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AUTOMORPHISMS OF UNIFORM LATTICES OF NILPOTENT LIE GROUPS UP TO DIMENSION FOUR

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Abstract. In this paper, when G is a connected and simply connected nilpotent Lie group of dimension less than or equal to four, we study the uniform lattices Γ of G up to isomorphism and then we study the structure of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ from the viewpoint of splitting as a natural extension.

1. Introduction

In this paper we study the group of automorphisms of any uniform lattice of a connected and simply connected nilpotent Lie group G up to dimension four. This work was motivated by the papers [3] and [5], in which the authors considered the discrete subgroup $\mathrm{Heis}(3,\mathbb{Z})$ of the three-dimensional Heisenberg group $\mathrm{Heis}(3,\mathbb{R})$ and proved that the automorphism group $\mathrm{Aut}(\mathrm{Heis}(3,\mathbb{Z}))$ admits a splitting as a natural extension of $\mathbb{Z}^2$ by $\mathrm{GL}(2,\mathbb{Z})$.

The connected and simply connected nilpotent Lie groups of dimension less than or equal to four are well understood. In dimension one or two, there is only one such Lie group, the abelian Lie group $\mathbb{R}$ or $\mathbb{R}^2$. There are two connected and simply connected three-dimensional nilpotent Lie groups $\mathbb{R}^3$ and Nil^3. The Lie group Nil^3 is the Heisenberg group

$$\mathrm{Heis}(3,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\}.$$

There are three connected and simply connected four-dimensional nilpotent Lie groups $\mathbb{R}^4, \mathrm{Nil}^3\times\mathbb{R}$ and Nil^4. The Lie groups $\mathrm{Nil}^3\times\mathbb{R}$ and Nil^4 are of the form

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$\mathbb{R}^3 \rtimes_{\varphi(s)} \mathbb{R}$, [4], where $\varphi(s)$ is respectively

$$\varphi(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & s & \frac{1}{2}s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

For solvable Lie groups, one may refer to [6, 7].

A discrete subgroup Γ of a Lie group G is called a *uniform lattice* of G if its orbit space $\Gamma\backslash G$ is compact. When G is an abelian Lie group $\mathbb{R}^n$, every uniform lattice Γ of G is isomorphic to $\mathbb{Z}^n \subset \mathbb{R}^n$ and hence $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Z})$.

In this paper, when G is Nil^3, $\mathrm{Nil}^3 \times \mathbb{R}$ or Nil^4, we first study the uniform lattices Γ of G up to isomorphism and then we study the structure of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ from the viewpoint of splitting as a natural extension.

2. The Lie group Nil^3

The Lie group Nil^3 is the Heisenberg group

$$\mathrm{Heis}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

We will study $\mathrm{Aut}(\Gamma)$ for any uniform lattice Γ of Nil^3. The groups

$$\Gamma_k = \left\{ \begin{pmatrix} 1 & n & \frac{\ell}{k} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mid \ell, m, n \in \mathbb{Z} \right\}$$

are uniform lattices of Nil^3. Note that $\Gamma_1 = \mathrm{Heis}(3, \mathbb{Z})$, and $\Gamma_{-k} \cong \Gamma_k$ for all $k \neq 0$. It is known that any uniform lattice of Nil^3 is isomorphic to exactly one Γ_k for some $k > 0$. In [3] and [5], it is shown that the automorphism group of Γ_1 is isomorphic to the group $\mathbb{Z}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})$. Utilizing the methods employed in [3] and [5], we can prove that $\mathrm{Aut}(\Gamma_k)$ is isomorphic to $\mathbb{Z}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})$ for every $k > 0$.

Lemma 2.1. *The lattice Γ_k of* Nil^3 *may be presented as*

$$\Gamma_k = \langle\, \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma^{-k}, [\gamma, \alpha] = [\gamma, \beta] = 1 \,\rangle$$

with α, β and γ corresponding to the generators

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because the center $\mathcal{Z}(\Gamma_k)$ of Γ_k is generated by γ, every automorphism of Γ_k induces an automorphism of the quotient group $\Gamma_k/\mathcal{Z}(\Gamma_k) \cong \mathbb{Z}^2$. Thus we have a natural homomorphism $\vartheta : \mathrm{Aut}(\Gamma_k) \to \mathrm{GL}(2, \mathbb{Z})$. Indeed it is well-known that

ϑ is surjective, see for example [3, Proposition 6]. The purpose of this section is, using the Lie algebra argument, to prove that $\vartheta : \mathrm{Aut}(\Gamma_k) \to \mathrm{GL}(2,\mathbb{Z})$ splits.

