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# GLOBAL UNIQUENESS FOR THE RADON TRANSFORM

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Abstract. In this article, we discuss the global uniqueness problem for the Radon transform. It is not sufficient for the global uniqueness for the Radon transform to assume that the Radon transform $Rf$ for a function $f$ absolutely converges on any hyperplane. It is also known that it is sufficient to assume that $f \in L^1$ for the global uniqueness to hold. There exists a big gap between the above two conditions, to fill which is our purpose in this paper. We shall give a better sufficient condition for the global uniqueness of the Radon transform.

## 1. Introduction

In this article, we discuss the global uniqueness problem for the Radon transform. Let us first review the definition of the Radon transform.

**Definition 1.1.** Let $f$ be a function defined on $\mathbb{R}^n$. Its Radon transform $Rf$ is defined by

$$Rf(H(\theta, s)) \equiv Rf(\theta, s) := \int_{\theta^\perp} f(s\theta + y)dy, \tag{1}$$

when it is well-defined, where $\theta \in S^{n-1}$, $s \in \mathbb{R}$, $\theta^\perp := \{x \in \mathbb{R}^n \ ; \ x \perp \theta\}$ and $y \in \theta^\perp$. Note that the pair $(\theta, s)$ is identified with the hyperplane $H = H(\theta, s) = \{x \in \mathbb{R}^n \ ; \ x \cdot \theta = s\}$. We also note that $H(\theta, s) = H(-\theta, -s)$.

We shall discuss the global uniqueness problem for this transform, which reads as:

**Problem 1.1.** *Let a function $f$ be defined on $\mathbb{R}^n$. Assume that for any hyperplane $H$, its Radon transform $Rf(H)$ absolutely converges to $0$. Does this condition imply that $f \equiv 0$?*

It is sufficient to study Problem 1.1 for the global uniqueness of the Radon transform, since the Radon transform is linear. The answer to Problem 1.1 is

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known to be negative without any global growth (or decay) condition on the function $f$. It is also known that if we assume that $f \in L^1(\mathbb{R}^n)$, then the answer to Problem 1.1 is positive (cf. Theorem 2.1 below). In the present article, we try to give a better global sufficient condition for the answer to Problem 1.1 to be positive than the known one, $f \in L^1(\mathbb{R}^n)$, which is the main purpose in this paper.

In this section, as the introduction of this article, we are introducing our problems and the context of this article.

In the next section, we shall introduce the known results on the global uniqueness problem and propose the main problem in this paper. It is well known that the condition $f \in L^1(\mathbb{R}^n)$ is sufficient for the global uniqueness of the Radon transform, and that the global uniqueness would not hold without any global condition. In Section 2, we pose a problem whether there exists a better sufficient global condition for the global uniqueness of the Radon transform.

In Section 3, we prove the main theorem in this paper. We give a better sufficient condition for the global uniqueness of the Radon transform. It is surprising that the global uniqueness for the Radon transform holds even if the function is not decreasing, that is, we shall prove that if the function $f$ is infra-exponentially increasing (Theorem 3.1 below), then the answer to Problem 1.1 is positive, in order to prove which, we apply the idea of the Fourier hyperfunctions.

In Section 4, we shall summarize the conclusion in this article and mention some open problems left for further development.

## 2. Known results

In this section, we shall review the known results on the global uniqueness of the Radon transform and propose the main problem to be discussed in this paper.

As was mentioned in Introduction, the answer to Problem 1.1 is negative without any global decay (or growth) condition on the function. It is also known that it is sufficient to assume that $f \in L^1(\mathbb{R}^n)$ for the answer to Problem 1.1 to be positive.

**Theorem 2.1** (A global uniqueness theorem for the Radon transform, [6]). *For $f \in L^1(\mathbb{R}^n)$, $Rf \equiv 0$ implies that $f \equiv 0$.*

The proof of this theorem is too easy to omit and we shall modify it in the third section to prove our main theorem, Theorem 3.1 below. Therefore, we shall review it.

*Proof of Theorem 2.1.* We first note that for a fixed $\theta \in S^{n-1}$, $Rf(\theta, s)$ is well-defined for almost all $s \in \mathbb{R}$ and $Rf(\theta, s) \in L^1(\mathbb{R})$ as a function of $s \in \mathbb{R}$.

