4^{14}$	31. $x_1x_2^3x_3^{14}$	32. $x_1x_2^{14}x_4^3$
33. $x_1x_2^2x_3^{14}$	34. $x_1x_3^6x_4^{11}$	35. $x_1x_2^6x_4^{11}$	36. $x_1x_2^6x_3^{11}$
37. $x_1x_3^7x_4^{10}$	38. $x_1x_2^7x_4^{10}$	39. $x_1x_2^7x_3^3$	40. $x_1x_2x_3^2x_4^{14}$
41. $x_1x_2x_3^{14}x_4^2$	42. $x_1x_2^2x_3x_4^{14}$	43. $x_1x_2^{14}x_3x_4^2$	44. $x_1x_2^2x_3^{13}x_4^2$
45. $x_1x_2x_3^6x_4^{10}$	46. $x_1x_2^6x_3x_4^{10}$	47. $x_1x_2^2x_3^3x_4^{12}$	48. $x_1x_2^2x_3^{12}x_4^3$
49. $x_1x_2^3x_3^2x_4^{12}$	50. $x_1x_2^3x_3^{12}x_4^2$	51. $x_1x_2^2x_3^4x_4^{11}$	52. $x_1x_2^2x_3^5x_4^{10}$
53. $x_1x_2^2x_3^7x_4^8$	54. $x_1x_2^7x_3^2x_4^8$	55. $x_1x_2^3x_3^4x_4^{10}$	56. $x_1x_2^3x_3^6x_4^8$
57. $x_1x_2x_3^3x_4^{13}$	58. $x_1x_2^3x_3x_4^{13}$	59. $x_1x_2^3x_3^3x_4$	60. $x_1x_2x_3x_4^{15}$
61. $x_1x_2x_3^{15}x_4$	62. $x_1x_2^2x_3x_4$	63. $x_1x_2^2x_3^5x_4^9$	64. $x_1x_2^3x_3^7x_4^7$
65. $x_1x_2^7x_3^2x_4^7$	66. $x_1x_2^7x_3^7x_4^3$	67. $x_1^3x_3x_4^{14}$	68. $x_1^3x_2x_4^{14}$
69. $x_1^3x_2x_3^{14}$	70. $x_1^3x_4^{15}$	71. $x_1^3x_3^{15}$	72. $x_1^3x_2^{15}$
73. $x_1^3x_3^{13}x_4^2$	74. $x_1^3x_2^{13}x_4^2$	75. $x_1^3x_2^{13}x_3^2$	76. $x_1^3x_3^5x_4^{10}$
77. $x_1^3x_2^5x_4^{10}$	78. $x_1^3x_2^5x_3^{10}$	79. $x_1^3x_3^3x_4^{12}$	80. $x_1^3x_2^3x_4^{12}$

81. $x_1^3x_2^3x_3^{12}$	82. $x_1^3x_2x_3^2x_4^{12}$	83. $x_1^3x_2x_3^{12}x_4^2$	84. $x_1^3x_2x_3^4x_4^{10}$
85. $x_1^3x_2x_3^6x_4^8$	86. $x_1^3x_2^5x_3^2x_4^8$	87. $x_1^3x_2^3x_3^4x_4^8$	88. $x_1^3x_2^5x_3^8x_4^2$
89. $x_1^3x_2x_3x_4^{13}$	90. $x_1^3x_2x_3^{13}x_4$	91. $x_1^3x_2^{13}x_3x_4$	92. $x_1^3x_2x_3^9x_4^9$
93. $x_1^3x_2^5x_3x_4^9$	94. $x_1^3x_2^5x_3^9x_4$	95. $x_1^3x_2x_3^7x_4^7$	96. $x_1^3x_2^7x_3x_4^7$
97. $x_1^3x_2^7x_3^7x_4$	98. $x_1^3x_2^3x_3^5x_4$	99. $x_1^3x_2^3x_3^5x_4^5$	100. $x_1^3x_2^5x_3^3x_4^7$
101. $x_1^3x_2^5x_3^7x_4^3$	102. $x_1^3x_2^7x_3^3x_4^5$	103. $x_1^3x_2^7x_5^3x_4^3$	104. $x_1^3x_2^5x_3^3x_4^5$
105. $x_1^7x_3x_4^{10}$	106. $x_1^7x_2x_4^{10}$	107. $x_1^7x_2x_3^{10}$	108. $x_1^7x_4^{11}$
109. $x_1^7x_3^{11}$	110. $x_1^7x_2^{11}$	111. $x_1^7x_2x_3^2x_4^8$	112. $x_1^7x_2x_3^3x_4^7$
113. $x_1^7x_2x_3^7x_4^3$	114. $x_1^7x_2^3x_3x_4^7$	115. $x_1^7x_2^3x_3^3x_4$	116. $x_1^7x_2^7x_3x_4^3$
117. $x_1^7x_2^7x_3^3x_4$	118. $x_1^7x_2^3x_3^3x_4^5$	119. $x_1^7x_2^3x_3^5x_4^3$	120. $x_1^{15}x_3x_4^2$
121. $x_1^{15}x_2x_4^2$	122. $x_1^{15}x_2x_3^2$	123. $x_1^{15}x_4^3$	124. $x_1^{15}x_3^3$
125. $x_1^{15}x_2^3$	126. $x_1^{15}x_2x_3x_4$		

4.2. The admissible monomials of degree 18 in P_5

We have $\mathcal{B}_5(18) = \mathcal{B}_5^0(18) \cup \mathcal{B}_5^+(18)$, where $\mathcal{B}_5^0(18) = \Phi^0(\mathcal{B}_4(18))$, $|\mathcal{B}_5^0(18)| = 450$, and $\mathcal{B}_5^+(18) = \mathcal{B}_5^+(2, 2, 1, 1) \cup \mathcal{B}_5^+(2, 2, 3) \cup \mathcal{B}_5^+(2, 4, 2) \cup \mathcal{B}_5^+(4, 1, 1) \cup \mathcal{B}_5^+(4, 1, 3) \cup \mathcal{B}_5^+(4, 3, 2)$.

$\mathcal{B}_5^+(2, 2, 1, 1)$ is the set of 25 monomials:

1. $x_1x_2x_3^2x_4^2x_5^{12}$	2. $x_1x_2x_3^2x_4^{12}x_5^2$	3. $x_1x_2^2x_3x_4^2x_5^{12}$	4. $x_1x_2^2x_3x_4^{12}x_5^2$
5. $x_1x_2^2x_3^{12}x_4x_5^2$	6. $x_1x_2x_3^2x_4^4x_5^{10}$	7. $x_1x_2^2x_3x_4^4x_5^{10}$	8. $x_1x_2^2x_3^4x_4x_5^{10}$
9. $x_1x_2x_3^2x_4^6x_5^8$	10. $x_1x_2x_3^6x_4^2x_5^8$	11. $x_1x_2^2x_3x_4^6x_5^8$	12. $x_1x_2^6x_3x_4^2x_5^8$
13. $x_1x_2^2x_3^5x_4^2x_5^8$	14. $x_1x_2^2x_3^5x_4^8x_5^2$	15. $x_1x_2^2x_3^3x_4^4x_5^8$	16. $x_1x_2^3x_3^2x_4^4x_5^8$
17. $x_1x_3^2x_3^4x_4^2x_5^8$	18. $x_1x_3^2x_4^4x_8x_5^2$	19. $x_1^3x_2x_3^2x_4^4x_5^8$	20. $x_1^3x_2x_3^4x_4^2x_5^8$
21. $x_1^3x_2x_3^4x_4^8x_5^2$	22. $x_1x_2^2x_3^4x_4^3x_5^8$	23. $x_1x_2^2x_3^4x_8x_5^3$	24. $x_1^3x_2^4x_3x_4^2x_5^8$
25. $x_1x_2^2x_3^9x_4^2x_5^2$			

$\mathcal{B}_5^+(2, 2, 3)$ is the set of 15 monomials:

1. $x_1x_2^2x_3^4x_4^4x_5^7$	2. $x_1x_2^2x_3^4x_4^7x_5^4$	3. $x_1x_2^2x_3^7x_4^4x_5^4$	4. $x_1x_2^7x_3^2x_4^4x_5^4$
5. $x_1^7x_2x_3^2x_4^4x_5^4$	6. $x_1x_2^2x_3^4x_5^4x_6^6$	7. $x_1x_2^2x_3^5x_4^4x_5^6$	8. $x_1x_2^2x_3^5x_4^6x_5^4$
9. $x_1x_2^2x_3^4x_4^4x_5^6$	10. $x_1x_2^3x_3^4x_6^4x_5^4$	11. $x_1x_3^2x_3^6x_4^4x_5^4$	12. $x_1^3x_2x_3^4x_4^4x_5^6$
13. $x_1^3x_2x_3^4x_4^4x_5^4$	14. $x_1^3x_2x_3^6x_4^4x_5^4$	15. $x_1^3x_2^5x_3^2x_4^4x_5^4$	

$\mathcal{B}_5^+(2, 4, 2)$ is the set of 10 monomials:

1. $x_1x_2^2x_3^3x_4^6x_5^6$	2. $x_1x_2^3x_3^2x_4^6x_5^6$	3. $x_1x_2^3x_3^6x_4^2x_5^6$	4. $x_1x_2^3x_3^6x_4^6x_5^2$
5. $x_1^3x_2x_3^3x_4^6x_5^6$	6. $x_1^3x_2x_3^6x_4^2x_5^6$	7. $x_1^3x_2x_3^6x_4^6x_5^2$	8. $x_1^3x_2^5x_3^2x_4^2x_5^6$
9. $x_1^3x_2^5x_3^2x_4^6x_5^2$	10. $x_1^3x_2^5x_3^6x_4^2x_5^2$		

$\mathcal{B}_5^+(4, 1, 1, 1)$ is the set of 40 monomials:

