SEMISYMMETRIC CUBIC GRAPHS OF ORDER $34p^3$

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Abstract. A simple graph is called semisymmetric if it is regular and edge transitive but not vertex transitive. Let $p$ be a prime. Folkman proved [J. Folkman, Regular line-symmetric graphs, Journal of Combinatorial Theory 3 (1967), no. 3, 215–232] that no semisymmetric graph of order $2p$ or $2p^2$ exists. In this paper an extension of his result in the case of cubic graphs of order $34p^3$, $p \neq 17$, is obtained.

1. Introduction

In this paper all graphs are finite, undirected and simple, i.e., without loops or multiple edges. A graph is called semisymmetric if it is regular and edge transitive but not vertex transitive. The class of semisymmetric graphs was first studied by Folkman [6], who found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric graphs of different orders. In [6], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime $p$. For prime $p$, cubic semisymmetric graphs of order $2p^3$ were investigated in [11], in which they proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph. Also in [2] and [1] the authors proved that there is no connected cubic semisymmetric graph of order $4p^3$ and of order $8p^2$ respectively.

In this paper we investigate connected cubic semisymmetric graphs of order $34p^3$ for prime $p$, and try to classify them. This investigation actually leads to a proof of their nonexistence for all primes $p \neq 17$.

2. Preliminaries

In this paper, the cardinality of a finite set $A$, is denoted by $|A|$. The symmetric and alternating groups of degree $n$, the dihedral group of order $2n$ and the cyclic group of order $n$ are respectively denoted by $S_n$, $A_n$, $D_{2n}$, $Z_n$. If
G is a group and \( H \leq G \), then \( Aut(G), G', Z(G), C_G(H) \) and \( N_G(H) \) denote respectively the group of automorphisms of \( G \), the commutator subgroup of \( G \), the center of \( G \), the centralizer and the normalizer of \( H \) in \( G \). We also write \( H \trianglelefteq G \) to denote \( H \) is a characteristic subgroup of \( G \). If \( H \trianglelefteq K \trianglelefteq G \), then \( H \trianglelefteq G \). For a prime \( p \) dividing the order of finite \( G \), \( O_p(G) \) will denote the largest normal \( p \)-subgroup of \( G \). It is easy to verify that \( O_p(G) \trianglelefteq G \). A function \( f \) acts on its argument from the left, i.e., we write \( f(x) \). The composition, \( fg \), of two functions \( f \) and \( g \), is defined as \( (fg)(x) = f(g(x)) \). For a group \( G \) and a nonempty set \( \Omega \), an action of \( G \) on \( \Omega \) is a function \( g.\omega \) from \( G \times \Omega \) to \( \Omega \), where \( 1.\omega = \omega \) and \( g.(h.\omega) = (gh)\omega \) for every \( g, h \in G \) and every \( \omega \in \Omega \). We write \( g.\omega \) instead of \( g.\omega \), if there is no fear of ambiguity. For \( \omega \in \Omega \), the stabilizer of \( \omega \) in \( G \) is defined as \( G_\omega = \{ g \in G : g.\omega = \omega \} \). The action is called semiregular if the stabilizer of each element in \( \Omega \) is trivial; it is called regular if it is semiregular and transitive.

Let \( \Gamma \) be a graph. The vertex set, the edge set and the set of all automorphisms of \( \Gamma \) are respectively denoted by \( V(\Gamma) \), \( E(\Gamma) \) and \( Aut(\Gamma) \). For two vertices \( u \) and \( v \), we write \( u \sim v \) to denote \( u \) is adjacent to \( v \). The set of all vertices adjacent to \( u \) is denoted by \( \Gamma(u) \). The degree or valency of a vertex \( u \) is \( |\Gamma(u)| \). The graph \( \Gamma \) is called regular if all of its vertices have the same valency. If \( \Gamma \) is a graph and \( N \leq Aut(\Gamma) \), then \( \Gamma_N \) will denote a simple undirected graph whose vertices are the orbits of \( N \) in its action on \( V(\Gamma) \), and where two vertices \( Nu \) and \( Nv \) are adjacent if and only if \( u \sim v \) in \( \Gamma \) for some \( n \in N \).

Let \( \Gamma_e \) and \( \Gamma \) be two graphs. Then \( \Gamma_e \) is said to be a covering graph for \( \Gamma \) if there is a surjection \( f : V(\Gamma_e) \rightarrow V(\Gamma) \) which preserves adjacency and for each \( u \in V(\Gamma_e) \), the restricted function \( f|_{\Gamma_e(u)} : \Gamma_e(u) \rightarrow \Gamma(f(u)) \) is a one to one correspondence. The function \( f \) is called a covering projection. Clearly, if \( \Gamma \) is bipartite, then so is \( \Gamma_e \). For each \( u \in V(\Gamma) \), the fibre on \( u \) is defined as \( fib_u = f^{-1}(u) \). The following important set is a subgroup of \( Aut(\Gamma_e) \) and is called the group of covering transformations for \( f \):

\[
CT(f) = \{ \sigma \in Aut(\Gamma) \mid \forall u \in V(\Gamma), \sigma(fib_u) = fib_u \}.
\]

It is known that \( K = CT(f) \) acts semiregularly on each fibre [9]. If this action is regular, then \( \Gamma_e \) is said to be a regular \( K \)-cover of \( \Gamma \).