Recall that the Lie algebra $\mathfrak{nil}^3$ of Nil^3 is

$$\mathfrak{nil}^3 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

Choose the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathfrak{nil}^3$ where

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $[\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_3$ and $[\mathbf{e}_3, \mathbf{e}_1] = [\mathbf{e}_3, \mathbf{e}_2] = \mathbf{0}$. With respect to the canonical basis, each automorphism of $\mathfrak{nil}^3$ has a matrix presentation.

Proposition 2.2 ([2, Proposition 2.2])**.** *The group* $\mathrm{Aut}(\mathfrak{nil}^3)$ *of all automorphisms of the Lie algebra* $\mathfrak{nil}^3$ *is isomorphic to the* **matrix group**

$$\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad-bc \end{pmatrix} \mid a, b, c, d, u, v \in \mathbb{R}, \ ad - bc \neq 0 \right\}.$$

Since the Lie group Nil^3 is simply connected, we have a canonical isomorphism $\mathrm{Aut}(\mathrm{Nil}^3) \cong \mathrm{Aut}(\mathfrak{nil}^3)$ defined by $\varphi \mapsto \mathrm{d}\varphi$ fitting in the following commuting diagram:

(2.1)
$$\begin{array}{ccc} \mathfrak{nil}^3 & \xrightarrow{\mathrm{d}\varphi} & \mathfrak{nil}^3 \\ \uparrow{\scriptstyle \log} & & \downarrow{\scriptstyle \exp} \\ \mathrm{Nil}^3 & \xrightarrow{\varphi} & \mathrm{Nil}^3 \end{array}$$

Using the diffeomorphism

$$\exp : \mathfrak{nil} \to \mathrm{Nil}, \begin{pmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & b & c + \frac{1}{2}ab \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

we can see that if

$$\mathrm{d}\varphi = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad-bc \end{pmatrix},$$

then

$$\varphi = \exp \circ \mathrm{d}\varphi \circ \log : \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & cx+dy & z^* \\ 0 & 1 & ax+by \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where $z^* = (ad-bc)z + \frac{1}{2}acx^2 + ux + bcxy + vy + \frac{1}{2}bdy^2$. In what follows, we shall identify $\phi = \mathrm{d}\varphi$ with φ.

Consider the uniform lattice Γ_k of Nil^3. Due to Mal'cev, every automorphism of Γ_k can be extended uniquely to a Lie group automorphism of Nil^3. This implies that we can regard $\mathrm{Aut}(\Gamma_k)$ as a subgroup of $\mathrm{Aut}(\mathrm{Nil}^3)$. Thus

$$\mathrm{Aut}(\Gamma_k) \subset \mathrm{Aut}(\mathrm{Nil}^3) = \mathrm{Aut}(\mathfrak{nil}^3)$$

as a subgroup of the matrix group $\mathrm{GL}(3,\mathbb{R})$. Furthermore, we have a commutative diagram between surjective homomorphisms

$$\begin{tikzcd}
\mathrm{Aut}(\mathfrak{nil}^3) \arrow[r, "\tilde\vartheta"] & \mathrm{Aut}(\mathbb{R}^2) = \mathrm{GL}(2,\mathbb{R}) \arrow[r] & 1 \\
\mathrm{Aut}(\Gamma_k) \arrow[u] \arrow[r, "\vartheta"] & \mathrm{GL}(2,\mathbb{Z}) \arrow[u] \arrow[r] & 1
\end{tikzcd}$$

where the vertical maps are inclusions, and

$$\tilde\vartheta : \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u & v & ad-bc \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now we recall that $\mathrm{GL}(2,\mathbb{Z})$ is presented as

$$\langle \rho,\tau,\kappa \mid \rho\tau\rho = \tau\rho\tau, (\rho\tau\rho)^4 = 1, \kappa\rho\kappa^{-1} = \rho^{-1}, \kappa\tau\kappa^{-1} = \tau^{-1}, \kappa^2 = 1\rangle,$$

where ρ,τ and κ may be taken to correspond to the matrices

$$\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \tau = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We seek an explicit section η of $\tilde\vartheta$. Consider the following elements of $\mathrm{Aut}(\mathfrak{nil}^3)$:

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ u_1 & v_1 & 1 \end{pmatrix},\ T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ u_2 & v_2 & 1 \end{pmatrix},\ K = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ u_3 & v_3 & -1 \end{pmatrix}.$$

As observed above, these elements are regarded as elements of $\mathrm{Aut}(\mathrm{Nil}^3)$. Assume that the elements R,T,K satisfy the defining relations of $\mathrm{GL}(2,\mathbb{Z})$.