1. $x_1x_2x_3x_4x_5^{14}$	2. $x_1x_2x_3x_4^{14}x_5$	3. $x_1x_2x_3^{14}x_4x_5$	4. $x_1x_2^{14}x_3x_4x_5$
5. $x_1x_2x_3x_4^2x_5^{13}$	6. $x_1x_2x_3^2x_4x_5^{13}$	7. $x_1x_2x_3^2x_4^3x_5^{13}$	8. $x_1x_2^2x_3x_4x_5^{13}$
9. $x_1x_2^2x_3x_4^{13}x_5$	10. $x_1x_2^2x_3^{13}x_4x_5$	11. $x_1x_2x_3x_4^3x_5^{12}$	12. $x_1x_2x_3^3x_4x_5^{12}$

- | | | | |
|------------------------------|------------------------------|------------------------------|------------------------------|
| 13. $x_1x_2x_3^3x_4^{12}x_5$ | 14. $x_1x_2^3x_3x_4x_5^{12}$ | 15. $x_1x_2^3x_3x_4^{12}x_5$ | 16. $x_1x_2^3x_3^{12}x_4x_5$ |
| 17. $x_1^3x_2x_3x_4x_5^{12}$ | 18. $x_1^3x_2x_3x_4^{12}x_5$ | 19. $x_1^3x_2x_3^{12}x_4x_5$ | 20. $x_1x_2x_3^2x_4^5x_5^2$ |
| 21. $x_1x_2^2x_3x_4^5x_5^9$ | 22. $x_1x_2^2x_3^5x_4x_5^9$ | 23. $x_1x_2^2x_3^5x_4^9x_5$ | 24. $x_1x_2x_3^4x_4^9x_5$ |
| 25. $x_1x_2^3x_3x_4^4x_5^9$ | 26. $x_1x_2^3x_3^4x_4x_5^9$ | 27. $x_1x_2^3x_3^4x_4^9x_5$ | 28. $x_1^3x_2x_3x_4^4x_5^9$ |
| 29. $x_1^3x_2x_3^4x_4x_5^9$ | 30. $x_1^3x_2x_3^4x_4^9x_5$ | 31. $x_1x_2x_3^3x_4^5x_5^8$ | 32. $x_1x_2^3x_3x_4^5x_5^8$ |
| 33. $x_1x_2^3x_3^5x_4x_5^8$ | 34. $x_1x_2^3x_3^5x_4^8x_5$ | 35. $x_1^3x_2x_3x_4^5x_5^8$ | 36. $x_1^3x_2x_3^5x_4x_5^8$ |
| 37. $x_1^3x_2x_3^5x_4^8x_5$ | 38. $x_1^3x_2x_3x_4x_5^8$ | 39. $x_1^3x_2^5x_3x_4^8x_5$ | 40. $x_1^3x_2^5x_3^8x_4x_5$ |

$\mathcal{B}_5^+(4, 1, 3)$ is the set of 10 monomials:

- | | | | |
|------------------------------|------------------------------|------------------------------|------------------------------|
| 1. $x_1x_2^2x_3^5x_4^5x_5^5$ | 2. $x_1x_2^3x_3^4x_4^5x_5^5$ | 3. $x_1x_2^3x_3^5x_4^4x_5^5$ | 4. $x_1x_2^3x_3^5x_4^5x_5^4$ |
| 5. $x_1^3x_2x_3^4x_4^5x_5^5$ | 6. $x_1^3x_2x_3^5x_4^4x_5^5$ | 7. $x_1^3x_2x_3^5x_4^5x_5^4$ | 8. $x_1^3x_2x_3x_4^4x_5^5$ |
| 9. $x_1^3x_2^5x_3x_4^5x_5^4$ | 10. $x_1^3x_2x_3^7x_4^7x_5$ | 11. $x_1x_2^7x_3x_4x_5^2$ | 12. $x_1x_2^7x_3x_4^2x_5$ |

$\mathcal{B}_5^+(4, 3, 2)$ is the set of 180 monomials:

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|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 1. $x_1x_2x_3^2x_4^7x_5^7$ | 2. $x_1x_2x_3^7x_4^2x_5^7$ | 3. $x_1x_2x_3^7x_4^7x_5^2$ | 4. $x_1x_2^2x_3x_4^7x_5^7$ |
| 5. $x_1x_2^2x_3^7x_4x_5^7$ | 6. $x_1x_2^2x_3^7x_4^7x_5$ | 7. $x_1x_2^7x_3x_4x_5^2$ | 8. $x_1x_2^7x_3x_4^2x_5^2$ |
| 9. $x_1x_2^2x_3^2x_4x_5^7$ | 10. $x_1x_2^7x_3^2x_4^7x_5$ | 11. $x_1x_2^7x_3^2x_4x_5^2$ | 12. $x_1x_2^7x_3^2x_4^2x_5$ |
| 13. $x_1^7x_2x_3x_4x_5^7$ | 14. $x_1^7x_2x_3x_4^7x_5^2$ | 15. $x_1^7x_2x_3^2x_4x_5^7$ | 16. $x_1^7x_2x_3^2x_4^7x_5$ |
| 17. $x_1^7x_2x_3^7x_4x_5^2$ | 18. $x_1^7x_2x_3^7x_4^2x_5$ | 19. $x_1^7x_2x_3x_4x_5^2$ | 20. $x_1^7x_2x_3x_4^2x_5$ |
| 21. $x_1x_2x_3^3x_4^6x_5^7$ | 22. $x_1x_2x_3^3x_4^7x_5^6$ | 23. $x_1x_2x_3^6x_4^3x_5^7$ | 24. $x_1x_2x_3x_4^6x_5^7$ |
| 25. $x_1x_2x_3^7x_4^3x_5^6$ | 26. $x_1x_2x_3^7x_4^6x_5^3$ | 27. $x_1x_2x_3x_4^6x_5^7$ | 28. $x_1x_2x_3x_4^7x_5^6$ |
| 29. $x_1x_2^3x_3^6x_4x_5^7$ | 30. $x_1x_2^3x_3^6x_4^7x_5$ | 31. $x_1x_2^3x_3^7x_4x_5^6$ | 32. $x_1x_2^3x_3x_4^6x_5^7$ |
| 33. $x_1x_2^6x_3x_4^3x_5^7$ | 34. $x_1x_2^6x_3x_4^7x_5^3$ | 35. $x_1x_2^6x_3^3x_4x_5^7$ | 36. $x_1x_2^6x_3^3x_4x_5^7$ |
| 37. $x_1x_2^6x_3^7x_4x_5^3$ | 38. $x_1x_2^6x_3^7x_4^3x_5$ | 39. $x_1x_2^7x_3x_4^3x_5^6$ | 40. $x_1x_2^7x_3x_4^6x_5^3$ |
| 41. $x_1x_2^7x_3^3x_4x_5^6$ | 42. $x_1x_2^7x_3^3x_4^6x_5^3$ | 43. $x_1x_2^7x_3^6x_4x_5^3$ | 44. $x_1x_2^7x_3x_4^6x_5^3$ |
| 45. $x_1^3x_2x_3x_4^6x_5^7$ | 46. $x_1^3x_2x_3x_4^7x_5^6$ | 47. $x_1^3x_2x_3^6x_4x_5^7$ | 48. $x_1^3x_2x_3^6x_4x_5^7$ |
| 49. $x_1^3x_2x_3^7x_4x_5^6$ | 50. $x_1^3x_2x_3^7x_4^6x_5^3$ | 51. $x_1^3x_2x_3^7x_4x_5^6$ | 52. $x_1^3x_2x_3x_4^6x_5^6$ |
| 53. $x_1^7x_2x_3x_4^3x_5^6$ | 54. $x_1^7x_2x_3x_4^6x_5^3$ | 55. $x_1^7x_2x_3^3x_4x_5^6$ | 56. $x_1^7x_2x_3x_4^6x_5^6$ |
| 57. $x_1^7x_2x_3^6x_4x_5^3$ | 58. $x_1^7x_2x_3^6x_4^3x_5^5$ | 59. $x_1^7x_2x_3x_4x_5^6$ | 60. $x_1^7x_2x_3x_4^6x_5^6$ |
| 61. $x_1x_2^2x_3^5x_4x_5^7$ | 62. $x_1x_2^2x_3^5x_4^7x_5^2$ | 63. $x_1x_2^2x_3^5x_4^3x_5^7$ | 64. $x_1x_2^2x_3x_4^5x_5^7$ |
| 65. $x_1x_2^2x_3^7x_4^3x_5^5$ | 66. $x_1x_2^2x_3^7x_4^5x_5^3$ | 67. $x_1x_2^3x_3^2x_4^5x_5^7$ | 68. $x_1x_2^3x_3x_4^2x_5^7$ |
| 69. $x_1x_2^3x_3^5x_4x_5^7$ | 70. $x_1x_2^3x_3^5x_4^7x_5^2$ | 71. $x_1x_2^3x_3^7x_4x_5^2$ | 72. $x_1x_2^3x_3x_4^2x_5^7$ |
| 73. $x_1x_2^7x_3^2x_4^3x_5^5$ | 74. $x_1x_2^7x_3^2x_4^5x_5^3$ | 75. $x_1x_2^7x_3^3x_4x_5^2$ | 76. $x_1x_2^7x_3x_4^3x_5^2$ |
| 77. $x_1^3x_2x_3^2x_4^5x_5^7$ | 78. $x_1^3x_2x_3^2x_4^7x_5^5$ | 79. $x_1^3x_2x_3^5x_4^2x_5^7$ | 80. $x_1^3x_2x_3^5x_4^7x_5^2$ |
| 81. $x_1^3x_2x_3^7x_4x_5^5$ | 82. $x_1^3x_2x_3^7x_4^5x_5^2$ | 83. $x_1^3x_2x_3^5x_4x_5^2$ | 84. $x_1^3x_2x_3x_4^5x_5^2$ |
| 85. $x_1^3x_2^5x_3x_4x_5^7$ | 86. $x_1^3x_2^5x_3^2x_4x_5^5$ | 87. $x_1^3x_2^5x_3^7x_4x_5^2$ | 88. $x_1^3x_2^5x_3x_4^2x_5^2$ |
| 89. $x_1^3x_2^7x_3x_4x_5^5$ | 90. $x_1^3x_2^7x_3x_4^2x_5^3$ | 91. $x_1^3x_2^7x_3x_4x_5^2$ | 92. $x_1^3x_2^7x_3^2x_4x_5^2$ |
| 93. $x_1^7x_2x_3^2x_4^3x_5^5$ | 94. $x_1^7x_2x_3^2x_4^5x_5^3$ | 95. $x_1^7x_2x_3^3x_4x_5^2$ | 96. $x_1^7x_2x_3x_4^3x_5^2$ |
| 97. $x_1^7x_2x_3x_4^2x_5^5$ | 98. $x_1^7x_2x_3x_4x_5^2$ | 99. $x_1^7x_2x_3^5x_4x_5^2$ | 100. $x_1^7x_2x_3^5x_4^2x_5$ |
| 101. $x_1x_2^3x_3^3x_4^4x_5^7$ | 102. $x_1x_2^3x_3^3x_4^7x_5^4$ | 103. $x_1x_2^3x_3^4x_4^3x_5^7$ | 104. $x_1x_2^3x_3^4x_4^7x_5^3$ |
| 105. $x_1x_2^3x_3^7x_4^3x_5^4$ | 106. $x_1x_2^3x_3^7x_4^4x_5^3$ | 107. $x_1x_2^3x_3^3x_4^4x_5^7$ | 108. $x_1x_2^3x_3^4x_4^3x_5^3$ |
| 109. $x_1^3x_2x_3^3x_4^4x_5^7$ | 110. $x_1^3x_2x_3^3x_4^7x_5^4$ | 111. $x_1^3x_2x_3^4x_4^3x_5^7$ | 112. $x_1^3x_2x_3^4x_4^7x_5^3$ |
| 113. $x_1^3x_2x_3^7x_4^3x_5^4$ | 114. $x_1^3x_2x_3^7x_4^4x_5^3$ | 115. $x_1^3x_2x_3^4x_4^3x_5^7$ | 116. $x_1^3x_2x_3x_4^7x_5^4$ |