Let \( X \leq Aut(\Gamma) \). We say \( \Gamma \) is \( X \)-vertex transitive or \( X \)-edge transitive if \( X \) acts transitively on \( V(\Gamma) \) or \( E(\Gamma) \) respectively. Also \( \Gamma \) is called \( X \)-semisymmetric if it is regular and \( X \)-edge transitive but not \( X \)-vertex transitive. For \( X = Aut(\Gamma) \), we omit \( X \) and simply talk about \( \Gamma \) being edge transitive, vertex transitive or semisymmetric. An \( X \)-edge transitive but not \( X \)-vertex transitive graph is necessarily bipartite, where the two partites are the orbits of the action of \( X \) on \( V(\Gamma) \). If \( \Gamma \) is regular, then the two partite sets have equal cardinality. So an \( X \)-semisymmetric graph is bipartite such that \( X \) is transitive on each partite but \( X \) carries no vertex from one partite set to the other. A census of all connected semisymmetric cubic graphs of orders up to 768 is given in [5].
Any minimal normal subgroup of a finite group, is the internal direct product of isomorphic copies of a simple group.

A finite simple group $G$ is called a $K_n$-group if its order has exactly $n$ distinct prime divisors, where $n \in \mathbb{N}$. The following two results determine all simple $K_3$-groups and $K_4$-groups $[4, 8, 13, 17]$.

**Theorem 2.1.** (i) If $G$ is a simple $K_3$-group, then $G$ is one of the following groups: $A_5$, $A_6$, $L_2(7)$, $L_2(2^4)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

(ii) If $G$ is a simple $K_4$-group, then $G$ is one of the following groups:

1. $A_7$, $A_8$, $A_9$, $A_{10}$, $M_{11}$, $M_{12}$, $J_2$, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^{15})$, $L_2(577)$, $L_2(2^{19})$, $L_2(3^5)$, $L_2(5^{11})$, $L_2(17)$, $L_2(3^3)$, $U_3(5)$, $U_4(7)$, $U_5(3^2)$, $U_5(3^3)$, $U_5(2)$, $S_4(2^2)$, $S_4(3)$, $S_4(7)$, $S_4(3^2)$, $S_6(2)$, $O_4^+(2)$, $S_4(3)$, $S_4(2)$, $L_3(3)$.

2. $L_2(r)$ where $r$ is a prime, $r^2 - 1 = 2^s \cdot 3^b$, $s > 3$ is a prime, $a, b \in \mathbb{N}$

3. $L_2(2^m)$ where $m$, $2^m - 1$, $\frac{2^m+1}{3}$ are primes greater than 3.

4. $L_2(3^m)$ where $m$, $3^m$, and $\frac{3^m+1}{2}$ are odd primes.

**Theorem 2.2** ([4]). If $H$ is a subgroup of a group $G$, then $C_G(H) \leq N_G(H)$ and $\frac{N_G(H)}{C_G(H)}$ is isomorphic to a subgroup of $\text{Aut}(H)$.

**Theorem 2.3** ([12]). Let $G$ be a finite group and $p$ a prime. If $G$ has an abelian Sylow $p$-subgroup, then $p$ does not divide $|G|/\text{Z}(G)|$.

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

**Theorem 2.4** ([4]). For any two distinct primes $p$ and $q$ and any two nonnegative integers $a$ and $b$, every finite group of order $p^aq^b$ is solvable.

Transitive permutation groups of prime degree are primitive. The following list can be seen e.g. in [3].

**Proposition 2.5.** A transitive permutation group of degree 17 is one of the following groups:

- $S_{17}$, $A_{17}$, $L_2(2^4)$, $L_2(2^4) \rtimes \mathbb{Z}_2$, $\mathbb{P}GL_2(2^4)$,
- $S_{17}$, $Z_{17} \rtimes \mathbb{Z}_2$, $Z_{17} \rtimes Z_4$, $Z_{17} \rtimes Z_8$, $Z_{17} \rtimes Z_{16}$.

In the following theorem, by the inverse of a pair $(a, b)$, we mean $(b, a)$.

Also for each $i$, $A_i$, $B_i$, $C_i$ and $D_i$ are noncyclic groups of order $i$ with known structures. We will not need their structures.

**Theorem 2.6** ([7]). If $\Gamma$ is a connected cubic $X$-semisymmetric graph, then the order of the stabilizer of any vertex is of the form $2^r \cdot 3$ for some $0 \leq r \leq 7$. More precisely if $\{u, v\}$ is any edge of $\Gamma$, then the pair $(X_u, X_v)$ can only be one of the following fifteen pairs or their inverses: $(\mathbb{Z}_3, \mathbb{Z}_3)$, $(S_3, S_3)$, $(S_3, \mathbb{Z}_6)$, $(D_{12}, D_{12})$, $(A_4, A_4)$, $(S_4, D_{24})$, $(S_4, \mathbb{Z}_4 \rtimes D_8)$, $(A_4 \rtimes \mathbb{Z}_2, D_{12} \times \mathbb{Z}_2)$, $(S_4 \times \mathbb{Z}_2, D_8 \times S_3)$, $(S_4, S_4)$, $(S_4 \times \mathbb{Z}_2, \mathbb{B}_2 \times \mathbb{Z}_2)$, $(A_{96}, B_{96})$, $(A_{192}, B_{192})$, $(C_{192}, D_{192})$, $(A_{384}, B_{384})$. 

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Proposition 2.7 ([15]). Let $\Gamma$ be a connected cubic $X$-semisymmetric graph for some $X \leq \text{Aut}(\Gamma)$ and let $N \leq X$. If $|\frac{X}{N}|$ is not divisible by 3, then $\Gamma$ is also $N$-semisymmetric.

Proposition 2.8 ([11]). Let $\Gamma$ be a connected cubic $X$-semisymmetric graph for some $X \leq \text{Aut}(\Gamma)$; then either $\Gamma \cong K_{3,3}$, the complete bipartite graph on 6 vertices, or $X$ acts faithfully on each of the bipartition sets of $\Gamma$.