$$RTR = TRT,\ (RTR)^4 = I_3,\ (RTR)^2 \neq I_3,$$
$$K^2 = I_3,\ KRK^{-1} = R^{-1},\ KTK^{-1} = T^{-1}.$$

By direct computation with (2.2) we have

$$R(\alpha) = \alpha\gamma^{u_1k},\ R(\beta) = \alpha\beta\gamma^{(v_1+1/2)k},\ R(\gamma) = \gamma,$$
$$T(\alpha) = \alpha\beta^{-1}\gamma^{(u_2-1/2)k},\ T(\beta) = \beta\gamma^{v_2k},\ T(\gamma) = \gamma,$$
$$K(\alpha) = \alpha^{-1}\gamma^{u_3k},\ K(\beta) = \beta\gamma^{v_3k},\ K(\gamma) = \gamma^{-1}.$$

Using $[\alpha,\gamma] = [\beta,\gamma] = 1$, it is immediate to check the implications

$$RTR(\alpha) = TRT(\alpha) \Rightarrow v_2 = 0,$$
$$K^2(\alpha) = \alpha \Rightarrow u_3 = 0,$$

$$KRK^{-1}(\alpha) = R^{-1}(\alpha) \Rightarrow u_1 = 0,$$
$$KTK^{-1}(\alpha) = T^{-1}(\alpha) \Rightarrow 2u_2 + v_3 = -1.$$

We can choose v_1, u_2, v_3 appropriately so that R, T, K also preserve Γ_k. For example, one can take

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix},\ T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix},\ K = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

and define $\eta : \mathrm{GL}(2,\mathbb{Z}) \to \mathrm{Aut}(\mathfrak{nil}^3)$ by $\rho \mapsto R,\ \tau \mapsto T,\ \kappa \mapsto K$. It is straightforward to check that the above R, T and K satisfy the remaining relations of $\mathrm{GL}(2,\mathbb{Z})$. Therefore, the subgroup of $\mathrm{Aut}(\Gamma_k) \subset \mathrm{Aut}(\mathfrak{nil}^3)$ generated by R, T and K is isomorphic to $\mathrm{GL}(2,\mathbb{Z})$. Consequently, η provides a desired splitting of ϑ. The splitting η is independent of Γ_k.

Remark 2.3. The group structure of $\mathrm{Aut}(\Gamma_k)$ is complicated. By considering the Lie algebra, we can embed $\mathrm{Aut}(\Gamma_k)$ as a subgroup of the matrix group $\mathrm{GL}(3,\mathbb{R})$. This makes easy to check whether a splitting function of the surjective homomorphism $\vartheta : \mathrm{Aut}(\Gamma_k) \to \mathrm{GL}(2,\mathbb{Z})$ is a "homomorphism".

Remark 2.4 (Automorphisms of Γ_k). We have shown that the homomorphism

$$\vartheta : \mathrm{Aut}(\Gamma_k)(\subset \mathrm{Aut}(\mathfrak{nil}^3) \subset \mathrm{GL}(3,\mathbb{R})) \to \mathrm{GL}(2,\mathbb{Z})$$

is surjective and splits. This in particular implies that

$$\mathrm{Aut}(\Gamma_k) = \left\{ H = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ p_1 & p_2 & p_3 \end{pmatrix} \in \mathrm{Aut}(\mathfrak{nil}^3) \;\middle|\; H \text{ preserves } \Gamma_k \right\}.$$

It is now straightforward to observe further that

$$\mathrm{Aut}(\Gamma_k) = \left\{ \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ p_1 & p_2 & m_{11}m_{22} - m_{21}m_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} m_{ij} \in \mathbb{Z}, \\ p_i + \frac{k}{2} m_{1i} m_{2i} \in \mathbb{Z} \end{array} \right\}.$$

Hence, the groups $\mathrm{Aut}(\Gamma_k)$ are identical for all $k \neq 0$.

In conclusion we have that

$$\ker(\vartheta) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_1 & p_2 & 1 \end{pmatrix} \;\middle|\; p_1, p_2 \in \mathbb{Z} \right\} \cong \mathbb{Z} \oplus \mathbb{Z},$$

and

$$\mathrm{Aut}(\Gamma_k) \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathrm{GL}(2,\mathbb{Z}),$$

where the $\mathrm{GL}(2,\mathbb{Z})$-action on $\ker(\vartheta)$ is

$$g \cdot \mathbf{p} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}. \tag{2.3}$$

3. The nilpotent Lie group $\mathrm{Nil}^3 \times \mathbb{R}$

There are three simply connected four-dimensional nilpotent Lie groups: $\mathbb{R}^4, \mathrm{Nil}^3 \times \mathbb{R}$ and Nil^4. The Lie groups $\mathrm{Nil}^3 \times \mathbb{R}$ and Nil^4 are of the form $\mathbb{R}^3 \rtimes_{\varphi(s)} \mathbb{R}$, [4], where $\varphi(s)$ is respectively

$$\varphi(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & s & \frac{1}{2}s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