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|--------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 117. $x_1^3x_2^3x_3^4x_4x_5^7$ | 118. $x_1^3x_2^3x_3^4x_4^7x_5$ | 119. $x_1^3x_2^3x_3^7x_4x_5^4$ | 120. $x_1^3x_2^3x_3^7x_4^4x_5$ |
| 121. $x_1^3x_2^3x_3x_4x_5^4$ | 122. $x_1^3x_2^7x_3x_4x_5^3$ | 123. $x_1^3x_2^7x_3^3x_4x_5^4$ | 124. $x_1^3x_2^7x_3^3x_4^4x_5$ |
| 125. $x_1^7x_2x_3x_4x_5^4$ | 126. $x_1^7x_2x_3^3x_4x_5^3$ | 127. $x_1^7x_2x_3x_4^3x_5^4$ | 128. $x_1^7x_2x_3x_4^4x_5^3$ |
| 129. $x_1^7x_2^3x_3x_4x_5^4$ | 130. $x_1^7x_2^3x_3^3x_4x_5^3$ | 131. $x_1x_2^3x_3^3x_4x_5^6$ | 132. $x_1x_2^3x_3^3x_4^6x_5^5$ |
| 133. $x_1x_2^3x_3^5x_4x_5^6$ | 134. $x_1x_2^3x_3x_4x_5^6$ | 135. $x_1x_2^3x_3^6x_4x_5^3$ | 136. $x_1x_2^3x_3x_4^5x_5^3$ |
| 137. $x_1x_2^6x_3^3x_4x_5^5$ | 138. $x_1x_2^6x_3^3x_4^5x_5^3$ | 139. $x_1^3x_2x_3^3x_4^5x_5^6$ | 140. $x_1^3x_2x_3^3x_4^6x_5^5$ |
| 141. $x_1^3x_2x_3^5x_4x_5^6$ | 142. $x_1^3x_2x_3^5x_4^6x_5^3$ | 143. $x_1^3x_2x_3^6x_4x_5^5$ | 144. $x_1^3x_2x_3^6x_4^5x_5^3$ |
| 145. $x_1^3x_2^3x_3x_4x_5^5$ | 146. $x_1^3x_2^3x_3x_4x_5^6$ | 147. $x_1^3x_2^3x_3^4x_4x_5^6$ | 148. $x_1^3x_2^3x_3^5x_4x_5$ |
| 149. $x_1^3x_2^5x_3x_4x_5^6$ | 150. $x_1^3x_2^5x_3x_4x_5^6$ | 151. $x_1^3x_2^5x_3^3x_4x_5^6$ | 152. $x_1^3x_2^5x_3^3x_4^6x_5$ |
| 153. $x_1^3x_2^5x_6x_4x_5^3$ | 154. $x_1^3x_2^5x_6^3x_4x_5$ | 155. $x_1^3x_2^5x_3^2x_4^5$ | 156. $x_1^3x_2^5x_3^5x_4^2x_5^2$ |
| 157. $x_1^3x_2^5x_3^2x_4x_5^5$ | 158. $x_1^3x_2^5x_3^2x_4^5x_5^3$ | 159. $x_1^3x_2^5x_3^3x_4^2x_5^5$ | 160. $x_1^3x_2^5x_3^3x_4^5x_2^2$ |
| 161. $x_1^3x_2^3x_3^3x_4x_5^5$ | 162. $x_1^3x_2^3x_3^3x_4^5x_5^4$ | 163. $x_1^3x_2^3x_3^4x_4^3x_5^5$ | 164. $x_1^3x_2^3x_3^4x_4^5x_5^3$ |
| 165. $x_1^3x_2^3x_3^5x_4x_5^4$ | 166. $x_1^3x_2^3x_3^5x_4^4x_5^3$ | 167. $x_1^3x_2^3x_3^4x_4^3x_5^4$ | 168. $x_1^3x_2^3x_3^4x_4^3x_5^3$ |
| 169. $x_1^3x_2^3x_3^3x_4x_5^7$ | 170. $x_1^3x_2^4x_3x_4x_5^7$ | 171. $x_1^3x_2^4x_3x_4x_5^7$ | 172. $x_1^3x_2^4x_3x_4x_5^7$ |
| 173. $x_1^3x_2^4x_3^7x_4x_5^3$ | 174. $x_1^3x_2^4x_3^7x_4x_5^3$ | 175. $x_1^3x_2^7x_3^4x_4x_5^3$ | 176. $x_1^3x_2^7x_3^4x_4^3x_5$ |
| 177. $x_1^7x_2^3x_3x_4x_5^3$ | 178. $x_1^7x_2^3x_3x_4x_5^3$ | 179. $x_1^3x_2^4x_3^3x_4x_5^5$ | 180. $x_1^3x_2^4x_3^3x_4^5x_5^3$ |

4.3. The admissible monomials of degree 39 in P_4

$\mathcal{B}_4(39)$ is the set of 225 monomials:

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|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 1. $x_2x_3^7x_4^{31}$ | 2. $x_2x_3^{15}x_4^{23}$ | 3. $x_2x_3^{31}x_4^7$ | 4. $x_2^3x_3^5x_4^{31}$ |
| 5. $x_2^3x_3x_4^{29}$ | 6. $x_2^3x_3^{13}x_4^{23}$ | 7. $x_2^3x_3^{15}x_4^{21}$ | 8. $x_2^3x_3^{29}x_4^7$ |
| 9. $x_2^3x_3^{31}x_4^5$ | 10. $x_2^7x_3x_4^{31}$ | 11. $x_2^7x_3^3x_4^{29}$ | 12. $x_2^7x_3^7x_4^{25}$ |
| 13. $x_2^7x_3^{11}x_4^{21}$ | 14. $x_2^7x_3^{27}x_4^5$ | 15. $x_2^7x_3^{31}x_4$ | 16. $x_2^{15}x_3x_4^{23}$ |
| 17. $x_2^{15}x_3^3x_4^{21}$ | 18. $x_2^{15}x_3^3x_4$ | 19. $x_2^{31}x_3x_4^7$ | 20. $x_2^{31}x_3^3x_4^5$ |
| 21. $x_2^{31}x_3^7x_4$ | 22. $x_1x_3^{15}x_4^{31}$ | 23. $x_1x_3^{15}x_4^{23}$ | 24. $x_1x_3^{31}x_4^7$ |
| 25. $x_1x_2x_3^6x_4^{31}$ | 26. $x_1x_2x_3^7x_4^{30}$ | 27. $x_1x_2x_3^{14}x_4^{23}$ | 28. $x_1x_2x_3^{15}x_4^{22}$ |
| 29. $x_1x_2x_3^{30}x_4^7$ | 30. $x_1x_2x_3^{31}x_4^6$ | 31. $x_1x_2^2x_3^5x_4^{31}$ | 32. $x_1x_2^2x_3^7x_4^{29}$ |
| 33. $x_1x_2^2x_3^{13}x_4^{23}$ | 34. $x_1x_2^2x_3^{15}x_4^{21}$ | 35. $x_1x_2^2x_3^{29}x_4^7$ | 36. $x_1x_2^2x_3^{31}x_4^5$ |
| 37. $x_1x_2x_3^4x_4^{31}$ | 38. $x_1x_2^3x_3^5x_4^{30}$ | 39. $x_1x_2^3x_3^6x_4^{29}$ | 40. $x_1x_2^3x_3^7x_4^{28}$ |
| 41. $x_1x_2^3x_3^{12}x_4^{23}$ | 42. $x_1x_2^3x_3^{13}x_4^{22}$ | 43. $x_1x_2^3x_3^{14}x_4^{21}$ | 44. $x_1x_2^3x_3^{15}x_4^{20}$ |
| 45. $x_1x_2^3x_3^{28}x_4^7$ | 46. $x_1x_2^3x_3^{29}x_4^6$ | 47. $x_1x_2^3x_3^{30}x_4^5$ | 48. $x_1x_2^3x_3^{31}x_4^4$ |
| 49. $x_1x_2^6x_3x_4^{31}$ | 50. $x_1x_2^6x_3^3x_4^{29}$ | 51. $x_1x_2^6x_3^7x_4^{25}$ | 52. $x_1x_2^6x_3^{11}x_4^1$ |
| 53. $x_1x_2^6x_3^{27}x_4^5$ | 54. $x_1x_2^6x_3^3x_4^4$ | 55. $x_1x_2^7x_3^{31}$ | 56. $x_1x_2^7x_3x_4^{30}$ |
| 57. $x_1x_2^7x_3^2x_4^{29}$ | 58. $x_1x_2^7x_3^3x_4^{28}$ | 59. $x_1x_2^7x_3^6x_4^{25}$ | 60. $x_1x_2^7x_3x_4^{24}$ |
| 61. $x_1x_2^7x_3^{10}x_4^{21}$ | 62. $x_1x_2^7x_3^{11}x_4^{20}$ | 63. $x_1x_2^7x_3^{26}x_4^5$ | 64. $x_1x_2^7x_3^{27}x_4^4$ |
| 65. $x_1x_2^7x_3^{30}x_4^4$ | 66. $x_1x_2^7x_3^{31}$ | 67. $x_1x_2^{14}x_3x_4^{23}$ | 68. $x_1x_2^{14}x_3^3x_4^4$ |
| 69. $x_1x_2^{14}x_3^{23}x_4$ | 70. $x_1x_2^{15}x_4^{23}$ | 71. $x_1x_2^{15}x_3x_4^{22}$ | 72. $x_1x_2^{15}x_3^2x_4^{21}$ |
| 73. $x_1x_2^{15}x_3^3x_4^{20}$ | 74. $x_1x_2^{15}x_3^{22}x_4$ | 75. $x_1x_2^{15}x_3^{23}$ | 76. $x_1x_2^{30}x_3x_4^7$ |
| 77. $x_1x_2^{30}x_3^3x_4^5$ | 78. $x_1x_2^{30}x_3^7x_4$ | 79. $x_1x_2^{31}x_4^7$ | 80. $x_1x_2^{31}x_3x_4^6$ |
| 81. $x_1x_2^{31}x_3^2x_4^5$ | 82. $x_1x_2^{31}x_3^3x_4^4$ | 83. $x_1x_2^{31}x_3^6x_4^4$ | 84. $x_1x_2^{31}x_3^7$ |
| 85. $x_1^3x_3^5x_4^{31}$ | 86. $x_1^3x_3^7x_4^{29}$ | 87. $x_1^3x_3^{13}x_4^{23}$ | 88. $x_1^3x_3^{15}x_4^{21}$ |