The following Proposition is part of Proposition 2.4 of [11] which is stated as we need it in this paper.

Proposition 2.9 ([11]). Let $\Gamma$ be a connected cubic $X$-edge transitive graph for some $X \leq \text{Aut}(\Gamma)$; if $u$ and $v$ are two arbitrary adjacent vertices, then $X_u \cap X_v$ is a common Sylow 2-subgroup of $X_u$ and $X_v$.

Theorem 2.10 ([10]). Let $\Gamma$ be a connected cubic $X$-semisymmetric graph. Let $\{U,W\}$ be a bipartition for $\Gamma$ and assume $N \triangleleft X$. If the actions of $N$ on both $U$ and $W$ are intransitive, then $N$ acts semiregularly on both $U$ and $W$, $\Gamma_N$ is $\frac{X}{N}$-semisymmetric, and $\Gamma$ is a regular $N$-covering of $\Gamma_N$.

This theorem has a nice result. For every normal subgroup $N \leq X$ either $N$ is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of $N$ is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.10, the induced action of $N$ on both $U$ and $W$ is semiregular and hence the order of $N$ divides $|U| = |W|$. So we have the following handy corollary.

Corollary 2.11. If $\Gamma$ is a connected cubic $X$-semisymmetric graph with $\{U,W\}$ as a bipartition and $N \leq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.

3. Main results

In this section, our goal is to prove the following important result:

Theorem 3.1. Let $p$ be a prime number other than 17. Then there is no connected cubic semisymmetric graph of order $34p^3$.

This theorem may be read like this: If there is a connected cubic edge transitive graph of order $34p^3$, $p \neq 17$ prime, then it will also be vertex transitive.

To prove this theorem, we need some lemmas that we now state and prove.

Lemma 3.2. Let $p$ be a prime where $3 < p \neq 17$, and let $0 \leq i \leq 7$.

(i) There is no simple group of order $2^i \cdot 3 \cdot 17 \cdot p^j$ for $j = 2,3$.

(ii) The group $L_2(2^i)$ is the only simple $K_4$-group whose order is of the form $2^i \cdot 3 \cdot 17 \cdot p$.

Proof. The order of every group in sub-items (2), (3) and (4) of item (ii) of Theorem 2.1 cannot be divisible by the square of $p$. Also the only simple $K_4$-groups in sub-item (1) of item (ii) of Theorem 2.1 whose orders are divisible by 17, are $L_2(2^i)$ of order $2^i \cdot 3 \cdot 5 \cdot 17$, $L_2(577)$ of order $2^9 \cdot 3^2 \cdot 577 \cdot 17^2$, $L_3(17)$ of
order $2^9 \cdot 3^2 \cdot 307 \cdot 17^3$ and $S_4(2^2)$ of order $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$. So part (i) follows. Also it follows that among the groups listed in sub-item (1) of item (ii) of Theorem 2.1, the only group whose order is of the form $2^i \cdot 3 \cdot 17 \cdot p$, is $L_2(2^i)$.

To complete the second part, first consider groups in sub-item (3) of item (ii) of Theorem 2.1. Let $L_2(2^m)$ be a group of order $2^i \cdot 3 \cdot 17 \cdot p$; then

$$2^m \cdot 3 \cdot (2^m - 1) \cdot \left(\frac{2^m + 1}{3}\right) = 2^i \cdot 3 \cdot 17 \cdot p,$$

where $m$, $2^m - 1$ and $\frac{2^m + 1}{3}$ are all primes according to Theorem 2.1. This equation has no answer as neither $2^m - 1 = 17$ nor $\frac{2^m + 1}{3} = 17$ has a solution for $m$.

Now let the group $L_2(r)$ in sub-item (2) be a candidate group. Then for odd prime $r$ and for prime $s > 3$ we have $r^2 - 1 = 2^a \cdot 3^b \cdot s$ and

$$2^{a-1} \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 17 \cdot p.$$

From these we obtain $b = 1$, $0 \leq a - 1 \leq 7$ and either $s = 17$ or $r = 17$. The group $L_2(17)$ is a $K_3$ group, so $r \neq 17$. Also the equality $s = 17$ is not possible, since the equation $r^2 - 1 = 2^a \cdot 3 \cdot 17$ has no plausible solution for $r$ when $a = 1, 2, \ldots, 8$.

Finally note that the order of a group in sub-item (4) is divisible by $3^m$ for $m > 1$.

**Lemma 3.3.** The group $\mathbb{Z}_{16} \times GL_2(7)$ does not have a subgroup isomorphic to $L_2(7)$.

**Proof.** Firstly $GL_2(7)$ does not have a subgroup isomorphic to $L_2(7)$. Suppose on the contrary that $L_2(7) \simeq K \leq GL_2(7)$. As $SL_2(7) \leq GL_2(7)$, we have $K \cap SL_2(7) \leq K$ and so $K \cap SL_2(7) = 1$ or $K$ since $K$ is simple. If $K \cap SL_2(7) = 1$, then $SL_2(7) \cap K$ is a subgroup of $GL_2(7)$ of order $|SL_2(7)| \cdot |L_2(7)|$. But this order is divisible by $7^2$ whereas $|GL_2(7)|$ is not. Therefore $K \cap SL_2(7) = K$ and so $K \leq SL_2(7)$ implying $K \leq SL_2(7)$ since $|K| = \frac{|SL_2(7)|}{2}$. Let $Z = Z(SL_2(7))$. Then

$$\frac{K}{K \cap Z} \simeq \frac{KZ}{Z} \leq \frac{SL_2(7)}{Z} \simeq K.$$

Again because $K$ is simple, this implies $\frac{K}{K \cap Z} = 1$ or $|\frac{K}{K \cap Z}| = |K|$. In the former case, $K \cap Z = K$ and so $K \leq Z$ which is impossible. In the latter case, $K \cap Z = 1$ and so $KZ$ is a subgroup of $SL_2(7)$ of order $|K| \cdot |Z| = |SL_2(7)|$, implying that $SL_2(7) = KZ$. Now we get $SL_2(7)' = (KZ)' = K' = K$.

