ON THE STABILITY OF THE FUNCTIONAL EQUATION

\[ g(x + y + xy) = g(x) + f(y) + xf(y) + yg(x) \]

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ABSTRACT. In this note, we investigate the Hyers-Ulam stability and the hyperstability of the Pexider type functional equation

\[ g(x + y + xy) = g(x) + f(y) + xf(y) + yg(x) \]

1. INTRODUCTION

The problem of stability of functional equations was originally raised by S.M. Ulam [10] in 1940: given a group \( V \), a metric group \( W \) with metric \( d(\cdot, \cdot) \), and a \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( f : V \to W \) satisfies \( d(f(xy), f(x)f(y)) \leq \delta \) for all \( x, y \in V \), then a homomorphism \( g : V \to W \) exists with \( d(f(x), g(x)) \leq \epsilon \) for all \( x \in V \)?

For Banach spaces the Ulam problem was first solved by D.H. Hyers [3] in 1941, which states that if \( \delta > 0 \) and \( f : X \to Y \) is a mapping with \( X, Y \) Banach spaces, such that

\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \]

for all \( x, y \in X \), then there exists a unique additive mapping \( T : X \to Y \) such that

\[ \|f(x) - T(x)\| \leq \delta \]

for all \( x, y \in X \).

Due to this fact, the additive functional equation \( f(x + y) = f(x) + f(y) \) is said to have the Hyers-Ulam stability property on \((X, Y)\). This terminology is also applied to other functional equations and the generalization of this property has been studied by many authors (see, for example, [1], [2], [4], [7]).

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During the 34th International Symposium on Functional Equations, Gy. Maksa [5] posed the problem concerning the Hyers-Ulam stability of the functional equation
\[ f(xy) = xf(y) + yf(x) \]
on the interval \((0, 1]\), which is usually called a derivation (or left derivation).

J. Tabor [9] gave an answer to the question of Maksa by proving the Hyers-Ulam stability of the functional equation (1.1) on the interval \((0, 1]\). Also Zs. Páles [6] remarked that the functional equation (1.1) for real-valued functions has a hyperstability on the interval \([1, \infty)\).

We here introduce the Pexider type functional equation motivated by the functional equation (1.1):
\[ g(x + y + xy) = g(x) + f(y) + xf(y) + yg(x). \]

In this note, we will solve the Pexider type functional equation (1.2) and then by referring the ideas of J. Tabor [9] and Zs. Páles [6], the Hyers-Ulam stability and the hyperstability of the equation (1.2) will be investigated, respectively.

2. Solutions of the Functional Equation (1.2)

In this section, we set the interval \(I = (-1, \infty)\). It is easy to see that the pair of the real-valued functions \(f(x) = (x + 1)\ln(x + 1)\) and \(g(x) = f(x) + a(x + 1)\) for some nonzero \(a \in \mathbb{R}\) is a solution of the functional equation (1.2) on \(I\). In the following theorem, we will find out the general solution of the functional equation (1.2) on \(I\).

**Theorem 2.1.** Let \(X\) be a real (or complex) vector space. The pair of mappings \(f, g : I \to X\) satisfies the functional equation (1.2) for all \(x, y \in I\) if and only if there exists a solution \(D : (0, \infty) \to X\) of the functional equation (1.1) such that
\[ f(x) = D(x + 1) \quad \text{and} \quad g(x) = D(x + 1) + (x + 1)a \]
hold for some \(a \in X\) and for all \(x \in I\).

**Proof.** (Necessity) Let us define a mapping \(D : (0, \infty) \to X\) by \(D(x) = f(x - 1)\) for all \(x \in (0, \infty)\). Let \(x = 0\) in the equation (1.2). Then we have
\[ g(y) = f(y) + (y + 1)g(0), \]
that is, \(g(x) = f(x) + (x + 1)a\) for all \(x \in I\), where \(a = g(0)\). We claim that \(D\) is a solution of the functional equation (1.1).
Indeed, for all $x, y \in (0, \infty)$, we have
\[
D(xy) = f(xy - 1) = g(xy - 1) - (xy)a \\
= g((x - 1) + (y - 1) + (x - 1)(y - 1)) - (xy)a \\
= g(x - 1) + f(y - 1) + (x - 1)f(y - 1) + (y - 1)g(x - 1) - (xy)a \\
= xf(y - 1) + yg(x - 1) - (xy)a \\
= xf(y - 1) + y\{g(x - 1) - xa\} = xD(y) + yD(x).
\]
Therefore, $D$ is a solution of the functional equation (1.1), as claimed, and
\[
f(x) = D(x + 1) \quad \text{and} \quad g(x) = D(x + 1) + (x + 1)a
\]
are true for all $x \in I$.