89. $x_1^3x_3^{29}x_4^7$	90. $x_1^3x_3^{31}x_4^5$	91. $x_1^3x_2x_3^4x_4^{31}$	92. $x_1^3x_2x_3^5x_4^{30}$
93. $x_1^3x_2x_3^6x_4^{29}$	94. $x_1^3x_2x_3^7x_4^{28}$	95. $x_1^3x_2x_3^{12}x_4^{23}$	96. $x_1^3x_2x_3^{13}x_4^{22}$
97. $x_1^3x_2x_3^{14}x_4^{21}$	98. $x_1^3x_2x_3^{15}x_4^{20}$	99. $x_1^3x_2x_3^{28}x_4^7$	100. $x_1^3x_2x_3^{29}x_4^6$
101. $x_1^3x_2x_3^{30}x_4^5$	102. $x_1^3x_2x_3^{31}x_4^4$	103. $x_1^3x_2x_3^{4}x_4^{29}$	104. $x_1^3x_2x_3^{5}x_4^{28}$
105. $x_1^3x_2x_3^{12}x_4^{21}$	106. $x_1^3x_2x_3^{13}x_4^{20}$	107. $x_1^3x_2x_3^{28}x_4^5$	108. $x_1^3x_2x_3^{29}x_4^4$
109. $x_1^3x_2x_3^{4}x_4^{31}$	110. $x_1^3x_2x_3^{4}x_4^{29}$	111. $x_1^3x_2x_3^{7}x_4^{25}$	112. $x_1^3x_2x_3^{11}x_4^{21}$
113. $x_1^3x_2x_3^{4}x_4^{27}$	114. $x_1^3x_2x_3^{4}x_4^{31}$	115. $x_1^3x_2x_4^{6}x_4^{31}$	116. $x_1^3x_2x_3x_4^{30}$
117. $x_1^3x_2x_3^{5}x_4^{29}$	118. $x_1^3x_2x_3^{5}x_4^{28}$	119. $x_1^3x_2x_3^{6}x_4^{25}$	120. $x_1^3x_2x_3x_4^{24}$
121. $x_1^3x_2x_3^{5}x_4^{21}$	122. $x_1^3x_2x_3^{5}x_4^{20}$	123. $x_1^3x_2x_3^{5}x_4^{26}$	124. $x_1^3x_2x_3^{5}x_4^{27}$
125. $x_1^3x_2x_3^{5}x_4^{30}$	126. $x_1^3x_2x_3^{5}x_4^{31}$	127. $x_1^3x_2x_4^{7}x_4^{29}$	128. $x_1^3x_2x_3x_4^{28}$
129. $x_1^3x_2x_3^{7}x_4^{25}$	130. $x_1^3x_2x_3^{5}x_4^{24}$	131. $x_1^3x_2x_3^{7}x_4^{21}$	132. $x_1^3x_2x_3^{7}x_4^{20}$
133. $x_1^3x_2x_3^{7}x_4^{25}$	134. $x_1^3x_2x_3^{7}x_4^{28}$	135. $x_1^3x_2x_3^{7}x_4^{29}$	136. $x_1^3x_2x_4^{13}$
137. $x_1^3x_2^{13}x_3x_4^{22}$	138. $x_1^3x_2^{13}x_3x_4^{21}$	139. $x_1^3x_2^{13}x_3x_4^{20}$	140. $x_1^3x_2^{13}x_3^{22}$
141. $x_1^3x_2^{13}x_3^{23}$	142. $x_1^3x_2^{15}x_4^{21}$	143. $x_1^3x_2^{15}x_3x_4^{20}$	144. $x_1^3x_2^{15}x_3^{21}$
145. $x_1^3x_2^{29}x_4^7$	146. $x_1^3x_2^{29}x_3x_4^6$	147. $x_1^3x_2^{29}x_3x_4^5$	148. $x_1^3x_2^{29}x_3x_4^4$
149. $x_1^3x_2^{29}x_6x_4$	150. $x_1^3x_2^{29}x_3^7$	151. $x_1^3x_2^{31}x_4^5$	152. $x_1^3x_2^{31}x_3x_4^4$
153. $x_1^3x_2^{31}x_3x_4$	154. $x_1^3x_2^{31}x_3^5$	155. $x_1^7x_3x_4^{31}$	156. $x_1^7x_3x_4^{29}$
157. $x_1^7x_3x_4^{25}$	158. $x_1^7x_3x_4^{21}$	159. $x_1^7x_3x_4^{25}$	160. $x_1^7x_3^{31}x_4$
161. $x_1^7x_2x_4^{31}$	162. $x_1^7x_2x_3x_4^{30}$	163. $x_1^7x_2x_3x_4^{29}$	164. $x_1^7x_2x_3x_4^{28}$
165. $x_1^7x_2x_3^6x_4^{25}$	166. $x_1^7x_2x_3^7x_4^{24}$	167. $x_1^7x_2x_3^{10}x_4^{21}$	168. $x_1^7x_2x_3^{11}x_4^{20}$
169. $x_1^7x_2x_3^{26}x_4^5$	170. $x_1^7x_2x_3^{27}x_4^4$	171. $x_1^7x_2x_3^{30}x_4$	172. $x_1^7x_2x_3^{31}$
173. $x_1^7x_2x_4^{29}$	174. $x_1^7x_2x_3x_4^{28}$	175. $x_1^7x_2x_3x_4^{25}$	176. $x_1^7x_2x_3x_4^{24}$
177. $x_1^7x_2x_3^8x_4^{21}$	178. $x_1^7x_2x_3^9x_4^{20}$	179. $x_1^7x_2x_3^{25}x_4^4$	180. $x_1^7x_2x_3^{28}x_4$
181. $x_1^7x_2x_3^3$	182. $x_1^7x_2x_4^7$	183. $x_1^7x_2x_3x_4^{24}$	184. $x_1^7x_2x_3x_4^{17}$
185. $x_1^7x_2x_3^7x_4^{16}$	186. $x_1^7x_2x_3^{24}x_4$	187. $x_1^7x_2x_3^{25}$	188. $x_1^7x_2^{11}x_4^{21}$
189. $x_1^7x_2^{11}x_3x_4^{20}$	190. $x_1^7x_2^{11}x_3^{21}$	191. $x_1^7x_2^{27}x_4^5$	192. $x_1^7x_2^{27}x_3x_4^4$
193. $x_1^7x_2^{27}x_3^4x_4$	194. $x_1^7x_2^{27}x_3^{25}$	195. $x_1^7x_2^{31}x_4$	196. $x_1^7x_2^{31}x_3$
197. $x_1^{15}x_3x_4^{23}$	198. $x_1^{15}x_3x_4^{21}$	199. $x_1^{15}x_3^{23}x_4$	200. $x_1^{15}x_2x_4^{23}$
201. $x_1^{15}x_2x_3x_4^{22}$	202. $x_1^{15}x_2x_3x_4^{21}$	203. $x_1^{15}x_2x_3x_4^{20}$	204. $x_1^{15}x_2x_3^{22}x_4$
205. $x_1^{15}x_2x_3^{23}$	206. $x_1^{15}x_2x_4^{21}$	207. $x_1^{15}x_2x_3x_4^{20}$	208. $x_1^{15}x_2^{3}x_3^{21}$
209. $x_1^{15}x_2^{23}x_4$	210. $x_1^{15}x_2^{23}x_3$	211. $x_1^{31}x_3x_4^7$	212. $x_1^{31}x_3x_4^5$
213. $x_1^{31}x_3x_4$	214. $x_1^{31}x_2x_4^7$	215. $x_1^{31}x_2x_3x_4^6$	216. $x_1^{31}x_2x_3^2x_4^5$
217. $x_1^{31}x_2x_3x_4^4$	218. $x_1^{31}x_2x_3^6x_4$	219. $x_1^{31}x_2x_3^7$	220. $x_1^{31}x_2x_3^5$
221. $x_1^{31}x_2x_3x_4^4$	222. $x_1^{31}x_2x_3^4x_4$	223. $x_1^{31}x_2x_3^5$	224. $x_1^{31}x_2x_4^7$
225. $x_1^{31}x_2^7x_3$.			