The group law of $\mathbb{R}^3 \rtimes_{\varphi(s)} \mathbb{R}$ is

$$(\mathbf{x}, s)(\mathbf{y}, t) = (\mathbf{x} + \varphi(s)\mathbf{y}, s + t),$$

and it can be embedded affinely in $\mathrm{GL}(4, \mathbb{R})$ as

$$\left\{ \begin{pmatrix} \varphi(s) & \mathbf{x} \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}(4, \mathbb{R}),$$

where $\varphi(s) \in \mathrm{GL}(3, \mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^3$ is a column vector. Remark that $\mathrm{Nil}^3 \times \mathbb{R}$ is 2-step and Nil^4 is 3-step. We will continue to study $\mathrm{Aut}(\Gamma)$ for a uniform lattice Γ of $\mathrm{Nil}^3 \times \mathbb{R}$ in this section and Nil^4 in the next section.

It is easy to see that any uniform lattice Γ of $\mathrm{Nil}^3 \times \mathbb{R}$ is isomorphic to the product $\Gamma_k \times \mathbb{Z}$, where Γ_k is a uniform lattice of Nil^3, see for example [1, Corollary 6.2.5]. Hence Γ can be presented as

$$\langle\, \alpha, \beta, \gamma, \delta \mid [\alpha, \beta] = \gamma^{-k}, [\gamma, \alpha] = [\gamma, \beta] = [\delta, \alpha] = [\delta, \beta] = [\delta, \gamma] = 1 \,\rangle$$

with α, β, γ and δ corresponding to the generators

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{k} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us denote $\Gamma_k \times \mathbb{Z}$ by $\Gamma_{k,0}$. Since $\mathcal{Z}(\Gamma_{k,0}) = \langle \gamma, \delta \rangle$, we obtain a canonical surjective homomorphism $\vartheta' : \mathrm{Aut}(\Gamma_{k,0}) \to \mathrm{GL}(2, \mathbb{Z})$. We will show that ϑ' also splits. It suffices to show that $\mathrm{Aut}(\Gamma_k)$ can be regarded as a subgroup of $\mathrm{Aut}(\Gamma_{k,0})$ so that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Aut}(\Gamma_{k,0}) & \xrightarrow{\tilde{\vartheta}} & \mathrm{GL}(2, \mathbb{Z}) & \longrightarrow & 1 \\ \uparrow & & \uparrow{=} & & \\ \mathrm{Aut}(\Gamma_k) & \xrightarrow{\vartheta} & \mathrm{GL}(2, \mathbb{Z}) & \longrightarrow & 1 \end{array} \tag{3.1}$$

Every element of $\Gamma_{k,0}$ can be written as $\alpha^m\beta^n\gamma^p\delta^q$. It can be seen easily that

$$(\alpha^m\beta^n\gamma^p\delta^q)^r = \alpha^{rm}\beta^{rn}\gamma^{rp+kr^{(2)}mn}\delta^{rq}, \tag{3.2}$$

$$\beta^n\alpha^m = \alpha^m\beta^n\gamma^{kmn} \tag{3.3}$$

for all $r \in \mathbb{Z}$. For any $h \in \mathrm{Aut}(\Gamma_{k,0})$, since $\mathcal{Z}(\Gamma_{k,0}) = \langle\gamma,\delta\rangle$, we must have

$$\begin{aligned} h(\alpha) &= \alpha^{m_{11}}\beta^{m_{21}}\gamma^{p_{11}}\delta^{p_{21}}, \\ h(\beta) &= \alpha^{m_{12}}\beta^{m_{22}}\gamma^{p_{12}}\delta^{p_{22}}, \\ h(\gamma) &= \gamma^e\delta^{e'},\ h(\delta) = \gamma^{e_1}\delta^{e_2}, \end{aligned}$$

for some integers m_{ij}, p_{ij} and e, e', e_1, e_2. Since h preserves the commutator relations

$$[\alpha,\beta] = \gamma^{-k}, [\gamma,\alpha] = [\gamma,\beta] = [\delta,\alpha] = [\delta,\beta] = [\delta,\gamma] = 1,$$

it follows from (3.2) and (3.3) that $e' = 0$ and $e = m_{11}m_{22} - m_{12}m_{21}$. Notice that

(1) $\tilde{\vartheta}(h) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$,

(2) any automorphism h of Γ_k can be regarded as an automorphism h of $\Gamma_{k,0}$ by taking

$$p_{21} = p_{22} = e' = e_1 = 0,\ e_2 = 1.$$

Consequently, we have shown that the diagram (3.1) is commutative and since ϑ splits by Section 2 it follows that $\tilde{\vartheta}$ splits.