(Sufficiency) For all $x, y \in I$, we get
\[
g(x + y + xy) = D(x + y + xy + 1) + (x + y + xy + 1)a \\
= D((x + 1)(y + 1)) + (x + 1)(y + 1)a \\
= (x + 1)D(y + 1) + (y + 1)D(x + 1) + (x + 1)(y + 1)a \\
= xD(y + 1) + D(y + 1) + (y + 1)\{D(x + 1) + (x + 1)a\} \\
= g(x) + f(y) + xf(y) + yg(x).
\]

3. HYERS-ULAM STABILITY OF THE FUNCTIONAL EQUATION (1.2)

In this section, we will denote $I = (-1, 0]$. We first need a theorem of F. Skof [8] concerning the stability of the additive functional equation $f(x + y) = f(x) + f(y)$ on a restricted domain:

**Theorem 3.1.** Let $X$ be a real (or complex) Banach space. Given $c > 0$, let a mapping $f : [0, c) \to X$ satisfy the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]
for some $\delta \geq 0$ and for all $x, y \in [0, c]$ with $x + y \in [0, c)$. Then there exists an additive mapping $A : \mathbb{R} \to X$ such that
\[
\|f(x) - A(x)\| \leq 3\delta
\]
for all $x \in [0, c)$. 

Theorem 3.2. Let $X$ be a real (or complex) Banach space, and let $g, f : I \to X$ be mappings satisfying the inequality
\begin{equation}
\|g(x + y + xy) - g(x) - f(y) - xf(y) - yg(x)\| \leq \delta
\end{equation}
for some $\delta > 0$ and for all $x, y \in I$. Then there exist mappings $h, p : I \to X$ satisfying the functional equation (1.2) such that
\begin{equation}
\|g(x) - h(x)\| \leq (12e + 1)\delta
\end{equation}
and
\begin{equation}
\|f(x) - p(x)\| \leq (12e + 2)\delta
\end{equation}
for all $x \in I$.

Proof. Let us define the mappings $G, H : I \to X$ by
\[ G(x) = \frac{g(x)}{x + 1} \quad \text{and} \quad H(x) = \frac{f(x)}{x + 1} \]
for all $x \in I$. Then, from (3.1), we see that $G$ and $H$ satisfy the inequality
\[ \|G(x + y + xy) - G(x) - H(y)\| \leq \frac{\delta}{(x + 1)(y + 1)} \]
for all $x, y \in I$. Define the mappings $U, V : [0, \infty) \to X$ by
\[ U(u) = G(e^{-u} - 1) \quad \text{and} \quad V(u) = H(e^{-u} - 1) \]
for all $u \in [0, \infty)$. Then we obtain
\begin{equation}
\|U(u + v) - U(u) - V(v)\| \leq \delta e^{u+v}
\end{equation}
for all $u, v \in [0, \infty)$. Putting $v = 0$ in (3.4), we have
\begin{equation}
\|V(0)\| \leq \delta e^u
\end{equation}
for all $u \in [0, \infty)$. Analogously, if we set $u = 0$ in (3.4), then we get
\begin{equation}
\|U(v) - U(0) - V(v)\| \leq \delta e^v
\end{equation}
for all $v \in [0, \infty)$. We now define a mapping $F : [0, \infty) \to X$ by
\begin{equation}
F(u) = U(u) - U(0) - V(0)
\end{equation}
for all $u \in [0, \infty)$. We claim that the inequality
\begin{equation}
\|F(u + v) - F(u) - F(v)\| \leq 3\delta e^{u+v}
\end{equation}
holds for all \( u, v \in [0, \infty) \). In fact, it follows from (3.4), (3.5), (3.6) and (3.7) that