4.4. The admissible monomials of degree 39 in P_5

We have $\mathcal{B}_5(39) = \mathcal{B}_5^0(39) \cup \varphi(\mathcal{B}_5(17)) \cup (\mathcal{B}_5^+(39) \cap \text{Ker}((\widetilde{Sq}_*)_{(5,39)}^0))$, where $\mathcal{B}_5^0(39) = \Phi^0(\mathcal{B}_4(39))$, $|\mathcal{B}_5^0(39)| = 915$, $|\varphi(\mathcal{B}_5(17))| = 566$ with

$$\varphi : P_5 \rightarrow P_5, \quad \varphi(u) = x_1x_2x_3x_4x_5u^2, \quad \forall u \in P_5,$$

and $\mathcal{B}_5^+(39) \cap \text{Ker}((\widetilde{S}q_*)_{(5,39)}^0) = \mathcal{B}_5^+(\omega_{(1)}) \cup \mathcal{B}_5^+(\omega_{(4)}) \cup \mathcal{B}_5^+(\omega_{(5)}) \cup \mathcal{B}_5^+(\omega_{(6)})$. Here $\omega_{(1)} = (3, 2, 2, 1, 1)$, $\omega_{(4)} = (3, 4, 1, 1, 1)$, $\omega_{(5)} = (3, 4, 1, 3)$, $\omega_{(6)} = (3, 4, 3, 2)$.

$\mathcal{B}_5^+(\omega_{(1)})$ is the set of 485 monomials $d_k = d_{39,k}$, $1 \leq k \leq 485$:

- | | | | |
|------------------------------------|------------------------------------|----------------------------------|------------------------------------|
| 1. $x_1x_2x_3x_4x_5^{30}$ | 2. $x_1x_2x_3x_4x_5^{30}$ | 3. $x_1x_2x_3x_4x_5^{30}$ | 4. $x_1x_2x_3x_4x_5^{30}$ |
| 5. $x_1x_2x_3^{30}x_4x_5^6$ | 6. $x_1x_2x_3^{30}x_4x_5^6$ | 7. $x_1x_2x_3x_4x_5^{30}$ | 8. $x_1x_2x_3x_4x_5^{30}$ |
| 9. $x_1x_2^{30}x_3x_4x_5^6$ | 10. $x_1x_2^{30}x_3x_4x_5^6$ | 11. $x_1x_2x_3x_4x_5^{22}$ | 12. $x_1x_2x_3^{14}x_4x_5^{22}$ |
| 13. $x_1x_2x_3^{14}x_4x_5^{22}$ | 14. $x_1x_2^{14}x_3x_4x_5^{22}$ | 15. $x_1x_2^{14}x_3x_4x_5^{22}$ | 16. $x_1x_2x_3^2x_4x_5^{31}$ |
| 17. $x_1x_2x_3^2x_4x_5^{31}$ | 18. $x_1x_2x_3^2x_4x_5^{31}$ | 19. $x_1x_2x_3x_4x_5^{31}$ | 20. $x_1x_2x_3x_4x_5^{31}$ |
| 21. $x_1x_2x_3^2x_4x_5^{31}$ | 22. $x_1x_2x_3^2x_4x_5^{31}$ | 23. $x_1x_2x_3^2x_4x_5^{31}$ | 24. $x_1x_2x_3^2x_4x_5^{31}$ |
| 25. $x_1x_2^3x_3x_4x_5^4$ | 26. $x_1x_2^3x_3x_4x_5^4$ | 27. $x_1x_2^3x_3x_4x_5^4$ | 28. $x_1^3x_2x_3x_4x_5^4$ |
| 29. $x_1^3x_2x_3^2x_4x_5^4$ | 30. $x_1^3x_2x_3^2x_4x_5^4$ | 31. $x_1x_2x_3^2x_4x_5^{30}$ | 32. $x_1x_2x_3^2x_4x_5^{30}$ |
| 33. $x_1x_2x_3^3x_4x_5^5$ | 34. $x_1x_2x_3x_4x_5^{30}$ | 35. $x_1x_2x_3x_4x_5^{30}$ | 36. $x_1x_2x_3x_4x_5^{30}$ |
| 37. $x_1x_2x_3^5x_4x_5^{30}$ | 38. $x_1x_2^3x_3x_4x_5^5$ | 39. $x_1x_2x_3^2x_4x_5^{29}$ | 40. $x_1x_2x_3^2x_4x_5^6$ |
| 41. $x_1x_2x_3^6x_4x_5^{29}$ | 42. $x_1x_2^2x_3x_4x_5^{29}$ | 43. $x_1x_2^2x_3x_4x_5^{29}$ | 44. $x_1x_2^2x_3x_4x_5^6$ |
| 45. $x_1x_2^2x_3^2x_4x_5^6$ | 46. $x_1x_2^6x_3x_4x_5^{29}$ | 47. $x_1x_2x_3^2x_4x_5^{28}$ | 48. $x_1x_2x_3^2x_4x_5^{28}$ |
| 49. $x_1x_2x_3^7x_4x_5^{28}$ | 50. $x_1x_2^2x_3x_4x_5^{28}$ | 51. $x_1x_2^2x_3x_4x_5^{28}$ | 52. $x_1x_2^2x_3x_4x_5^{28}$ |
| 53. $x_1x_2^2x_3x_4x_5^{28}$ | 54. $x_1x_2^2x_3x_4x_5^7$ | 55. $x_1x_2^2x_3x_4x_5^7$ | 56. $x_1x_2^2x_3x_4x_5^7$ |
| 57. $x_1x_2^2x_3x_4x_5^{28}$ | 58. $x_1x_2^2x_3x_4x_5^{28}$ | 59. $x_1^7x_2x_3x_4x_5^{28}$ | 60. $x_1^7x_2x_3x_4x_5^{28}$ |
| 61. $x_1x_2x_3^2x_4x_5^{28}$ | 62. $x_1x_2x_3x_4x_5^{12}$ | 63. $x_1x_2x_3x_4x_5^{12}$ | 64. $x_1x_2x_3^2x_4x_5^{23}$ |
| 65. $x_1x_2x_3^2x_4x_5^{23}$ | 66. $x_1x_2x_3^2x_4x_5^{13}$ | 67. $x_1x_2x_3x_4x_5^{13}$ | 68. $x_1x_2x_3^2x_4x_5^{22}$ |
| 69. $x_1x_2x_3^3x_4x_5^{22}$ | 70. $x_1x_2x_3^2x_4x_5^{21}$ | 71. $x_1x_2x_3^{14}x_4x_5^{21}$ | 72. $x_1x_2x_3x_4x_5^{14}$ |
| 73. $x_1x_2^{14}x_3x_4x_5^{21}$ | 74. $x_1x_2x_3^2x_4x_5^{20}$ | 75. $x_1x_2x_3^{15}x_4x_5^{20}$ | 76. $x_1x_2x_3x_4x_5^{15}$ |
| 77. $x_1x_2x_3^2x_4x_5^{20}$ | 78. $x_1x_2x_3^2x_4x_5^{20}$ | 79. $x_1x_2^{15}x_3x_4x_5^{20}$ | 80. $x_1x_2^{15}x_3x_4x_5^{20}$ |
| 81. $x_1x_2^{15}x_3x_4x_5^{20}$ | 82. $x_1^{15}x_2x_3x_4x_5^{20}$ | 83. $x_1^{15}x_2x_3x_4x_5^{20}$ | 84. $x_1^{15}x_2x_3x_4x_5^{20}$ |
| 85. $x_1x_2x_3^3x_4x_5^{30}$ | 86. $x_1x_2x_3^3x_4x_5^{30}$ | 87. $x_1x_2x_3^3x_4x_5^{30}$ | 88. $x_1x_2x_3^3x_4x_5^{30}$ |
| 89. $x_1x_2^3x_3x_4x_5^{30}$ | 90. $x_1x_2^3x_3x_4x_5^{30}$ | 91. $x_1x_2^3x_3x_4x_5^{30}$ | 92. $x_1x_2^3x_3x_4x_5^{30}$ |
| 93. $x_1x_2x_3^3x_4x_5^4$ | 94. $x_1x_2^3x_3x_4x_5^4$ | 95. $x_1x_2^3x_3x_4x_5^4$ | 96. $x_1x_2^3x_3x_4x_5^4$ |
| 97. $x_1^3x_2x_3x_4x_5^{30}$ | 98. $x_1^3x_2x_3x_4x_5^{30}$ | 99. $x_1^3x_2x_3x_4x_5^{30}$ | 100. $x_1^3x_2x_3x_4x_5^{30}$ |
| 101. $x_1^3x_2x_3^{30}x_4x_5^4$ | 102. $x_1^3x_2x_3^{30}x_4x_5^4$ | 103. $x_1^3x_2^4x_3x_4x_5^{30}$ | 104. $x_1^3x_2^4x_3x_4x_5^{30}$ |
| 105. $x_1x_2x_3^3x_4x_5^{28}$ | 106. $x_1x_2x_3^3x_4x_5^{28}$ | 107. $x_1x_2x_3^6x_4x_5^{28}$ | 108. $x_1x_2x_3^6x_4x_5^{28}$ |
| 109. $x_1x_2^3x_3x_4x_5^{28}$ | 110. $x_1x_2^3x_3x_4x_5^{28}$ | 111. $x_1x_2x_3^6x_4x_5^{28}$ | 112. $x_1x_2x_3^6x_4x_5^{28}$ |
| 113. $x_1x_2^3x_3^2x_4x_5^6$ | 114. $x_1x_2^3x_3x_4x_5^6$ | 115. $x_1x_2^6x_3x_4x_5^{28}$ | 116. $x_1x_2^6x_3x_4x_5^{28}$ |
| 117. $x_1^3x_2x_3x_4x_5^{28}$ | 118. $x_1^3x_2x_3x_4x_5^{28}$ | 119. $x_1^3x_2x_3x_4x_5^{28}$ | 120. $x_1^3x_2x_3x_4x_5^{28}$ |
| 121. $x_1^3x_2x_3^2x_4x_5^6$ | 122. $x_1^3x_2x_3^2x_4x_5^6$ | 123. $x_1x_2x_3^3x_4x_5^{12}$ | 124. $x_1x_2x_3x_4x_5^{12}$ |
| 125. $x_1x_2^3x_3^{12}x_4x_5^{22}$ | 126. $x_1x_2^3x_3^{12}x_4x_5^{22}$ | 127. $x_1^3x_2x_3x_4x_5^{12}$ | 128. $x_1^3x_2x_3^{12}x_4x_5^{22}$ |
| 129. $x_1^3x_2x_3^{12}x_4x_5^{22}$ | 130. $x_1x_2x_3^3x_4x_5^{14}$ | 131. $x_1x_2x_3^{14}x_4x_5^{20}$ | 132. $x_1x_2^3x_3x_4^{14}$ |
| 133. $x_1x_2^3x_3^{14}x_4x_5^{20}$ | 134. $x_1x_2^3x_3^{14}x_4x_5^{20}$ | 135. $x_1x_2^{14}x_3x_4x_5^{20}$ | 136. $x_1x_2^{14}x_3x_4x_5^{20}$ |
| 137. $x_1x_2^3x_3x_4x_5^{14}$ | 138. $x_1x_2^3x_3x_4x_5^{14}$ | 139. $x_1x_2^3x_3x_4x_5^{14}$ | 140. $x_1x_2x_3^6x_4x_5^{27}$ |
| 141. $x_1x_2^6x_3x_4x_5^{27}$ | 142. $x_1x_2^6x_3x_4x_5^{27}$ | 143. $x_1x_2^6x_3x_4x_5^{27}$ | 144. $x_1x_2x_3^7x_4x_5^{26}$ |
| 145. $x_1x_2^7x_3x_4x_5^{26}$ | 146. $x_1x_2^7x_3x_4x_5^{26}$ | 147. $x_1x_2^7x_3x_4x_5^{26}$ | 148. $x_1x_2x_3x_4x_5^{26}$ |
| 149. $x_1^7x_2x_3x_4x_5^{26}$ | 150. $x_1^7x_2x_3x_4x_5^{26}$ | 151. $x_1x_2x_3x_4x_5^{26}$ | 152. $x_1x_2x_3x_4x_5^{26}$ |
| 153. $x_1x_2x_3x_4x_5^{25}$ | 154. $x_1x_2^6x_3x_4x_5^{25}$ | 155. $x_1x_2x_3x_4x_5^{24}$ | 156. $x_1x_2x_3x_4x_5^{24}$ |