By using the well-known fact that $SL_2(q)' = SL_2(q)$ for $q > 3$, we obtain $K = SL_2(7)$, a contradiction to $K \simeq L_2(7)$.

Now let $G = \mathbb{Z}_{16} \times GL_2(7) = N_1N_2$ be an internal direct product of $N_1 \simeq \mathbb{Z}_{16}$ and $N_2 \simeq GL_2(7)$ and suppose on the contrary that $L_2(7) \simeq H \leq G$. Since $H \cap N_2 \trianglelefteq H$, either $H \cap N_2 = 1$ or $H \cap N_2 = H$. If $H \cap N_2 = H$, then $H \leq N_2$ which is not possible as we already showed that $GL_2(7)$ does not have subgroups isomorphic to $L_2(7)$. Accordingly $H \cap N_2 = 1$ and therefore the
order of the subgroup $HN_2 \leq G$ is $|H||N_2|$ which should divide $|G| = |N_1||N_2|$. This requires $|N_1| = 16$ to be divisible by $|H|$ which is not the case. \hfill \Box

**Lemma 3.4.** Let $\Gamma$ be a connected cubic $X$-semisymmetric graph. Let $G \leq X$ and suppose for every vertex $u$ the stabilizer $G_u$ is a 2-group. Then for each vertex $u$, $G_u = 1$.

**Proof.** Since $G \leq X$, for every vertex $u$, $G_u \leq X_u$. Fix an arbitrary vertex $u$ of $\Gamma$ and take $v$ to be an arbitrary neighbor of $u$. Also suppose $g \in G_u$ is arbitrary. By Sylow's Theorem, $G_u$ is contained in a Sylow 2-subgroup of $X_u$. According to Proposition 2.9 $X_u \cap X_v$ is a Sylow 2-subgroup of $X_u$. Hence $G_u$ is contained in a conjugate of $X_u \cap X_v$. So assume $G_u \leq x^{-1}(X_u \cap X_v)x$ for some $x \in X_u$. Then $G_u = xGux^{-1} \leq X_u \cap X_v$. This yields $g \in X_v$ and hence $g$ also stabilizes $v$.

The conclusion is that if $g \in G$ stabilizes a vertex $u$, then it will also stabilize every neighbor of $u$. By connectedness of $\Gamma$, $g$ will stabilize every vertex of $\Gamma$ and so $g$ is the identity automorphism. \hfill \Box

For every prime power $q$ subgroups of $L_2(q)$ have been classified (see Chapter 3 of [14]). It can be verified that $L_2(17)$ has no proper subgroup of order $2^4 \cdot s$ for any integer $s > 1$.

**Lemma 3.5.** There is no connected cubic semisymmetric graph of order $34 \cdot 3^3$.

**Proof.** Suppose on the contrary that $\Gamma$ is a connected cubic semisymmetric graph of order $34 \cdot 3^3$ with a bipartition $\{U, W\}$. Each of the bipartition sets has cardinality $17 \cdot 3^3$ and if $A = Aut(\Gamma)$, then $|A| = 2^r \cdot 3^4 \cdot 17$ for some $0 \leq r \leq 7$.

If $P \leq A$ is of order $3^3$, then $P$ is intransitive on both $U$ and $W$ and hence according to Theorem 2.10 $\Gamma_P$ is connected cubic $\frac{A}{P}$-semisymmetric with a bipartition $\{U_P, W_P\}$ where $|U_P| = |W_P| = 17$ and where $|\frac{A}{P}| = 2^r \cdot 3 \cdot 17$. Therefore $\frac{A}{P}$ is transitive on $U_P$ and also by Proposition 2.8 the action of $\frac{A}{P}$ on $U_P$ is faithful. So $\frac{A}{P}$ is a transitive permutation group of degree 17. All such groups are known and are listed in Proposition 2.5. The order of none of these groups is of the form $2^r \cdot 3 \cdot 17$. So $\frac{A}{P}$ cannot be a transitive permutation group of degree 17, a contradiction. Therefore in the rest of the proof, we assume that $A$ does not have any normal subgroup of order $3^3$.

Let $N \simeq T^k$ be a minimal normal subgroup of $A$, where $T$ is simple. If $T$ is nonabelian, then in view of Corollary 2.11 and Theorem 2.4, $|N|$ is divisible by $17 \cdot 3^3$. So $T$ is a simple $K_3$-group. Since the power of $17$ in $|A|$ is 1, we conclude that $k = 1$ and $N \simeq T$. Hence $N$ is a simple group of order $2^j \cdot 3^j \cdot 17$ where $j = 3$ or 4. But there is no such group as the only simple $K_3$-group whose order is divisible by 17, is $L_2(17)$ of order $2^4 \cdot 3^2 \cdot 17$. Therefore $T$ is abelian and hence $N$ is elementary abelian. This means that $|N|$ is not divisible by $|U| = 17 \cdot 3^3$ and so by Corollary 2.11 $|N|$ divides $|U| = 17 \cdot 3^3$. Because we
have assumed $A$ does not have normal subgroups of order $3^i$, it follows that $N \cong \mathbb{Z}_{17}$ or $\mathbb{Z}_i^3$ for some $1 \leq i \leq 2$.