\[
\|F(u + v) - F(u) - F(v)\| \\
= \|U(u + v) - U(u) - U(v) + U(0) + V(0)\| \\
\leq \|U(u + v) - U(u) - V(v)\| + \|V(0)\| + \|V(v) - U(v) + U(0)\| \\
\leq \delta e^{u+v} + \delta e^u + \delta e^v \\
\leq 3\delta e^{u+v}.
\]

for all \( u, v \in [0, \infty) \). The inequality (3.8) means that the inequality

\[
\|F(u + v) - F(u) - F(v)\| \leq 3\delta e^c
\]

is valid for all \( u, v \in [0, c) \) with \( u + v < c \), where \( c > 1 \) is an arbitrary given constant.

According to Theorem 3.1, there exists an additive mapping \( A : \mathbb{R} \to X \) such that

\[
\|F(u) - A(u)\| \leq 9\delta e^c
\]

for all \( u \in [0, c) \). If we let \( c \to 1 \) in the last inequality, we then get

\[
(3.9) \quad \|F(u) - A(u)\| \leq (9\delta) e^c
\]

for all \( u \in [0, 1] \). Moreover, it follows from (3.8) that

\[
\|F(u + 1) - F(u) - F(1)\| \leq 3\delta e^{u+1} \\
\|F(u + 2) - F(u + 1) - F(1)\| \leq 3\delta e^{u+2} \\
\vdots \\
\|F(u + k) - F(u + k - 1) - F(1)\| \leq 3\delta e^{u+k}
\]

for all \( u \in [0, 1] \) and \( k \in \mathbb{N} \). Summing up these inequalities, we obtain

\[
(3.10) \quad \|F(u + k) - F(u) - kF(1)\| \leq (3e)\delta \cdot e^{u+k}
\]

for all \( u \in [0, 1] \) and \( k \in \mathbb{N} \). We assert that

\[
(3.11) \quad \|F(v) - A(v)\| \leq (12\delta e) \cdot e^v
\]

for all \( v \in [0, \infty) \). For, let \( v \geq 0 \) and let \( k \in \{0\} \cup \mathbb{N} \) be given with \( v - k \in [0, 1] \).

Then, by (3.9) and (3.10), we have
\[ \| F(v) - A(v) \| \leq \| F(v) - F(v - k) - kF(1) \| \\
+ \| F(v - k) - A(v - k) \| + \| A(k) - kF(1) \| \\
\leq (3e)\delta \cdot e^v + (9e)\delta + \| A(k) - kF(1) \| \\
\leq (3e)\delta \cdot e^v + (9e)\delta + k\| A(1) - F(1) \| \\
\leq (3e)\delta \cdot e^v + (9e)\delta + (9e)v \\
\leq (3e)(e^v + 3(1 + v)) \\
\leq (12e)\delta \cdot e^v. \]

Now, from (3.11) and the definitions of \( F, G, H \) and \( V \), it follows that

\[ \| G(x) - G(0) - H(0) - A(-\ln(x + 1)) \| \leq (12e)\delta \cdot e^{-\ln(x+1)} = \frac{(12e)\delta}{x + 1} \]

for all \( x \in I \). Using (3.5), we have

\[ \| H(0) = \| V(0) \| \leq \delta e^{-\ln(x+1)} = \frac{\delta}{x + 1} \]

for all \( x \in I \). Therefore, we obtain

\[ \| G(x) - A(-\ln(x + 1)) - G(0) \| \]

\[ \leq \| G(x) - G(0) - H(0) - A(-\ln(x + 1)) \| + \| H(0) \| \\
\leq \frac{(12e + 1)\delta}{x + 1}, \]

i.e.,

(3.12)

\[ \left\| \frac{g(x)}{x + 1} - g(0) - A(-\ln(x + 1)) \right\| \leq \frac{(12e + 1)\delta}{x + 1} \]

for all \( x \in I \).