4. The nilpotent Lie group Nil^4

The nilpotent Lie group Nil^4 is the matrix group

$$\mathrm{Nil}^4 = \left\{ \begin{pmatrix} 1 & s & \frac{1}{2}s^2 & x \\ 0 & 1 & s & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x, y, z, s \in \mathbb{R} \right\}$$

and so its Lie algebra is

$$\mathfrak{nil}^4 = \left\{ \begin{pmatrix} 0 & s & 0 & a \\ 0 & 0 & s & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c, s \in \mathbb{R} \right\}.$$

The commutator subgroup $[\mathrm{Nil}^4, \mathrm{Nil}^4]$ is $\mathbb{R}^2$ (with $s = z = 0$), and the center $\mathcal{Z}(\mathrm{Nil}^4)$ of Nil^4 is $\mathbb{R}$ (with $s = y = z = 0$). Since $\mathrm{Nil}^4/\mathcal{Z}(\mathrm{Nil}^4)$ is a three-dimensional nilpotent Lie group, it follows that it is isomorphic to Nil^3. That is,

$$\begin{tikzcd}
 & & & \mathrm{Nil}^3 & \\
1 \arrow[r] & \mathcal{Z}(\mathrm{Nil}^4) \arrow[r] & \mathrm{Nil}^4 \arrow[r] & \mathrm{Nil}^4/\mathcal{Z}(\mathrm{Nil}^4) \arrow[r] \arrow[u, "\pi", "\cong"'] & 1
\end{tikzcd}$$

where the explicit isomorphism is given by

$$\pi : \begin{pmatrix} 1 & s & \frac{1}{2}s^2 & * \\ 0 & 1 & s & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & s & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

We can see further that $[\mathrm{Nil}^4, \mathrm{Nil}^4]/\mathcal{Z}(\mathrm{Nil}^4)$ is isomorphic to $\mathcal{Z}(\mathrm{Nil}^3)$. Consequently, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \mathcal{Z}(\mathrm{Nil}^3) \arrow[r] \arrow[u] & \mathrm{Nil}^3 \arrow[r] \arrow[u] & \mathrm{Nil}^3/\mathcal{Z}(\mathrm{Nil}^3) \arrow[r] & 1 \\
1 \arrow[r] & {[\mathrm{Nil}^4, \mathrm{Nil}^4]} \arrow[r] \arrow[u] & \mathrm{Nil}^4 \arrow[r] \arrow[u, "\pi"'] & \mathrm{Nil}^4/[\mathrm{Nil}^4, \mathrm{Nil}^4] \arrow[r] \arrow[u, "\cong"'] & 1 \\
 & \mathcal{Z}(\mathrm{Nil}^4) \arrow[r, "="] \arrow[u] & \mathcal{Z}(\mathrm{Nil}^4) \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

4.1. The lattices of $\mathbf{Nil^4}$

Noting that the subset of elements of Nil^4 with $s, x, y, z \in \mathbb{Z}$ is not a group, we shall describe the uniform lattices of Nil^4.

Let Γ be a uniform lattice of Nil^4. Then $\mathcal{Z}(\mathrm{Nil}^4) \cap \Gamma$ is a uniform lattice of $\mathcal{Z}(\mathrm{Nil}^4) = \mathbb{R}$, and $\Gamma/(\mathcal{Z}(\mathrm{Nil}^4) \cap \Gamma)$ is a uniform lattice of $\mathrm{Nil}^4/\mathcal{Z}(\mathrm{Nil}^4) = \mathrm{Nil}^3$.

Hence Γ fits in the following short exact sequences:

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u] & \bar{\Gamma} \arrow[r] \arrow[u] & \mathbb{Z}^2 \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[u] & \Gamma \arrow[r] \arrow[u, "\pi"'] & \mathbb{Z}^2 \arrow[r] \arrow[u, "\cong"'] & 1 \\
 & \mathbb{Z} \arrow[r, "="] \arrow[u] & \mathbb{Z} \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd} \tag{4.1}$$

Since $\bar{\Gamma}$ is a uniform lattice of Nil^3, it is isomorphic to some Γ_k. Thus we can choose a set of generators $\alpha, \beta, \gamma, \delta$ of Γ so that $\delta \in \mathcal{Z}(\mathrm{Nil}^4)$, and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ generate $\bar{\Gamma}$. We can assume that $[\bar{\alpha}, \bar{\beta}] = \bar{\gamma}^{-k}$ with $k > 0$. This implies that α, β, γ and δ may be taken to correspond to matrices in Nil^4 of the form

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 1 & \frac{1}{2} & b \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.2}$$
$$\gamma = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & \frac{1}{k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \delta = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A direct computation shows that the commutators among $\alpha, \beta, \gamma, \delta$ are independent of a and b, hence we may choose $a = b = 0$. Moreover, $[\alpha, \gamma] = 1$. Therefore, Γ can be presented as

$$\begin{aligned} \langle \alpha, \beta, \gamma, \delta \mid [\alpha, \beta] = \gamma^{-k}\delta^m, [\alpha, \gamma] = 1, \\ [\beta, \gamma] = \delta^n, [\alpha, \delta] = [\beta, \delta] = [\gamma, \delta] = 1 \rangle. \end{aligned} \tag{4.3}$$