Denote $x = (x', x_n),\ x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}$, then we have

$$\widehat{f}(0, \xi_n) = \int_{\mathbb{R}} e^{-ix_n\xi_x} dx_n \int_{\mathbb{R}^{n-1}} f(x', x_n) dx' = 0, \tag{2}$$

where for $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x) dx \tag{3}$$

is the Fourier transform of the function $f$. In the same way, we can prove that for any $\theta \in S^{n-1}$, the condition $Rf(H) = 0$ for any hyperplane $H \perp \theta$ implies $\widehat{f}(r\theta) = 0$ for any $r \in \mathbb{R}$. Since $\widehat{f}$ is continuous by Riemann-Lebesgue theorem, we conclude that $\widehat{f} \equiv 0$. Therefore $f \equiv 0$. □

By Theorem 2.1, it is sufficient to assume $f \in L^1$ for the global uniqueness of the Radon transform. We shall next introduce an example to show that the global uniqueness for the Radon transform would not hold without any global condition on the function. It was L. Zalcman [6] who first constructed such an example, however, we shall introduce another counterexample by the author [4]. We also note that another type of example to show the non-uniqueness of the Radon transform is constructed by D. H. Armitage [2].

**Theorem 2.2** (cf. [4] and [6])**.** *There exists a continuous function $f$ defined on $\mathbb{R}^2$ satisfying the following conditions*;

(i) *$Rf(l)$ absolutely converges to* 0 *for any line $l \subset \mathbb{R}^2$.*
(ii) *$f \not\equiv 0$.*

Let us introduce the proof by the author [4];
Let us regard $\mathbb{R}^2 \cong \mathbb{C}$. We constructed an entire function $f \not\equiv 0$ on $\mathbb{C}$ satisfying the following conditions.

(a) $f(z)$ rapidly decays uniformly outside $\{1/4 < (\mathrm{Re}\ z)^2 - (\mathrm{Im}\ z)^2 < 4,\ \mathrm{Re}\ z < 0,\ \mathrm{Im}\ z > 0\}$ as $|z| \to \infty$.
(b) The Radon transform $Rf(l)$ of $f$ absolutely converges to 0 for any line $l$ in $\mathbb{C}$.

For this purpose, we constructed an entire function $g(z) \not\equiv 0$ defined on $\mathbb{C}$ satisfying the following conditions.

$$\int_l |g'(z)||dz| < \infty \quad \text{for } \forall l, \tag{4}$$

$$\begin{gathered} |z|^k g(z) \to 0 \text{ uniformly in } z \text{ as } |z| \to \infty,\ \forall k > 0, \\ \text{for } z \in \mathbb{C} \setminus \left\{1/4 < (\mathrm{Re}\ z)^2 - (\mathrm{Im}\ z)^2 < 4,\ \mathrm{Re}\ z < 0,\ \mathrm{Im}\ z > 0\right\}. \end{gathered} \tag{5}$$

Let $f(z) := g'(z)$. By (4), $\int_l f(z)dz$ absolutely converges for any $l$. By (5) and Cauchy's integral theorem we obtain

$$\int_l f(z)dz = 0 \quad \text{for any } l, \tag{6}$$

which proves Theorem 2.2.

For later discussion, it is very important to know what functions $f(z)$ and $g(z)$ are, which let us review.

**Lemma 2.1** (N. U. Arakelian, 1965, [1]). *Let $M$ be a closed set in $\mathbb{C}$ such that $\hat{\mathbb{C}} \setminus M$ is connected and arcwise connected at infinity in $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$ is the one point compactification of the complex plane $\mathbb{C}$, for whose model, we take the unit sphere $S^1 \subset \mathbb{R}^2$ with the north pole as the point at infinity. Assume that $\varepsilon(t) > 0$ is a decreasing function in $t$ satisfying*

$$\int_1^\infty \frac{\log \varepsilon(t)}{t^{\frac{3}{2}}} dt > -\infty. \tag{7}$$

*Then for any $h(z) \in C(M) \cap \mathcal{A}(M^{\mathrm{int}})$ there exists an entire function $g(z)$ such that*

$$\Big| h(z) - g(z) \Big| < \varepsilon(|z|) \quad \text{for } \forall z \in M, \tag{8}$$

*where $C(M)$ the set of continuous functions on $M$ and $\mathcal{A}(M^{\mathrm{int}})$ is the set of the analytic (or holomorphic) functions in the interior of $M$.*