Let \( D : (0,1] \to X \) be a mapping defined by \( D(x) = xA(-\ln x) \) for all \( x \in (0,1] \). Then \( D \) satisfies the functional equation (1.1). If we set \( p(x) = D(x + 1) \) and \( h(x) = p(x) + (x + 1)g(0) \) for all \( x \in I \), then \( h \) and \( p \) satisfy the functional equation (1.2), i.e.,

\[ h(x + y + xy) = h(x) + p(y) + xp(y) + yh(x) \]

and we obtain the inequality (3.2) by (3.12). Using (3.6), (3.7) and (3.11), we have

\[ \| V(v) - A(v) - V(0) \| = \| V(v) - A(v) + F(v) - U(v) + U(0) \| \\
\leq \| F(v) - A(v) \| + \| U(v) - U(0) - V(v) \| \\
\leq (12e)\delta e^v + \delta e^v = (12e + 1)\delta e^v \]
for all $v \in [0, \infty)$ and so employing the inequality (3.5), we get
\begin{equation}
\|V(v) - A(v)\| \leq \|V(v) - A(v) - V(0)\| + \|V(0)\| \\
\leq (12e + 1)\delta e^v + \delta e^v = (12e + 2)\delta e^v
\end{equation}
for all $v \in [0, \infty)$. From (3.13) and the definitions of $V$ and $H$, it follows that
\begin{equation}
\left\| \frac{f(x)}{x+1} - A(-\ln(x+1)) \right\| \leq (12e + 2)\delta e^{-\ln(x+1)} = \frac{(12e + 2)\delta}{x+1}
\end{equation}
for all $x \in I$. Hence we arrive at the inequality (3.3). This completes the proof. \qed

4. Hyperstability of the Functional Equation (1.2)

In this section, we will investigate the hyperstability of the functional equation (1.2) on the interval $I$, where $I = [0, \infty)$.

**Theorem 4.1.** Let $X$ be a real (or complex) Banach space. If there exist mappings $g, f : I \to X$ satisfying the inequality
\begin{equation}
\|g(x + y + xy) - g(x) - f(y) - xf(y) - yg(x)\| \leq \delta
\end{equation}
for some $\delta > 0$ and for all $x, y \in I$ under the condition $g(0) = f(0)$, then $g = f$ on $I$ and $g(= f)$ satisfies the functional equation (1.2) for all $x, y \in I$.

**Proof.** Defining the mappings $G, H : I \to X$ by
\[ G(x) = \frac{g(x)}{x+1} \quad \text{and} \quad H(x) = \frac{f(x)}{x+1} \]
for all $x \in I$ and defining the mappings $U, V : I \to X$ by
\[ U(u) = G(e^u - 1) \quad \text{and} \quad V(u) = H(e^u - 1) \]
for all $u \in I$, it follows from (4.1) that
\begin{equation}
\|U(u + v) - U(u) - V(v)\| \leq \delta e^{-(u+v)}
\end{equation}
for all $u, v \in I$. We will show that $U(u + v) = U(u) + V(v)$ for all $u, v \in I$.

Let $\varepsilon > 0$ be given. Then the inequality (4.2) implies that there exists a $c > 0$ satisfying
\begin{equation}
\|U(u + v) - U(u) - V(v)\| < \frac{\varepsilon}{36}
\end{equation}
for all $u, v \in I$ with $u + v > c$. 
Suppose that positive real numbers $u$ and $v$ are given. Let $m$ and $n$ be integers greater than 1. Then it is easy to see that

\begin{equation}
V(nu) - V((n-1)u) - V(u) = U(mv + u) - U(mv) - V(u)
+ U((mv + u) + (n-1)u) - U(mv + u) - V((n-1)u)
- (U(mv + nu) - U(mv) - V(nu))
\end{equation}

If $m$ is so large that $m > (c + u)/v$, the last equality implies

\begin{equation}
\|V(nu) - V((n-1)u) - V(u)\| < \frac{\varepsilon}{12}
\end{equation}

for all $u \in I$ and all integers $n > 1$. The relation

$$V(nu) - nV(u) = \sum_{k=2}^{n} (V(ku) - V((k-1)u) - V(u))$$

together with (4.5), produces the inequality

\begin{equation}
\|V(nu) - nV(u)\| < \frac{n-1}{12} \varepsilon
\end{equation}

for all $u \in I$ and all integers $n > 1$. Let $u = 0$ in (4.3). Then we obtain the inequality

$$\|U(v) - U(0) - V(v)\| < \frac{\varepsilon}{36}$$

for all $v \in I$ with $v > c$ which gives the inequality

\begin{equation}
\|U(v) - V(v)\| \leq \|U(v) - V(v) - U(0)\| + \|U(0)\| < \frac{\varepsilon}{36} + \|U(0)\|
\end{equation}

for all $v \in I$ with $v > c$.