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|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 157. $x_1x_2^6x_3x_4x_5^{24}$ | 158. $x_1x_2^6x_3x_4x_5^{24}$ | 159. $x_1x_2^6x_3x_4x_5^{24}$ | 160. $x_1x_2^7x_3x_4x_5^{24}$ |
| 161. $x_1x_2^7x_3x_4x_5^{24}$ | 162. $x_1x_2^7x_3x_4x_5^{24}$ | 163. $x_1^7x_2x_3x_4x_5^{24}$ | 164. $x_1^7x_2x_3x_4x_5^{24}$ |
| 165. $x_1^7x_2x_3x_4x_5^{24}$ | 166. $x_1x_2x_3x_4x_5^{10,21}$ | 167. $x_1x_2x_3x_4x_5^{10,21}$ | 168. $x_1x_2x_3x_4x_5^{11,20}$ |
| 169. $x_1x_2^6x_3x_4x_5^{11,20}$ | 170. $x_1x_2^7x_3x_4x_5^{10,20}$ | 171. $x_1x_2x_3x_4x_5^{10,20}$ | 172. $x_1x_2^7x_3x_4x_5^{10,20}$ |
| 173. $x_1x_2x_3^7x_4x_5^{20}$ | 174. $x_1x_2x_3^7x_4x_5^{20}$ | 175. $x_1^7x_2x_3x_4x_5^{10,20}$ | 176. $x_1^7x_2x_3^7x_4x_5^{20}$ |
| 177. $x_1x_2x_3^7x_4x_5^{10,20}$ | 178. $x_1x_2x_3^7x_4x_5^{29,2}$ | 179. $x_1x_2x_3^7x_4x_5^{29,2}$ | 180. $x_1x_2x_3^7x_4x_5^{29,2}$ |
| 181. $x_1x_2^3x_3x_4x_5^{4,29}$ | 182. $x_1x_2^3x_3x_4x_5^{4,29}$ | 183. $x_1x_2^3x_3x_4x_5^{4,29}$ | 184. $x_1x_2^3x_3x_4x_5^{4,29}$ |
| 185. $x_1x_2^3x_3x_4x_5^{4,29}$ | 186. $x_1x_2^3x_3x_4x_5^{4,29}$ | 187. $x_1x_2^3x_3x_4x_5^{4,29}$ | 188. $x_1x_2^3x_3x_4x_5^{4,29}$ |
| 189. $x_1^3x_2x_3x_4x_5^{2,4,29}$ | 190. $x_1^3x_2x_3x_4x_5^{2,29,4}$ | 191. $x_1^3x_2x_3x_4x_5^{4,2,29}$ | 192. $x_1^3x_2x_3^7x_4x_5^{2,4}$ |
| 193. $x_1^3x_2x_3x_4x_5^{4,29}$ | 194. $x_1^3x_2x_3x_4x_5^{29,4}$ | 195. $x_1^3x_2x_3x_4x_5^{29,4}$ | 196. $x_1^3x_2x_3x_4x_5^{28,3,5}$ |
| 197. $x_1x_2^2x_3x_4x_5^{3,5,28}$ | 198. $x_1x_2^2x_3x_4x_5^{3,28,5}$ | 199. $x_1x_2^2x_3x_4x_5^{3,28}$ | 200. $x_1x_2^2x_3x_4x_5^{28,3,5}$ |
| 201. $x_1x_2^2x_3x_4x_5^{2,5,28}$ | 202. $x_1x_2^2x_3x_4x_5^{28,5}$ | 203. $x_1x_2^2x_3x_4x_5^{5,2,28}$ | 204. $x_1x_2^2x_3x_4x_5^{28,2,5}$ |
| 205. $x_1^3x_2x_3x_4x_5^{2,5,28}$ | 206. $x_1^3x_2x_3x_4x_5^{28,5}$ | 207. $x_1^3x_2x_3x_4x_5^{5,2,28}$ | 208. $x_1^3x_2x_3x_4x_5^{28,2,5}$ |
| 209. $x_1^3x_2x_3x_4x_5^{2,28}$ | 210. $x_1^3x_2x_3x_4x_5^{28}$ | 211. $x_1^3x_2x_3x_4x_5^{5,2,28}$ | 212. $x_1^3x_2x_3x_4x_5^{12,21}$ |
| 213. $x_1x_2^2x_3^2x_4x_5^{12,3,21}$ | 214. $x_1x_2^3x_3x_4x_5^{12,2,21}$ | 215. $x_1x_2^3x_3x_4x_5^{12,2,21}$ | 216. $x_1^3x_2x_3x_4x_5^{12,2,21}$ |
| 217. $x_1^3x_2x_3^2x_4x_5^{12,2,21}$ | 218. $x_1x_2^2x_3^3x_4x_5^{13,2,20}$ | 219. $x_1x_2^2x_3^3x_4x_5^{13,3,20}$ | 220. $x_1x_2^3x_3^2x_4x_5^{13,2,20}$ |
| 221. $x_1x_2^3x_3^2x_4x_5^{13,2,20}$ | 222. $x_1^3x_2x_3^2x_4x_5^{13,2,20}$ | 223. $x_1^3x_2x_3^2x_4x_5^{13,2,20}$ | 224. $x_1^3x_2x_3^2x_4x_5^{13,2,20}$ |
| 225. $x_1x_2^3x_3^2x_4x_5^{13,2,20}$ | 226. $x_1x_2^3x_3^2x_4x_5^{20}$ | 227. $x_1x_2^2x_3x_4x_5^{4,27,5}$ | 228. $x_1x_2^2x_3x_4x_5^{5,27,4}$ |
| 229. $x_1x_2^2x_3x_4x_5^{7,25}$ | 230. $x_1x_2^2x_3x_4x_5^{25,7}$ | 231. $x_1x_2^2x_3x_4x_5^{7,4,25}$ | 232. $x_1x_2^2x_3x_4x_5^{25,4}$ |
| 233. $x_1x_2^2x_3x_4x_5^{7,24,25}$ | 234. $x_1x_2^2x_3x_4x_5^{25,4}$ | 235. $x_1x_2^2x_3x_4x_5^{7,2,25}$ | 236. $x_1x_2^2x_3x_4x_5^{25,4}$ |
| 237. $x_1x_2^2x_3x_4x_5^{9,23}$ | 238. $x_1x_2^2x_3x_4x_5^{11,21}$ | 239. $x_1x_2^2x_3x_4x_5^{14,15,17}$ | 240. $x_1x_2^2x_3x_4x_5^{15,4,17}$ |
| 241. $x_1x_2^{15}x_3x_4x_5^{4,17}$ | 242. $x_1^{15}x_2x_3x_4x_5^{4,17}$ | 243. $x_1x_2^2x_3x_4x_5^{5,26,5}$ | 244. $x_1x_2^2x_3x_4x_5^{6,25}$ |
| 245. $x_1x_2^2x_3x_4x_5^{25,6}$ | 246. $x_1x_2^2x_3x_4x_5^{7,24}$ | 247. $x_1x_2^2x_3x_4x_5^{24,7}$ | 248. $x_1x_2^2x_3x_4x_5^{24}$ |
| 249. $x_1x_2^2x_3x_4x_5^{24,5}$ | 250. $x_1x_2^2x_3x_4x_5^{24,5}$ | 251. $x_1x_2^2x_3x_4x_5^{24,5}$ | 252. $x_1x_2^2x_3x_4x_5^{24}$ |
| 253. $x_1^7x_2x_3x_4x_5^{24,5}$ | 254. $x_1x_2^2x_3x_4x_5^{8,23}$ | 255. $x_1x_2^2x_3x_4x_5^{9,22}$ | 256. $x_1x_2^2x_3x_4x_5^{10,21}$ |
| 257. $x_1x_2^2x_3x_4x_5^{5,11,20}$ | 258. $x_1x_2^2x_3x_4x_5^{14,17}$ | 259. $x_1x_2^2x_3x_4x_5^{15,16}$ | 260. $x_1x_2^2x_3x_4x_5^{15,5,16}$ |