Let

$$K := \big\{ z \in \mathbb{C} \ ; \ |z| < 5 \big\}, \tag{9}$$

$$S := \{1/4 < (\mathrm{Re}\ z)^2 - (\mathrm{Im}\ z)^2 < 4, \ \mathrm{Re}\ z < 0, \ \mathrm{Im}\ z > 0\}, \tag{10}$$

$$M := \mathbb{C} \setminus (K \cup S). \tag{11}$$

Note that $M$ is a closed subset in $\mathbb{C}$ and $\hat{\mathbb{C}} \setminus M$ is connected and arcwise connected at infinity. We put

$$\varphi(z) := iz^2 - i. \tag{12}$$

Note that we can define $0 < \arg \varphi(z) < 4\pi$ on $M$, which makes $\log \varphi(z)$ a single-valued holomorphic function in $M^{\mathrm{int}}$. Let

$$h(z) := \frac{1}{\varphi(z)^{\log \varphi(z)}} = e^{-(\log \varphi(z))^2} \in C(M) \cap \mathcal{A}(M^{\mathrm{int}}), \tag{13}$$

$$\varepsilon(t) := \frac{1}{(t^2 - 1)^{\log(t^2 - 1)}}. \tag{14}$$

Since $M$, $h$ and $\varepsilon$ defined by (11), (13) and (14) satisfy the assumption of Lemma 2.1, there exists an entire function $g(z)$ satisfying

$$\Big| \frac{1}{\varphi(z)^{\log \varphi(z)}} - g(z) \Big| < \frac{1}{(|z|^2 - 1)^{\log(|z|^2 - 1)}} \quad \text{for } z \in M. \tag{15}$$

Since

$$\Big| \frac{1}{\varphi(z)^{\log \varphi(z)}} \Big| = \frac{e^{(\arg \varphi(z))^2}}{|\varphi(z)|^{\log |\varphi(z)|}}, \tag{16}$$

we have $g(z) \not\equiv 0$. In fact, if we assume $g \equiv 0$, then taking $z \in \mathbb{R}$ contradicts to (16), since, for $z \in \mathbb{R}$, $|\varphi(z)^{\log \varphi(z)}| = |\varphi(z)|^{\log |\varphi(z)|}$ and $\arg \varphi(z) > 1$. By (16), we have

$$|g(z)| \leq \frac{e^{16\pi^2} + 1}{|\varphi(z)|^{\log |\varphi(z)|}} \quad \text{for } z \in M. \tag{17}$$

Therefore, $g(z)$ is rapidly decreasing in $M$, which implies (5). Let $z \in M^{\text{int}}$ and

$$d = d(z) := \frac{1}{2}\text{dist}(z, \partial M), \tag{18}$$

where $\partial M$ is the boundary of $M$. Let $L(z) := \max_{|\zeta - z| = d} |g(\zeta)|$. Then we have

$$\frac{1}{2}|z| \leq |z| - d \leq |\zeta| \quad \text{for } |\zeta - z| = d, \tag{19}$$

since $d(z) \leq \frac{1}{2}|z|$. Hence it holds by virtue of (17), (19) and $|z| \geq 5$ that

$$L(z) \leq \max_{|\zeta - z| = d} \frac{e^{16\pi^2} + 1}{(|\zeta|^2 - 1)^{\log(|\zeta|^2 - 1)}} \leq (e^{16\pi^2} + 1) e^{-(\log(\frac{|z|^2}{4} - 1))^2}. \tag{20}$$

Cauchy's integral formula yields

$$\begin{aligned} |g'(z)| &= \left| \frac{1}{2\pi i} \int_{|\zeta - z| = d} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{L(z)}{d(z)} \\ &\leq \frac{e^{16\pi^2} + 1}{d(z)} e^{-(\log(\frac{|z|^2}{4} - 1))^2}. \end{aligned} \tag{21}$$

Since $d(z) = O(1/|z|)$ on the most critical line $\{\text{Im } z = -\text{Re } z\}$ as $-\text{Re } z = \text{Im } z \to \infty$, $|g'(z)|$ is integrable on all lines in $\mathbb{C}$ by (21). Thus we have (4).

By Theorem 2.1, it is sufficient to assume $f \in L^1$ for the global uniqueness of the Radon transform, while Theorem 2.2 claims that the global uniqueness for the Radon transform would not hold without any global condition on the function. The function $g$ constructed in (15), consequently as well as the function $f(z) = g'(z)$, super-exponentially increases as $|z| \to \infty$ in $S$ defined by (10). Another counterexample constructed by L. Zalcman [6] also super-exponentially grows in some domain.