We choose a, b, c and $d > 0$ so that they satisfy

$$a = b = 0, \ dn = \frac{1}{k}, \ kc = dm + \frac{1}{2}. \tag{4.4}$$

Then we can assume that $k, n > 0$ and the choice (4.2) of matrices for α, β, γ and δ with the conditions (4.4) gives rise to a realization of an abstract group with presentation (4.3) as a uniform lattice of Nil^4. We shall denote this Γ by

$\Gamma_{k,m,n}$ with $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$. For example, we can see that

$$\Gamma_{1,-1,2} = \left\{ \begin{pmatrix} 1 & n_2 & \frac{n_2^2}{2} & \frac{n_4}{2} \\ 0 & 1 & n_2 & n_3 \\ 0 & 0 & 1 & n_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\Big|\; n_1, n_2, n_3, n_4 \in \mathbb{Z} \right\}.$$

We remark also that $\Gamma_{k,m,n} \cong \Gamma_{k',m',n'}$ if and only if

$$k = k',\ n = n',\ m' = \pm m \mod (k, n),$$

see for example [1, Corollary 6.2.7].

4.2. Automorphisms of $\boldsymbol{\Gamma_{k,m,n}}$

Let $\Gamma = \Gamma_{k,m,n}$ and let $h : \Gamma \to \Gamma$ be an automorphism of Γ. Then h preserves the diagram (4.1). This implies that the diagram (4.1) induces the following commutative diagram:

$$\begin{tikzcd} \mathrm{Aut}(\Gamma_k) \arrow[r, "\vartheta"] & \mathrm{GL}(2,\mathbb{Z}) \\ \mathrm{Aut}(\Gamma) \arrow[u, "\sigma"] \arrow[r, "\Theta"] & \mathrm{GL}(2,\mathbb{Z}) \arrow[u, "="] \end{tikzcd}$$

Here $\sigma : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma_k)$ is the homomorphism induced by π. We remark also that ϑ is surjective and split by Section 2.

Consider the elements $\mathbf{e}_i \in \mathfrak{nil}^4$

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{m}{k^2 n} + \frac{1}{2k} \\ 0 & 0 & 0 & \frac{1}{k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{e}_4 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{kn} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\exp \mathbf{e}_1 = \alpha,\ \exp \mathbf{e}_2 = \beta,\ \exp \mathbf{e}_3 = \gamma,\ \exp \mathbf{e}_4 = \delta$$

generate Γ, and the nontrivial Lie brackets in $\mathfrak{nil}^4$ between $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ are

$$[\mathbf{e}_1, \mathbf{e}_2] = -k\mathbf{e}_3 + \left(m + \frac{kn}{2}\right)\mathbf{e}_4,\ [\mathbf{e}_2, \mathbf{e}_3] = n\mathbf{e}_4. \tag{4.5}$$

Consider a Lie algebra automorphism $\phi : \mathfrak{nil}^4 \to \mathfrak{nil}^4$ of $\mathfrak{nil}^4$. Then ϕ is a linear transformation of the linear space $\mathfrak{nil}^4$ preserving all the Lie brackets between the linear basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of $\mathfrak{nil}^4$. Because ϕ must preserve the lower central series of $\mathfrak{nil}^4$, it is of the form

$$\phi(\mathbf{e}_1) = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + p_{11}\mathbf{e}_3 + p_{21}\mathbf{e}_4,$$

$$\begin{aligned}
\phi(\mathbf{e}_2) &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + p_{12}\mathbf{e}_3 + p_{22}\mathbf{e}_4,\\
\phi(\mathbf{e}_3) &= a_{33}\mathbf{e}_3 + a_{43}\mathbf{e}_4,\\
\phi(\mathbf{e}_4) &= a_{44}\mathbf{e}_4.
\end{aligned}$$

Because φ must preserve all the Lie brackets (4.5) including the trivial ones, we obtain that

$$\begin{aligned}
a_{21} &= 0,\\
a_{33} &= a_{11}a_{22},\\
a_{44} &= a_{22}a_{33} = a_{11}a_{22}^2,\\
a_{43}k &= a_{22}p_{11}n + a_{11}a_{22}(a_{22}-1)\left(m + \frac{kn}{2}\right).
\end{aligned}$$

Consequently, we have that $\mathrm{Aut}(\mathfrak{nil}^4)$ is isomorphic to a subgroup of the matrix group $\mathrm{GL}(4,\mathbb{R})$:

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ p_{11} & p_{12} & a_{11}a_{22} & 0 \\ p_{21} & p_{22} & p_{23} & a_{11}a_{22}^2 \end{pmatrix} \;\middle|\; \begin{array}{l} p_{23}k = a_{22}p_{11}n \\ \quad +a_{11}a_{22}(a_{22}-1)\left(m+\frac{kn}{2}\right) \end{array} \right\}. \tag{4.6}$$

By a result of Mal'cev again, we can regard

$$\mathrm{Aut}(\Gamma) \subset \mathrm{Aut}(\mathrm{Nil}^4) = \mathrm{Aut}(\mathfrak{nil}^4)$$

via the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{nil}^4 \arrow[r, "\phi = \mathrm{d}\varphi"] & \mathfrak{nil}^4 \arrow[d, "\exp"] \\
\mathrm{Nil}^4 \arrow[u, "\log"] \arrow[r, "\varphi"] & \mathrm{Nil}^4 \\
\Gamma \arrow[u] \arrow[r, "\varphi"] & \Gamma \arrow[u]
\end{tikzcd} \tag{4.7}$$

Definition 4.1. Denote by $\mathrm{UT}(2,-)$ the subgroup of $\mathrm{GL}(2,-)$ consisting of upper triangular matrices.

Thus we have commutative diagrams:

$$\begin{tikzcd}
\mathrm{Aut}(\mathfrak{nil}^4) \arrow[r] & \mathrm{UT}(2,\mathbb{R}) \arrow[r] & 1 \\
\mathrm{Aut}(\Gamma) \arrow[u] \arrow[r, "\Theta"] & \mathrm{UT}(2,\mathbb{Z}) \arrow[u] \arrow[r] & 1
\end{tikzcd} \tag{4.8}$$

and

$$\begin{tikzcd}
\mathrm{Aut}(\Gamma_k) \arrow[r, "\vartheta"] & \mathrm{GL}(2,\mathbb{Z}) \arrow[r] & 1 \\
\mathrm{Aut}(\Gamma_{k,m,n}) \arrow[u, "\sigma"] \arrow[r, "\Theta"] & \mathrm{UT}(2,\mathbb{Z}) \arrow[u] \arrow[r] & 1.
\end{tikzcd} \tag{4.9}$$

This induces the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \ker(\vartheta) \arrow[r] & \vartheta^{-1}(\mathrm{UT}(2,\mathbb{Z})) \arrow[r, "\vartheta"] & \mathrm{UT}(2,\mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \ker(\Theta) \arrow[u] \arrow[r] & \mathrm{Aut}(\Gamma_{k,m,n}) \arrow[u, "\sigma"] \arrow[r, "\Theta"] & \mathrm{UT}(2,\mathbb{Z}) \arrow[u, "="] \arrow[r] & 1
\end{tikzcd}$$

Remark that the top extension $1 \to \ker(\vartheta) \to \vartheta^{-1}(\mathrm{UT}(2,\mathbb{Z})) \xrightarrow{\vartheta} \mathrm{UT}(2,\mathbb{Z}) \to 1$ splits by Section 2. If Θ splits then ϑ splits as well. We will study whether the bottom extension

$$1 \to \ker(\Theta) \to \mathrm{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \mathrm{UT}(2,\mathbb{Z}) \to 1$$

splits.

4.3. Splitting of $\mathrm{Aut}(\Gamma_{k,m,n}) \xrightarrow{\Theta} \mathrm{UT}(2,\mathbb{Z})$

Remark that the subgroup $\mathrm{UT}(2,\mathbb{Z})$ of $\mathrm{GL}(2,\mathbb{Z})$ is generated by

$$\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \eta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the relations

$$\kappa^2 = \eta^2 = 1, \ \kappa\eta = \eta\kappa, \ \kappa\rho\kappa^{-1} = \rho^{-1}, \ \eta\rho\eta^{-1} = \rho.$$

To discuss whether Θ splits, we first lift ρ, κ and η to elements of $\mathrm{Aut}(\mathfrak{nil}^4)$ using (4.6) as follows:

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r_{11} & r_{12} & 1 & 0 \\ r_{21} & r_{22} & r_{23} & 1 \end{pmatrix} \text{ with } r_{23}k = r_{11}n,$$

$$K = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k_{11} & k_{12} & -1 & 0 \\ k_{21} & k_{22} & k_{23} & -1 \end{pmatrix} \text{ with } k_{23}k = k_{11}n,$$

$$N = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ n_{11} & n_{12} & 1 & 0 \\ n_{21} & n_{22} & n_{23} & -1 \end{pmatrix} \text{ with } n_{23}k = -n_{11}n - 2\left(m + \frac{kn}{2}\right).$$