In the following we consider two general cases and discuss that both result in contradictions:

(a) Suppose $A$ has at least one normal subgroup of order $17 \cdot 3^i$ for some $i \geq 0$. Let $M$ be the largest such subgroup, i.e., $M \leq A$, $|M| = 17 \cdot 3^i$ for some $j \geq 0$, and if $K \triangleleft A$ and $|K| = 17 \cdot 3^i$, then $|K| \leq |M|$.

Let $P$ be a Sylow 3-subgroup of $M$. The number of Sylow 3-subgroups of $M$ is 1 and so $P \leq M \leq A$ which leads to $P \leq A$. Now according to Corollary 2.11 $|P|$ must divide $|U| = 17 \cdot 3^3$ which results in $j \leq 3$. Also since we have assumed $A$ does not have any normal subgroup of order $3^3$, it follows that $j < 3$. Now due to its order, $M$ is intransitive on both $U$ and $W$ and hence according to Theorem 2.10 $\Gamma_M$ is a connected cubic $\Delta/\overline{3}$-semisymmetric graph with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = 3^{3-j}$ and where $|\Delta/\overline{3}| = 2^i \cdot 3^{j-i}$. If $\Delta/\overline{3}$ is a minimal normal subgroup of $\Gamma_M$, by Theorem 2.4 it is solvable and hence elementary abelian of order $q^i$ for some prime $q$ and some $i \geq 1$. By applying Corollary 2.11 to $\Gamma_M$, it follows that $q = 3$ and so $K \triangleleft A$ is of order $17 \cdot 3^{3-i}$, contradicting the choice of $M$.

(b) Now suppose $A$ does not have any normal subgroup of order $17 \cdot 3^i$ for any $i \geq 0$. Let $N$ be a minimal normal subgroup of $A$. As we showed earlier, $N \cong \mathbb{Z}_{17}$ or $\mathbb{Z}_i^3$ for some $1 \leq i \leq 2$. In this case $N$ cannot be isomorphic to $\mathbb{Z}_{17}$ since $|N| = 17 \cdot 3^i$. It follows that $O_3(A) \neq 1$. Corollary 2.11 implies that $|O_3(A)| \leq 3^i$. Also by our assumption, $|O_3(A)| \neq 3^i$; So $|O_3(A)| = 3^i$ for $i = 1$ or 2. Let $M = O_3(A)$. According to Theorem 2.10 $\Gamma_M$ is a connected cubic $\overline{4}/\overline{3}$-semisymmetric graph of order $34 \cdot 3^{3-i}$ with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = 17 \cdot 3^{3-i}$ and where $|\overline{4}/\overline{3}| = 2^i \cdot 3^{j-i} \cdot 17$. Let $\Delta/\overline{3}$ be a minimal normal subgroup of $\Gamma_M$.

If $\Delta/\overline{3}$ is solvable, it is elementary abelian and hence it follows from Corollary 2.11 that $|\Delta/\overline{3}|$ divides $|U_M| = |W_M| = 17 \cdot 3^{3-i}$. Consequently $|\Delta/\overline{3}| = 17$ or $3^i$ for some $1 \leq j \leq 3-i$. If $|\Delta/\overline{3}| = 17$, then $K \triangleleft A$ is of order $17 \cdot 3^j$; but we have assumed $A$ does not have any such normal subgroup. On the other hand, if $|\Delta/\overline{3}| = 3^i$, then $|K| = 3^{i+j} > 3^j$, contradicting the assumption that $|O_3(A)| = 3^i$.

Now let $G = \Delta/\overline{3}$ be unsolvable. Then it follows from Theorem 2.4 and Theorem 2.10 that $G$ is transitive on at least one of the bipartition sets; Suppose $G$ is transitive on $U_M$. So $|G|$ is divisible by $|U_M| = 17 \cdot 3^{3-i}$. The stabilizer $G_u$ of a vertex $u \in U_M$ has cardinality $\left\lfloor \frac{|G|}{|U_M|} \right\rfloor = 2^i \cdot 3^{j-i} \cdot 17$. Since the power of 17 in $|G|$ is 1, $G$ must be a simple $K_3$-group. The only simple $K_3$-group whose order is divisible by 17, is $L_2(17)$ of order $2^4 \cdot 3^2 \cdot 17$. Therefore $G \cong L_2(17)$.

If $i = 2$, then for a vertex $u \in U_M$ we have $|G_u| = \frac{|G|}{|U_M|} = \frac{2^4 \cdot 3^2 \cdot 17}{4 \cdot 3^3} = 2^4 \cdot 3$. But $L_2(17)$ has no subgroup of order $2^4 \cdot 3$. So the case $i = 2$ results in a contradiction.
If \( i = 1 \), then for every vertex \( u \in U_M \) we have \(|G_u| = \frac{2^4 \cdot 3^2 \cdot 17}{17 \cdot 3^2} = 2^4\). Now there are two possibilities; either \( G \) is also transitive on \( W_M \) or \( G \) is intransitive on \( W_M \). In the former case, for each vertex \( w \in W_M \) we have \(|G_w| = \frac{|G|}{|W_M|} = 2^4\) and therefore we can invoke Lemma 3.4 to conclude that for each vertex \( v \) of the graph \( \Gamma_M \) the stabilizer size is \(|G_v| = 1\) which is obviously a contradiction.