We claim that the gap between the known sufficient growth condition, $f \in L^1$, for the global uniqueness and the growth of a counterexample, super-exponential one, is too big and must be filled. Therefore, we pose the following problem, which is the main problem to be studied in this paper.

**Problem 2.1.** *Under the assumption that $Rf(H)$ absolutely converges to 0 for any (or for almost all) hyperplane $H$, is there any better sufficient condition for the global uniqueness of the Radon transform to hold?*

## 3. Main theorem

In this section, we give an answer to Problem 2.1. In our main theorem, Theorem 3.1, we claim that under the assumption that $Rf(H)$ absolutely converges to 0 for any (or for almost all) hyperplane $H$, it is sufficient to assume that $f$ is measurable and that $f(x)$ globally and infra-exponentially grows as $|x| \to \infty$, for the answer to Problem 1.1 to be positive. In the main theorem, Theorem 3.1, we treat functions with infra-exponential growth ((28) below), in order of which, it is useful to introduce the idea of the Fourier hyperfunctions.

**Definition 3.1.** A Fourier hyperfunction $f(x)$ on $\mathbb{R}^n$ is defined by the boundary value of holomorphic functions

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \tag{22}$$

where $\Gamma_j \subset \mathbb{R}^n$ is an open cone and for any relatively compact cone $\Gamma'_j \Subset \Gamma_j$ there exists a constant $c(\Gamma'_j) > 0$ such that $F_j(z)$ is holomorphic on $(\mathbb{R}^n + i\Gamma'_j) \cap \{|\mathrm{Im}\ z| < c(\Gamma'_j)\}$, and for any $\delta > 0$ it satisfies that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|F_j(z)| \le C_\varepsilon e^{\varepsilon|\mathrm{Re} z|} \tag{23}$$

uniformly on $(\mathbb{R}^n + i\Gamma'_j) \cap \{\delta < |\mathrm{Im}\ z| < c(\Gamma'_j)\}$.

Roughly speaking, a Fourier hyperfunction is defined as a sum of the boundary values as $\mathrm{Im}\ z \to 0$ of holomorphic functions $F_j$'s which are holomorphic in the domain $(\mathbb{R}^n + i\Gamma_j) \cap \{\mathrm{Im}\ z < k_j\}$ with some $k_j > 0$ and have the infra-growth estimate (23).

It is important in this paper that the space of Fourier hyperfunctions is defined as the strong dual space of the space

$$\mathcal{P}_*(\mathbb{R}^n) = \varinjlim_{0 \in I} \varinjlim_{\delta \to +0} \mathcal{O}^{-\delta}(\mathbb{R}^n + iI), \tag{24}$$

where $g \in \mathcal{O}^{-\delta}(\mathbb{R}^n + iI)$ for an interval $I \subset \mathbb{R}^n$ containing the origin is defined by the following two conditions;

(3.1) $g(x + iy)$ is holomorphic in $\mathbb{R}^n + iI$,

(3.2) For any $K \Subset I$ and for any $\varepsilon > 0$, there exists some constant $C_{K,\varepsilon} > 0$ such that the following estimate uniformly holds

$$|f(x + iy)| \le C_{K,\varepsilon} e^{-(\delta-\varepsilon)|x|}. \tag{25}$$

Roughly speaking, $\mathcal{P}_*(\mathbb{R}^n)$ is the space of exponentially decreasing analytic functions which are extended holomorphically and exponentially decreasingly to some strip neighborhood of the real axis. For a pair of a Fourier hyperfunction $f$ and a function $g \in \mathcal{P}_*(\mathbb{R}^n)$, its duality is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x + iy) g(x + iy) dx \tag{26}$$

for any $y$ with small $|y|$, and the Fourier transform $\widehat{f}$ is defined via the duality as follows:

$$\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle. \tag{27}$$

It is known that the Fourier transform is the automorphism on the space of the Fourier hyperfunctions (Theorem 8.3.4 in [3]). For general properties of the Fourier hyperfunctions as well as more details of $P_*$, $O^{-\delta}$ and so on, we refer the readers to [3].

The following theorem gives a new and better sufficient condition for the global uniqueness of the Radon transform.

**Theorem 3.1.** *Let $f \in C(\mathbb{R}^n)$ satisfy that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$|f(x)| \le C_\varepsilon e^{\varepsilon|x|} \tag{28}$$

*for any $x \in \mathbb{R}^n$. Then the condition that the Radon transform $Rf(H)$ absolutely converges to $0$ for all hyperplanes $H \subset \mathbb{R}^n$ yields that $f(x) \equiv 0$.*

*Proof.* Theorem 3.1 is proved by modifying the proof of Theorem 2.1.