| 261. $x_1x_2^{15}x_3x_4x_5^{5,16}$ | 262. $x_1^{15}x_2x_3x_4x_5^{5,16}$ | 263. $x_1x_2^2x_3x_4x_5^{7,8,21}$ | 264. $x_1x_2^2x_3x_4x_5^{8,21}$ |
| 265. $x_1^7x_2x_3x_4x_5^{8,21}$ | 266. $x_1x_2^2x_3x_4x_5^{9,20}$ | 267. $x_1x_2^2x_3x_4x_5^{9,20}$ | 268. $x_1x_2^2x_3x_4x_5^{9,20}$ |
| 269. $x_1x_2^2x_3x_4x_5^{12,17}$ | 270. $x_1x_2^2x_3x_4x_5^{12,17}$ | 271. $x_1x_2^2x_3x_4x_5^{12,17}$ | 272. $x_1x_2^2x_3x_4x_5^{7,13,16}$ |
| 273. $x_1x_2^2x_3x_4x_5^{13,16}$ | 274. $x_1^7x_2x_3x_4x_5^{13,16}$ | 275. $x_1x_2^2x_3x_4x_5^{4,28}$ | 276. $x_1x_2^2x_3x_4x_5^{28,4}$ |
| 277. $x_1x_2^3x_3x_4x_5^{3,28}$ | 278. $x_1x_2^3x_3x_4x_5^{28,3,4}$ | 279. $x_1x_2^3x_3x_4x_5^{4,28}$ | 280. $x_1x_2^3x_3x_4x_5^{28,4}$ |
| 281. $x_1^3x_2x_3x_4x_5^{3,28}$ | 282. $x_1^3x_2x_3x_4x_5^{28,3,4}$ | 283. $x_1x_2^3x_3x_4x_5^{4,28}$ | 284. $x_1x_2^3x_3x_4x_5^{28,4}$ |
| 285. $x_1^3x_2^3x_3x_4x_5^{4,28}$ | 286. $x_1^3x_2^3x_3x_4x_5^{4,28}$ | 287. $x_1^3x_2^3x_3x_4x_5^{4,28}$ | 288. $x_1^3x_2^3x_3x_4x_5^{28,4}$ |
| 289. $x_1^3x_2x_3x_4x_5^{3,28}$ | 290. $x_1^3x_2x_3x_4x_5^{3,28}$ | 291. $x_1x_2^2x_3x_4x_5^{12,20}$ | 292. $x_1x_2^2x_3x_4x_5^{12,3,20}$ |
| 293. $x_1^3x_2x_3x_4x_5^{12,20}$ | 294. $x_1^3x_2x_3^{12}x_4x_5^{20}$ | 295. $x_1^3x_2x_3x_4x_5^{12,20}$ | 296. $x_1^3x_2x_3x_4x_5^{12,20}$ |
| 297. $x_1^3x_2x_3^{12}x_4x_5^{20}$ | 298. $x_1x_2^3x_3x_4x_5^{27,4}$ | 299. $x_1x_2x_3x_4x_5^{27,4}$ | 300. $x_1x_2^4x_3x_4x_5^{27,4}$ |
| 301. $x_1^3x_2^4x_3x_4x_5^{27,4}$ | 302. $x_1x_2^3x_3x_4x_5^{26,5}$ | 303. $x_1x_2^3x_3x_4x_5^{26,4}$ | 304. $x_1^3x_2x_3x_4x_5^{26,5}$ |
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| 309. $x_1x_2^3x_3x_4x_5^{26,4}$ | 310. $x_1x_2^3x_3x_4x_5^{26,5}$ | 311. $x_1x_2^3x_3x_4x_5^{25,6}$ | 312. $x_1x_2^3x_3x_4x_5^{26,4}$ |
| 313. $x_1x_2^3x_3x_4x_5^{6,25,4}$ | 314. $x_1x_2^3x_3x_4x_5^{6,25}$ | 315. $x_1x_2^3x_3x_4x_5^{6,25}$ | 316. $x_1x_2^3x_3x_4x_5^{6,25}$ |
| 317. $x_1^3x_2x_3x_4x_5^{4,25,6}$ | 318. $x_1^3x_2x_3x_4x_5^{6,25}$ | 319. $x_1^3x_2x_3x_4x_5^{6,25}$ | 320. $x_1^3x_2x_3x_4x_5^{6,25}$ |
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 430. $x_1^7 x_2^3 x_4 x_5^{16}$
 431. $x_1^7 x_2^3 x_4 x_5^{25}$
 432. $x_1^7 x_2^3 x_4 x_5^{25}$
 433. $x_1^7 x_2^3 x_4 x_5^{17}$
 434. $x_1^7 x_2^3 x_4 x_5^{24}$
 435. $x_1^7 x_2^3 x_4 x_5^{24}$
 436. $x_1^7 x_2^3 x_4 x_5^{21}$
 437. $x_1^7 x_2^3 x_4 x_5^{20}$
 438. $x_1^7 x_2^3 x_4 x_5^{17}$
 439. $x_1^7 x_2^3 x_4 x_5^{16}$
 440. $x_1^7 x_2^3 x_4 x_5^{16}$
 441. $x_1^7 x_2^3 x_4 x_5^{17}$
 442. $x_1^7 x_2^3 x_4 x_5^{16}$
 443. $x_1^7 x_2^3 x_4 x_5^{25}$
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 445. $x_1^7 x_2^3 x_4 x_5^{25}$
 446. $x_1^7 x_2^3 x_4 x_5^{24}$
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 453. $x_1^7 x_2^3 x_4 x_5^{21}$
 454. $x_1^7 x_2^3 x_4 x_5^{20}$
 455. $x_1^7 x_2^3 x_4 x_5^{20}$
 456. $x_1^7 x_2^3 x_4 x_5^{17}$
 457. $x_1^7 x_2^3 x_4 x_5^{17}$
 458. $x_1^7 x_2^3 x_4 x_5^{16}$
 459. $x_1^7 x_2^3 x_4 x_5^{16}$
 460. $x_1^7 x_2^3 x_4 x_5^{16}$
 461. $x_1^7 x_2^3 x_4 x_5^{20}$
 462. $x_1^7 x_2^3 x_4 x_5^{20}$
 463. $x_1^7 x_2^3 x_4 x_5^{16}$
 464. $x_1^7 x_2^3 x_4 x_5^{16}$
 465. $x_1^7 x_2^3 x_4 x_5^{16}$
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 467. $x_1^7 x_2^3 x_4 x_5^{17}$
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 472. $x_1^7 x_2^3 x_4 x_5^{16}$
 473. $x_1^7 x_2^3 x_4 x_5^{16}$
 474. $x_1^7 x_2^3 x_4 x_5^{16}$
 475. $x_1^7 x_2^3 x_4 x_5^{16}$
 476. $x_1^7 x_2^3 x_4 x_5^{16}$
 477. $x_1^7 x_2^3 x_4 x_5^{16}$
 478. $x_1^7 x_2^3 x_4 x_5^{16}$
 479. $x_1^7 x_2^3 x_4 x_5^{16}$
 480. $x_1^7 x_2^3 x_4 x_5^{16}$
 481. $x_1^7 x_2^3 x_4 x_5^{17}$
 482. $x_1^7 x_2^3 x_4 x_5^{16}$
 483. $x_1^7 x_2^3 x_4 x_5^{16}$
 484. $x_1^7 x_2^3 x_4 x_5^{16}$
 485. $x_1^7 x_2^3 x_4 x_5^{16}$