Now consider the latter case, where \( G \) is not transitive on \( W_M \). Since \( \Gamma_M \) is \( \frac{\Delta}{\Delta} \)-semisymmetric, \( \frac{\Delta}{\Delta} \) is transitive on \( W_M \) and also according to Proposition 2.8, \( \frac{\Delta}{\Delta} \) is faithful on \( W_M \). Because \( G \not\subseteq \frac{\Delta}{\Delta} \), it follows from Proposition 6.3 and Proposition 7.1 of [16] that all the orbits of the action of \( G \) on \( W_M \) are of equal size \( t \) and \( t \) divides \(|W_M| = 17 \cdot 3^2\). Therefore \( s = \frac{17 \cdot 3^2}{t} \) is an integer. Let \( \Delta \) be an orbit of the action of \( G \) on \( W_M \) and let \( w \in \Delta \). Since \( \frac{\Delta}{\Delta} \) is faithful on \( W_M \), we conclude that \( G_w \) is a proper subgroup of \( G \) (equivalently \( t \neq 1 \)).

Also since \( G \) is not transitive on \( W_M \), we have \( t < 17 \cdot 3^2 \) and so \( s > 1 \). Now \(|G_w| = \frac{|G|}{|\Delta|} = 2^4 \cdot s \); but \( L_2(17) \) does not have any proper subgroup of such order. So the case \( i = 1 \) leads to a contradiction too.

\[ \square \]

**Lemma 3.6.** Let \( p > 3 \) be a prime number other than 17. If \( \Gamma \) is a connected cubic semisymmetric graph of order \( 34p^3 \), then \( \text{Aut}(\Gamma) \) has a normal subgroup of order \( p^3 \).

**Proof.** Let \( A = \text{Aut}(\Gamma) \) and take \( \{U, W\} \) to be a bipartition for \( \Gamma \). Each of the two partite sets has cardinality \( 17p^3 \) and since \( A \) is transitive on the partite sets, we have \(|A| = 2^r \cdot 3 \cdot 17 \cdot p^3 \) for some \( 0 \leq r \leq 7 \). We prove the result by showing that the assumption \(|O_p(A)| < p^3 \) leads to a contradiction. Let \( N \simeq T^8 \) be a minimal normal subgroup of \( A \), where \( T \) is simple.

If \( T \) is nonabelian, since the power of 3 and 17 in \(|A|\) is 1, we should have \( k = 1 \) and \( N \simeq T \) is nonabelian simple. In this case the order of \( N \) cannot divide \(|U| = 17p^3 \) according to Theorem 2.4. So according to Corollary 2.11 \(|N| \) should be divisible by \(|U| = 17p^3 \). The order of every simple \( K_3 \)-group, all listed in part (i) of Theorem 2.1, is divisible by 2 and 3. Therefore \( N \) cannot be a \( K_3 \)-group and hence it must be a simple \( K_3 \)-group whose order is of the form \( 2^r \cdot 3 \cdot 17 \cdot p^3 \) for some \( 1 \leq i \leq 7 \). But no such group exists according to Lemma 3.2.

Therefore \( N \) should be elementary abelian and hence by Corollary 2.11 \(|N| \) divides \( 17p^3 \). As a result, \( N \simeq Z_{17} \) or \( Z_{p^3}^3 \) for some \( 1 \leq i \leq 3 \). In any cases, \( \Gamma_N \) would itself be a connected cubic \( \frac{\Delta}{\Delta} \)-semisymmetric graph of order \( \frac{34p^3}{17p^3} \).

**Case 1.** \( O_p(A) = 1 \). In this case, the minimal normal subgroup of \( A \) is \( N \simeq Z_{17} \) and \( \Gamma_N \) is \( \frac{\Delta}{\Delta} \)-semisymmetric of order \( 2p^3 \). We have \(|\frac{\Delta}{\Delta}| = 2^r \cdot 3 \cdot p^3 \) and \(|U_N| = |W_N| = p^3 \) where \( \{U_N, W_N\} \) is a partition for \( \Gamma_N \). Let \( \frac{M}{N} \) be a minimal normal subgroup of \( \frac{\Delta}{\Delta} \). If \( \frac{M}{N} \) is unsolvable, then it must be a simple \( \{2, 3, p\} \)-group whose order is divisible by \( p^3 \) according to Corollary 2.11. But no simple \( \{2, 3, p\} \)-group (all listed in Theorem 2.1) has order divisible by \( p^3 \). So \( \frac{M}{N} \) is elementary abelian and hence \( \frac{M}{N} \simeq Z_{p^i}^i \) for \( i = 1, 2 \) or 3. Consequently
M \leq A$ is of order $17p^i$ for $i = 1, 2$ or 3. The number of Sylow $p$-subgroups of $M$ divides 17 and hence must be 1 since $p \neq 2$. Now the Sylow $p$-subgroup of $M$ is characteristic in $M$ and hence normal in $A$, a contradiction.

**Case 2.** $|O_p(A)| = p$. Let $M = O_p(A)$. According to Theorem 2.10, $\Gamma_M$ is a connected cubic $\frac{A}{M}$-semisymmetric graph with the bipartition $\{U_M, W_M\}$, where $|U_M| = |W_M| = 17p^2$ and $|\frac{A}{M}| = 2^r \cdot 3 \cdot 17 \cdot p^2$. Take $\frac{A}{M}$ to be a minimal normal subgroup of $\frac{A}{M}$.

If $\frac{A}{M}$ is unsolvable, then its order should be divisible by $|U_M| = 17p^2$ according to Corollary 2.11 and Theorem 2.4, and hence it is a simple $\{2, 3, 17, p\}$-group of order $2^r \cdot 3 \cdot 17 \cdot p^2$ for some $1 \leq i \leq 7$. But there is no such group according to Lemma 3.2.