On the other hand, let $u \in [0, \infty)$ with $u > c$. Then the inequality (4.3) with such an $u$ and $v = 0$ yields

\begin{equation}
\|V(0)\| < \frac{\varepsilon}{36}.
\end{equation}

Note that we have $\|U(0)\| = \|V(0)\|$ since $g(0) = f(0)$. Applying (4.3), (4.7) and (4.8) with this equality, we see that the inequality

$$\|U(u + v) - U(u) - U(v)\| \leq \|U(u + v) - U(u) - V(v)\| + \|V(v) - U(v)\|
< \frac{\varepsilon}{36} + \varepsilon + \|U(0)\| < \frac{\varepsilon}{12}$$

holds for all $u, v \in I$ with $v > c$. Replacing $V$ by $U$ in (4.4) and then following the same process to obtain the inequality (4.6), we have

\begin{equation}
\|U(nu) - nU(u)\| < \frac{n-1}{4} \varepsilon
\end{equation}
for all \( u \in I \) and all integers \( n > 1 \). Obviously, it follows from (4.6) and (4.9) that
\[
\| (U(nu + nv) - U(nu) - V(nv)) - n(U(u + v) - U(u) - V(v)) \|
\leq \| U(nu + nv) - nU(u + v) \| + \| U(nu) - nU(u) \| + \| V(nv) - nV(v) \|
< \frac{n - 1}{4} \varepsilon + \frac{n - 1}{4} \varepsilon + \frac{n - 1}{12} \varepsilon = \frac{7}{12} (n - 1) \varepsilon
\]
for all \( u, v \in [0, \infty) \) with \( (u, v) \neq (0, 0) \). Dividing by \( n \) both sides of the last inequality and then letting \( n \to \infty \) and considering the fact that \( (1/n)(U(nu + nv) - U(nu) - V(nv)) \to 0 \) as \( n \to \infty \) on account of (4.3), we get
\[
\| U(u + v) - U(u) - V(v) \| < \frac{7}{12} \varepsilon
\]
for all \( u, v \in I \) with \( (u, v) \neq (0, 0) \). Also, we see that
\[
\| U(0 + 0) - U(0) - V(0) \| = \| V(0) \| < \frac{\varepsilon}{36}
\]
by using (4.8). Therefore, the inequality (4.3) holds for all \( u, v \in I \). Since \( \varepsilon > 0 \) was arbitrary, we conclude that the relation \( U(u + v) = U(u) + V(v) \) is established for all \( u, v \in I \).

Now, according to the definitions of \( U \) and \( G \), we get
\[
\frac{g(x)}{x + 1} = G(x) = U(\ln(x + 1))
\]
for all \( x \in I \), i.e.,
\[
g(x) = (x + 1)U(\ln(x + 1))
\]
for all \( x \in I \). From the definitions of \( V \) and \( H \), we have
\[
\frac{f(x)}{x + 1} = H(x) = V(\ln(x + 1))
\]
for all \( x \in I \), i.e.,
\[
f(x) = (x + 1)V(\ln(x + 1))
\]
for all \( x \in I \). Since \( U(u + v) = U(u) + V(v) \) for all \( u, v \in I \), we see that \( g \) and \( f \) satisfy the functional equation (1.2) for all \( x, y \in I \). In fact, \( g = f \) on \( I \).

Letting \( x = 0 \) in (1.2), we obtain
\[
(4.10) \quad g(y) = f(y) + (y + 1)g(0)
\]
for all \( y \in I \). Since \( g(0) = f(0) \), it follows from (4.10) that \( g(0) = 0 \). Hence the equality (4.10) gives \( g(x) = f(x) \) for all \( x \in I \). The proof of the theorem is completed. \( \Box \)
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