$\mathcal{B}_5^+(\omega_{(4)})$ is the set of 119 monomials $d_k = d_{39,k}$, $486 \leq k \leq 604$:

$$486. x_1 x_2^2 x_3^2 x_4^3 x_5^{31} \quad 487. x_1 x_2^2 x_3^2 x_4^3 x_5^3 \quad 488. x_1 x_2^2 x_3^3 x_4^2 x_5^{31} \quad 489. x_1 x_2^2 x_3^3 x_4^3 x_5^2$$

490. $x_1x_2^2x_3^{31}x_4x_5^3$	491. $x_1x_2^2x_3^{31}x_4^2x_5^2$	492. $x_1x_2^3x_3^2x_4^2x_5^{31}$	493. $x_1x_2^3x_3^2x_4^{31}x_5^2$
494. $x_1x_2^3x_3^{31}x_4^2x_5^2$	495. $x_1x_2^3x_3^2x_4^2x_5^3$	496. $x_1x_2^3x_3^2x_4^3x_5^2$	497. $x_1x_2^3x_3^2x_4^2x_5^2$
498. $x_1^3x_2x_3x_4x_5^{31}$	499. $x_1^3x_2x_3x_4^2x_5^{31}$	500. $x_1^3x_2x_3^2x_4x_5^2$	501. $x_1^3x_2x_3x_4x_5^{31}$
502. $x_1^3x_2x_3x_4x_5^2$	503. $x_1^3x_2x_3x_4x_5^2$	504. $x_1^3x_2x_3x_4x_5^2$	505. $x_1^3x_2x_3x_4x_5^2$
506. $x_1x_2^2x_3^2x_4x_5^{27}$	507. $x_1x_2^2x_3^2x_4x_5^{27}$	508. $x_1x_2^2x_3^2x_4x_5^{27}$	509. $x_1x_2^2x_3^2x_4x_5^{27}$
510. $x_1x_2^7x_3^2x_4x_5^{27}$	511. $x_1x_2^7x_3^2x_4x_5^{27}$	512. $x_1x_2^7x_3^2x_4x_5^{27}$	513. $x_1x_2^7x_3^2x_4x_5^{27}$
514. $x_1^7x_2x_3x_4x_5^{27}$	515. $x_1^7x_2^2x_3x_4x_5^{27}$	516. $x_1x_2^2x_3x_4x_5^{33}$	517. $x_1x_2^2x_3x_4x_5^{30}$
518. $x_1x_2^3x_3x_4x_5^{30}$	519. $x_1x_2^3x_3x_4x_5^{30}$	520. $x_1x_2^3x_3x_4x_5^{30}$	521. $x_1x_2^3x_3x_4x_5^{30}$
522. $x_1x_2^3x_3^3x_4x_5^2$	523. $x_1x_2^3x_3^3x_4x_5^2$	524. $x_1x_2x_3^2x_4x_5^{30}$	525. $x_1x_2x_3^2x_4x_5^{30}$
526. $x_1^3x_2x_3x_4x_5^{30}$	527. $x_1^3x_2x_3x_4x_5^{30}$	528. $x_1x_2x_3^2x_4x_5^{30}$	529. $x_1x_2x_3^2x_4x_5^{30}$
530. $x_1^3x_2x_3x_4x_5^{30}$	531. $x_1^3x_2x_3x_4x_5^{30}$	532. $x_1x_2x_3x_4x_5^{36}$	533. $x_1x_2x_3x_4x_5^{27}$
534. $x_1x_2^3x_3x_4x_5^{27}$	535. $x_1x_2^3x_3x_4x_5^{27}$	536. $x_1x_2^3x_3x_4x_5^{27}$	537. $x_1x_2x_3x_4x_5^{27}$
538. $x_1^3x_2x_3x_4x_5^{27}$	539. $x_1x_2^3x_3x_4x_5^{27}$	540. $x_1x_2^2x_3x_4x_5^{26}$	541. $x_1x_2^3x_3x_4x_5^{26}$
542. $x_1x_2^3x_3x_4x_5^{26}$	543. $x_1x_2^3x_3x_4x_5^{26}$	544. $x_1x_2^2x_3x_4x_5^{26}$	545. $x_1x_2^3x_3x_4x_5^{26}$
546. $x_1^3x_2x_3x_4x_5^{26}$	547. $x_1^3x_2x_3x_4x_5^{26}$	548. $x_1^3x_2x_3x_4x_5^{26}$	549. $x_1x_2x_3x_4x_5^{26}$
550. $x_1^3x_2x_3x_4x_5^{26}$	551. $x_1^3x_2x_3x_4x_5^{26}$	552. $x_1^7x_2x_3x_4x_5^{26}$	553. $x_1x_2x_3x_4x_5^{26}$
554. $x_1^7x_2x_3x_4x_5^{26}$	555. $x_1^7x_2x_3x_4x_5^{26}$	556. $x_1^7x_2x_3x_4x_5^{26}$	557. $x_1x_2x_3x_4x_5^{26}$
558. $x_1x_2x_3x_4x_5^{26}$	559. $x_1x_2x_3x_4x_5^{26}$	560. $x_1x_2x_3x_4x_5^{26}$	561. $x_1x_2x_3x_4x_5^{26}$
562. $x_1^3x_2x_3x_4x_5^{26}$	563. $x_1^3x_2x_3x_4x_5^{26}$	564. $x_1x_2^3x_3x_4x_5^{19}$	565. $x_1x_2x_3x_4x_5^{19}$
566. $x_1x_2^3x_3x_4x_5^{18}$	567. $x_1^3x_2x_3x_4x_5^{18}$	568. $x_1x_2^3x_3x_4x_5^{18}$	569. $x_1x_2^7x_3x_4x_5^{18}$
570. $x_1^3x_2x_3x_4x_5^{10,18}$	571. $x_1^3x_2x_3x_4x_5^{10,18}$	572. $x_1^7x_2x_3x_4x_5^{10,18}$	573. $x_1^7x_2x_3x_4x_5^{10,18}$
574. $x_1^3x_2x_3^{29}x_4x_5^2$	575. $x_1^3x_2^{29}x_3x_4x_5^2$	576. $x_1^3x_2^{29}x_3x_4x_5^2$	577. $x_1^3x_2^{29}x_3x_4x_5^2$
578. $x_1^3x_2^5x_3^2x_4x_5^{27}$	579. $x_1^3x_2^5x_3^2x_4x_5^{27}$	580. $x_1^3x_2^5x_3^2x_4x_5^{27}$	581. $x_1^3x_2^7x_3^2x_4x_5^{27}$
582. $x_1^7x_2x_3^{25}x_4x_5^2$	583. $x_1^3x_2x_3x_4x_5^{26}$	584. $x_1^3x_2x_3x_4x_5^{26}$	585. $x_1^3x_2x_3x_4x_5^{26}$
586. $x_1^3x_2^5x_3x_4x_5^{26}$	587. $x_1^3x_2^5x_3x_4x_5^{26}$	588. $x_1^3x_2^5x_3x_4x_5^{26}$	589. $x_1^3x_2x_3x_4x_5^{26}$
590. $x_1^3x_2^5x_3^2x_4x_5^2$	591. $x_1^3x_2^5x_3x_4x_5^{10,19}$	592. $x_1^3x_2^5x_3x_4x_5^{10,19}$	593. $x_1^3x_2x_3^{10}x_4x_5^2$
594. $x_1^3x_2^5x_3x_4x_5^{11,18}$	595. $x_1^3x_2^5x_3^{11}x_4x_5^{18}$	596. $x_1^3x_2^5x_3^{11}x_4x_5^{18}$	597. $x_1^3x_2^7x_3x_4x_5^{18}$
598. $x_1^3x_2^7x_3x_4x_5^{18,2}$	599. $x_1^7x_2x_3x_4x_5^{18,2}$	600. $x_1^7x_2x_3x_4x_5^{18,2}$	601. $x_1^3x_2x_3^5x_4x_5^{10}$
602. $x_1^3x_2^5x_3x_4x_5^{10,18}$	603. $x_1^3x_2^5x_3^{10}x_4x_5^{18}$	604. $x_1^3x_2^5x_3^{10}x_4x_5^{18}$	

$\mathcal{B}_5^+(\omega_{(5)})$ is the set of 5 monomials $d_k = d_{39,k}$, $605 \leq k \leq 609$:

$$\begin{array}{lll} 605. x_1^3x_2^5x_3^{10}x_4^{10}x_5^{11} & 606. x_1^3x_2^5x_3^{10}x_4^{11}x_5^{10} & 607. x_1^3x_2^5x_3^{11}x_4^{10}x_5^{10} \\ 608. x_1^3x_2^7x_3x_4^{10}x_5^{10} & 609. x_1^7x_2^3x_3x_4^{10}x_5^{10} & \end{array}$$

$\mathcal{B}_5^+(\omega_{(6)})$ is the set of 40 monomials $d_k = d_{39,k}$, $610 \leq k \leq 649$:

610. $x_1x_2^3x_3^7x_4^{14}x_5^{14}$	611. $x_1x_2^7x_3^3x_4^{14}x_5^{14}$	612. $x_1^3x_2x_3x_4^{14}x_5^{14}$	613. $x_1^3x_2^7x_3x_4^{14}x_5^{14}$
614. $x_1^7x_2x_3^3x_4^{14}x_5^{14}$	615. $x_1^7x_2^3x_3x_4^{14}x_5^{14}$	616. $x_1x_2^7x_3^{11}x_4^6x_5^{14}$	617. $x_1x_2^7x_3^{11}x_4^{14}x_5^6$
618. $x_1^7x_2x_3^{11}x_4^6x_5^{14}$	619. $x_1^7x_2x_3^{11}x_4^{14}x_5^6$	620. $x_1^7x_2^{11}x_3x_4^6x_5^{14}$	621. $x_1^7x_2^{11}x_3x_4^{14}x_5^6$
622. $x_1^3x_2x_3^7x_4^{13}x_5^{14}$	623. $x_1^3x_2x_3^7x_4^{13}x_5^2$	624. $x_1^7x_2x_3^7x_4^{13}x_5^2$	625. $x_1^7x_2x_3^7x_4^{13}x_5^2$
626. $x_1^7x_2^7x_3x_4^{14}x_5^{14}$	627. $x_1^7x_2^7x_3^5x_4^{14}x_5^2$	628. $x_1^7x_2^7x_3^{13}x_4^2x_5^6$	629. $x_1^7x_2^7x_3^{13}x_4^2x_5^6$
630. $x_1^3x_2^3x_3^5x_4^{14}x_5^{14}$	631. $x_1^3x_2^3x_3^5x_4^{14}x_5^{14}$	632. $x_1^3x_2^3x_3^{13}x_4^6x_5^{14}$	633. $x_1^3x_2^3x_3^{13}x_4^{14}x_5^6$
634. $x_1^3x_2^5x_3^{11}x_4^{14}x_5^6$	635. $x_1^3x_2^5x_3^{11}x_4^{14}x_5^6$	636. $x_1^3x_2^7x_3^5x_4^{10}x_5^{14}$	637. $x_1^3x_2^7x_3^5x_4^{14}x_5^{10}$
638. $x_1^7x_2^3x_3^5x_4^{10}x_5^{14}$	639. $x_1^7x_2^3x_3^5x_4^{14}x_5^{10}$	640. $x_1^3x_2^7x_3^9x_4^6x_5^{14}$	641. $x_1^3x_2^7x_3^9x_4^{14}x_5^6$

$$\begin{array}{cccc} 642. x_1^7 x_2^3 x_3^9 x_4^6 x_5^{14} & 643. x_1^7 x_2^3 x_3^9 x_4^{14} x_5^6 & 644. x_1^3 x_2^7 x_3^{13} x_4^6 x_5^{10} & 645. x_1^3 x_2^7 x_3^{13} x_4^{10} x_5^6 \\ 646. x_1^7 x_2^3 x_3^{13} x_4^6 x_5^{10} & 647. x_1^7 x_2^3 x_3^{13} x_4^{10} x_5^6 & 648. x_1^7 x_2^{11} x_3^5 x_4^6 x_5^{10} & 649. x_1^7 x_2^{11} x_3^5 x_4^{10} x_5^6 \end{array}$$

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