Now assume $\frac{A}{M}$ is solvable. In this case, $\frac{A}{M}$ is elementary abelian and by Corollary 2.11 its order divides $|U_M| = 17p^2$. So $\frac{A}{M} \simeq \mathbb{Z}_{17}$ or $\mathbb{Z}_p$ for $i = 1$ or 2. The isomorphism $\frac{A}{M} \simeq \mathbb{Z}_{17}$ results in $|L| = p^{i+1}$ which contradicts the assumption that $|O_p(A)| = p$. Hence $\frac{A}{M} \simeq \mathbb{Z}_{17}$ and so $|L| = 17p$. The normal subgroup $L \leq A$, is intransitive on both $U$ and $W$ due to its order. So we can consider the graph $\Gamma_L$ which is connected cubic $\frac{A}{L}$-semisymmetric (Theorem 2.10) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = p^2$ and where $|\frac{A}{L}| = 2^r \cdot 3 \cdot p^2$.

Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$. If it is solvable, then it follows from Corollary 2.11 that $\frac{T}{L} \simeq \mathbb{Z}_p$ for $j = 1$ or 2, and therefore $|T| = 17p^{j+1}$. A Sylow $p$-subgroup of $T$ is normal in $A$, contradicting our current assumption on $|O_p(A)|$. On the other hand, if $\frac{T}{L}$ is unsolvable, then by Corollary 2.11 its order is divisible by $|U_L| = p^2$ and hence it is a simple group whose order is of the form $2^r \cdot 3 \cdot p^2$ for some $1 \leq i \leq 7$. But there is no such simple $K_3$-group.

Therefore every assumption on $\frac{T}{L}$ and hence every assumption on $\frac{A}{M}$ results in a contradiction from which we conclude that $|O_p(A)| = p$ is not possible.

**Case 3.** $|O_p(A)| = p^2$. Let $M = O_p(A)$. According to Proposition 2.10, $\Gamma_M$ is a connected cubic $\frac{4A}{M}$-semisymmetric graph with the bipartition $\{U_M, W_M\}$, where $|U_M| = |W_M| = 17p$ and $|\frac{A}{M}| = 2^r \cdot 17 \cdot p$. Take $\frac{A}{M}$ to be a minimal normal subgroup of $\frac{A}{M}$.

If $\frac{A}{M}$ is unsolvable, then it is a simple group whose order is divisible by $|U_M| = 17p$ and hence it is a simple group of order $2^i \cdot 17 \cdot p$ for some $0 \leq i \leq 7$. According to Lemma 3.2 the only such group is $L_2(2^i)$ which of course yields $p = 5$. So assume $p = 5$ and $\frac{A}{M} \simeq L_2(2^i)$. Since 3 does not divide the order of $\frac{A}{M}$, according to Proposition 2.7, $\Gamma_M$ is $G$-semisymmetric where $G = \frac{A}{M} \simeq L_2(2^i)$. Now $G$ is transitive on $U_M$ and $W_M$, each with $17 \cdot 5$ points. So the stabilizer $G_u$ is of order 48 for any vertex $u$ of $\Gamma_M$. According to Proposition 2.6, if $\{u, w\}$ is an edge, then there are two possibilities for a pair $(G_u, G_w)$ of stabilizers of size 48. One possibility is $(S_4 \times S_2, D_8 \times S_3)$ or its inverse, and the other possibility is $(S_4 \times Z_2, S_4 \times Z_2)$ or its inverse. Therefore
at least one of $G_u$ and $G_w$ must be isomorphic to $S_4 \times \mathbb{Z}_2$ and hence $S_4$ should be a subgroup of $G$. But according to ([14], Chapter 3) for a prime power $q$, the group $L_2(q)$ has a subgroup isomorphic to $S_4$ only when $q^2 \equiv 1 \pmod{16}$ which obviously does not hold for $L_2(2^4)$.

Now assume $\frac{L}{T}$ is solvable. In this case, $\frac{L}{T}$ is elementary abelian and hence intransitive on both $U_M$ and $W_M$. So $\frac{L}{T} \cong \mathbb{Z}_{17}$ or $\mathbb{Z}_p$. The isomorphism $\frac{L}{T} \cong \mathbb{Z}_p$ results in $|L| = p^3$ which contradicts the assumption that $|O_p(A)| = p^2$. Hence $\frac{L}{T} \cong \mathbb{Z}_{17}$ and so $|L| = 17p^2$. The normal subgroup $L$, is intransitive on both $U$ and $W$ due to its order. So we can consider the graph $\Gamma_L$ which is connected cubic $\frac{A}{L}$-semisymmetric (Proposition 2.10) with the bipartition $\{U_L,W_L\}$, where $|U_L| = |W_L| = p$ and where $|A| = 2^r \cdot 3 \cdot p$. Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$. If it is solvable, then $\frac{T}{L} \cong \mathbb{Z}_p$, and therefore $|T| = 17p^3$. A Sylow $p$-subgroup of $T$ is characteristic in $T$ and hence normal in $A$, contradicting our current assumption on $|O_p(A)|$. On the other hand, if $\frac{T}{L}$ is unsolvable, then it is a simple $\{2,3,p\}$-group and hence $\frac{T}{L} \cong A_5, L_2(7)$.

In the following we show that these two cases will result in contradiction.

(1) If $\frac{T}{L} \cong A_5$, then $p = 5$ and $\Gamma_L$ is also $\frac{A}{L}$-semisymmetric according to Proposition 2.7, as 3 does not divide the order of $\frac{A}{L}$. So $G = \frac{A}{L}$ is transitive on $U_L$ and on $W_L$, each with 5 points. For any vertices $u \in U_L$ and $w \in W_L$, the stabilizers $G_u$ and $G_w$ are of order 12 and hence both are isomorphic to $A_4$, the only subgroup of $A_5$ of order 12. But the pair $(G_u,G_w) = (A_4,A_4)$ is not possible for an edge $(u,w)$ of a cubic $G$-semisymmetric graph according to Theorem 2.6.