In view of the above discussion, any function $f$ satisfying the assumption of Theorem 3.1 can be taken for a Fourier hyperfunction, since its duality with any $g \in \mathcal{P}_*(\mathbb{R}^n)$ is simply defined by the following absolutely convergent integral;

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)dx. \tag{29}$$

Let us denote $\xi = (\xi', \xi_n)$, $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$. The fact that the Radon transform, for example $\int_{\mathbb{R}^{n-1}} f(x', x_n)dx'$, absolutely converges to 0 for any $\xi_n \in \mathbb{R}$, yields that the duality

$$\langle f, g\rangle = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} f(x', x_n)dx' \right) \varphi(x_n)dx_n \tag{30}$$

is well defined to be equal to 0, where $g(x) = 1_{\mathbb{R}^{n-1}} \otimes \varphi(x_n)$, $1_{\mathbb{R}^{n-1}}$ is a function identically equal to 1 for any $x' \in \mathbb{R}^{n-1}$ and $\varphi(x_n) \in \mathcal{P}_*(\mathbb{R})$. In the same way, we obtain that the duality

$$\langle \widehat{f}(0, \xi_n), 1_{\mathbb{R}^{n-1}} \otimes \varphi(\xi_n)\rangle = \int_{\mathbb{R}} \left( \left( \int_{\mathbb{R}^{n-1}} f(x', x_n)dx' \right) e^{-ix_n\xi_n} dx_n \right) \varphi(\xi_n)d\xi_n \tag{31}$$

is well defined to be 0, by virtue of the fact that the Radon transform

$$\int_{\mathbb{R}^{n-1}} f(x', x_n)dx'$$

absolutely converges to 0 for any $x_n \in \mathbb{R}$. Since we have assumed that the Radon transform $Rf(H)$ absolutely converges to 0 for all hyperplanes $H \subset \mathbb{R}^n$, we conclude that the Fourier transform $\widehat{f}$ of the function $f$ is identically 0, which proves the theorem by the uniqueness of the Fourier transform of the Fourier hyperfunctions (Theorem 8.3.4 in [3]). $\square$

*Remark* 3.1. Roughly speaking, the Fourier transform *exchanges* the regularity of the function and its decay (or growth). The smoother the function is, the more rapidly its Fourier transform decreases. The faster the function decreases, the more regular its Fourier transform is. If the function grows infra-exponentially, then its Fourier transform is a Fourier hyperfunctions (cf. [3]). Therefore, we have applied the idea of the Fourier hyperfunctions in the proof of Theorem 3.1. It is very difficult to prove our main theorem in the space of the distributions.

Theorem 3.1 claims that the global uniqueness for the Radon transform holds even if the function is increasing (not decreasing), if its increasing order is an infra-exponential one defined in (28). The author being afraid that the proof of Theorem 3.1 looks too complicated, what is important is that the essential idea to proof Theorem 2.1 is applicable in the frame of the Fourier hyperfunctions. The idea to prove the main theorem, Theorem 3.1, being very simple, its conclusion is very important.

## 4. Conclusion and open problems

As the final section of this article, we shall summarize the conclusions of this article and mention open problems left to be solved for further development.

Let us first summarize the conclusions of this article.

**Conclusion 4.1.**

- *We have proved a generalized global uniqueness theorem* (*Theorem 3.1*). *The essential idea to prove our main theorem is to assume the global infra-exponential growth condition on the function.*
- *It is interesting and surprising that the main theorem, Theorem 3.1, claims that the global uniqueness for the Radon transform holds even if the function is increasing, if the growth is infra-exponential one.*
- *We have introduced a counterexample constructed in Section 2.2 to Theorem 2.1. It grows super-exponentially in the narrow domain $S$ defined in* (11) *as* $|x+iy| = |z| \to \infty$.

In view of the last conclusion of Conclusion 4.1, it is left open to study the case, for example, where the global growth condition is assumed to be an exponential one, for the complete study of the global uniqueness. The exact global growth of the counterexamples introduced in Section 2 has not been completely studied. Therefore there still exist open problems to complete the study of the sufficient global growth condition for the global uniqueness of the Radon transform. Our main theorem (Theorem 3.1) also holds for Fourier hyperfunctions. We refer the readers to [5] for the Radon transform for hyperfunctions.

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