We first find the conditions on R, N and K to generate a subgroup of $\mathrm{Aut}(\mathfrak{nil}^4)$ isomorphic to $\mathrm{UT}(2,\mathbb{Z})$. They must satisfy the identities

$$K^2 = N^2 = I_2,\ KN = NK,\ KRK^{-1} = R^{-1},\ NRN^{-1} = R \tag{4.10}$$

or equivalently they satisfy

$$\begin{aligned}
&r_{11} = r_{21} = r_{23} = 0,\ k_{11} = k_{21} = k_{23} = 0,\\
&n_{11} = -2r_{12},\ n_{12} = -k_{12},\\
&n_{23} = -\frac{2m + kn + nn_{11}}{k},\ n_{21} = \frac{n_{11}n_{23}}{2},\ n_{22} = \frac{n_{12}n_{23}}{2}.
\end{aligned}$$

Next we find the conditions on R, K and N to preserve the lattice $\Gamma_{k,m,n}$. For this purpose, we need to recall that a Lie algebra automorphism of $\mathfrak{nil}^4$ is regarded as a Lie group automorphism of Nil^4 via the diagram (4.7). In fact, by considering Nil^4 as the Mal'cev completion of $\Gamma_{k,m,n}$, i.e., by considering the elements of Nil^4 as $\alpha^{r_1}\beta^{r_2}\gamma^{r_3}\delta^{r_4}$ for $r_i \in \mathbb{R}$, we can see that

$$\begin{aligned}
&R(\alpha)=\alpha,\ R(\beta)=\alpha\beta\gamma^{\frac{k-n_{11}}{2}}\delta^{-\frac{6m+7kn-3nn_{11}-12r_{22}}{12}},\ R(\gamma)=\gamma,\ R(\delta)=\delta,\\
&K(\alpha)=\alpha^{-1},\ K(\beta)=\beta\gamma^{-n_{12}}\delta^{\frac{2k_{22}+nn_{12}}{2}},\ K(\gamma)=\gamma^{-1},\ K(\delta)=\delta^{-1},\\
&N(\alpha)=\alpha^{-1}\gamma^{n_{11}}\delta^{\frac{n_{11}n_{23}}{2}},\ N(\beta)=\beta^{-1}\gamma^{n_{12}}\delta^{\frac{n_{12}(n_{23}+n)}{2}}, N(\gamma)=\gamma\delta^{n_{23}}, N(\delta)=\delta^{-1}.
\end{aligned}$$

Thus R, K and N preserve the lattice $\Gamma_{k,m,n}$ if and only if

$$\begin{aligned}
&n_{11},\ n_{12},\ n_{23} \in \mathbb{Z},\\
&n_{11} - k,\ nn_{12},\ n_{11}n_{23},\ n_{12}n_{23} \in 2\mathbb{Z},\\
&6m + 7kn - 3nn_{11} \in 12\mathbb{Z},\\
&n + n_{23} = -\frac{2m + nn_{11}}{k}.
\end{aligned}$$

With $n_{11} = k + 2p$ ($p \in \mathbb{Z}$) and $n_{12} = q, n_{23} = -r \in \mathbb{Z}$, we see that the identity $n + n_{23} = -\frac{2m+nn_{11}}{k}$ reduces to

$$r = 2n + \frac{2(m + pn)}{k}, \tag{4.11}$$

the condition $6m + 7kn - 3nn_{11} \in 12\mathbb{Z}$ reduces to

$$3m + 2kn - 3pn \in 6\mathbb{Z}, \tag{4.12}$$

and the remaining conditions reduce to the conditions

$$qn,\ rk,\ qr \in 2\mathbb{Z}. \tag{4.13}$$

By (4.11), rk is even. If n is even then by (4.11) and (4.12), $m+rk-5pn \in 6\mathbb{Z}$, which implies that m must be even.

Consequently, we have proven the following main result.

Theorem 4.2. *Given $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$, the natural surjective homomorphism $\Theta : \mathrm{Aut}(\Gamma_{k,m,n}) \to \mathrm{UT}(2, \mathbb{Z})$ splits if and only if there exists an integer p such that*

$$3m + 2kn - 3pn \in 6\mathbb{Z}, \ \frac{m + pn}{k} \in \mathbb{Z}.$$

If n is even, then so is m.

Example 4.3. For the lattice $\Gamma_{1,-1,2}$ of Nil^4, the corresponding homomorphism Θ cannot split because $n = 2$ is even and $m = -1$ is odd.

Consider the lattice $\Gamma_{3,1,3}$ of Nil^4. The corresponding homomorphism Θ cannot split, because $3m + 2kn - 3pn = 21 - 9p \notin 6\mathbb{Z}$.

The homomorphism Θ corresponding to the lattice $\Gamma_{1,-1,3}$ splits because the above conditions are satisfied if we take $p = -1$.

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