(2) If $\frac{T}{L} \cong L_2(7)$, then $p = 7$ and $\Gamma$ is also $T$-semisymmetric according to Proposition 2.7 as the order of $\frac{A}{L} \cong \frac{T}{L}$ is not divisible by 3. In this case $|L| = 17 \cdot 7^2$. The number of Sylow 17-subgroups of $L$ divides $7^2$ and hence equals 1. Similarly the number of Sylow 7-subgroups of $L$ is 1. So if $P$ and $Q$ are respectively the Sylow $17$-subgroup and the Sylow 7-subgroup of $L$, then $L \cong P \times Q$ and hence $L$ is abelian. Now $T$ is nonabelian and $L$ is a maximal normal subgroup of $T$ since $\frac{T}{L}$ is nonabelian simple. We have $L \leq C_T(L) \leq N_T(L) = T$ which implies $C_T(L) = L$ or $T$.

If $C_T(L) = L$, then by Theorem 2.2, $\frac{T}{L} \leq Aut(L) \cong Aut(P) \times Aut(Q) \cong \mathbb{Z}_{16} \rtimes Aut(Q)$. Now $|Q| = 7^2$ implies either $Q \cong \mathbb{Z}_{2^2}$ or $Q \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. If $Q \cong \mathbb{Z}_{2^2}$, then $Aut(Q) \cong \mathbb{U}_{2^2}$, the multiplicative group of integers modulo $2^2$ which is abelian of order $\varphi(2^2) = 4$. In this case $\frac{T}{L}$ should be isomorphic to a subgroup of $\mathbb{Z}_{16} \rtimes \mathbb{U}_{2^2}$ which is not possible since $\frac{T}{L}$ is nonabelian. On the other hand if $Q \cong \mathbb{Z}_7 \times \mathbb{Z}_7$, then $Aut(Q) \cong \mathbb{U}_{7^2}$, the multiplicative group of integers modulo $7^2$ which is abelian of order $\varphi(7^2) = 42$. In this case $\frac{T}{L}$ should be isomorphic to a subgroup of $\mathbb{Z}_{16} \times GL_2(7)$ which is not possible according to Lemma 3.3.

If $C_T(L) = T$, then $L \leq Z(T)$ and so either $Z(T) = L$ or $Z(T) = T$. As $T$ is not abelian, we should have $Z(T) = L$. Since $|T| = |L||L_2(7)| = 2^3 \cdot 3 \cdot 17 \cdot 7^2$, Sylow 17-subgroups of $T$ are abelian and so according to Theorem 2.3 $|T| \cap
Z(T) is not divisible by 17. By taking |Z(T)| = 17 · 7^2 into consideration, this yields T' ∩ Z(T) = 1, 7 or 7^2. Again since Z(T) = L is maximal normal in T, the relations Z(T) ≤ T'Z(T) ≤ T imply either T'Z(T) = Z(T) or T'Z(T) = T. The equality T''Z(T) = Z(T) yields T' ≤ Z(T) and so T'' = T' ∩ Z(T) = 1, 7 or 7^2. Since T is not abelian, T' = 1 does not hold. If |T'| = 7 for i = 1 or 2, then T_i is abelian of order 2^i · 3 · 17 · 7^{i-1}. So all the subgroups of T_i are normal in it. If P_3 and P_{17} are the Sylow 3-subgroup and the Sylow 17-subgroup of T respectively, then P_3P_{17} ≤ T_i is of order 21 and so there exists some K ≤ T with T_i = P_3P_{17}. The order of K is 3 · 17 · 7^i. According to Corollary 2.11, one of |U| = 17 · 7^3 and |T| should divide the other. But since i < 3, none of them divides the other, a contradiction.

Now suppose T'Z(T) = T. By taking cardinalities, we obtain 2^3 · 3 · 17 · 7^3 = |T'Z(T)| = (17 · 7^2) where here i = 0, 1 or 2. So |T'| = 2^3 · 3 · 7^{i-1+i}. With this order |T'| neither divides |U| = 17 · 7^3 nor is divisible by it, a contradiction.

We conclude that T_i cannot be isomorphic to L_2(7) as it leads to a contradiction. So the case |O_p(A)| = p^3 is impossible.

**Proof of Theorem 3.1.** The result for p = 2 follows from [5] and for p = 3 follows from Lemma 3.5. Now for any prime 3 < p ≠ 17, if Γ is a connected cubic semisymmetric graph of order 34p^3 with the bipartition {U, W}, then by Lemma 3.6 A = Aut(Γ) has a normal subgroup P of order p^3 which is obviously intransitive on both U and W. Therefore according to Theorem 2.10 Γ_P must be a connected cubic G-semisymmetric graph of order 34 with the bipartition {U_P, W_P}, where G = A_Γ and |U_P| = |W_P| = 17. We have |G| = 2^r · 3 · 17 and G is transitive on both U_P and W_P. Also according to Proposition 2.8 the action of G on each of U_P and W_P is faithful. So G is a transitive permutation group of degree 17. All such groups are listed in Proposition 2.5. But the order of any of the groups S_{17}, A_{17}, L_2(2^4), L_2(2^4) ∅ Z_2, PTL_2(2^4) or Z_{17} ∅ Z_n for n = 1, 2, 4, 8, 16, is not of the form 2^r · 3 · 17. This contradiction proves that there is no connected semisymmetric cubic graph of order 34p^3 for any prime p ≠ 17.

**References**


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