COMMON FIXED POINT RESULTS ON FUZZY METRIC SPACES AND MODULAR METRIC SPACES VIA SIMULATION FUNCTION

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ABSTRACT. In this paper, we prove common fixed point theorems for two mappings by using simulation function on fuzzy metric spaces. We also deduce some consequences in modular metric spaces.

1. INTRODUCTION AND PRELIMINARIES


Recently, the notion of simulation function was given by Khojasteh et al. [7]. In [9] and [11] authors revised the definition of simulation function introduced by Khojasteh et al. [7].

Definition 1.1 ([9, 11]). A mapping \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) is a simulation function if it satisfies the following conditions:

\((\zeta_1)\) \( \zeta(t, s) < s - t \) for all \( t, s > 0 \).

\((\zeta_2)\) If \( \{t_n\}, \{s_n\} \) are sequences in \( (0, \infty) \) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \) and \( t_n < s_n \) for all \( n \in \mathbb{N} \), then \( \lim \sup_{n \to \infty} \sup \zeta(t_n, s_n) < 0 \).

The set of all simulation functions is denoted by \( Z \).

Several examples of simulation function are given in [4], [7], [9], [10], [13], [15].

It is clear from \((\zeta_1)\) that \( \zeta(t, t) < 0 \) when \( t > 0 \).

Definition 1.2 (Schweizer and Sklar [12]). A binary operation \( \ast : [0, 1] \times [0, 1] \to [0, 1] \) is continuous \( t \)-norm if it satisfies the following conditions:

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(1) * is commutative and associative,
(2) * is continuous,
(3) \(a * 1 = a\) for all \(a \in [0, 1]\),
(4) \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).
A few examples of continuous t-norm are
\[a * b = ab, \quad a * b = \min\{a, b\}, \quad a * b = \max\{a + b - 1, 0\}.
\]

Definition 1.3 (George and Veeramani [5]). A fuzzy metric space is an ordered triple \((X, M, \cdot)\) such that \(X\) is an arbitrary non-empty set, \(\cdot\) is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions, for all \(x, y, z \in X\) and \(t > 0\):

\[
(FM - 1) \quad M(x, y, 0) > 0,
(FM - 2) \quad M(x, y, t) = 1 \text{ iff } x = y,
(FM - 3) \quad M(x, y, t) = M(y, x, t),
(FM - 4) \quad M(x, y, t) \cdot M(y, z, t) \leq M(x, z, t + s),
(FM - 5) \quad M(x, y, t) : [0, \infty) \to [0, 1] \text{ is continuous.}
\]

then the triple \((X, M, \cdot)\) is called a fuzzy metric space. If we replace \((FM - 4)\) by

\[
(FM - 6) M(x, y, t) \cdot M(y, z, t) \leq M(x, z, t)
\]

then the triple \((X, M\cdot)\) is called a non-Archimedean fuzzy metric space. We note that if \((X, M, \cdot)\) is nondecreasing for all \(x, y, z \in X\) then \((FM - 6)\) is equivalent to

\[
(FM - 6) M(x, y, t) \cdot M(y, z, s) \leq M(x, z, \max\{t, s\})
\]

that implies \((FM - 4)\). Thus each non-Archimedean fuzzy metric space is a fuzzy metric space, if \((X, M, \cdot)\) is nondecreasing for all \(x, y, z \in X\).

Definition 1.4. Let \((X, M, \cdot)\) be a fuzzy metric space. Then

(i) a sequence \(\{x_n\}\) converges to \(x_0 \in X\) iff for all \(t > 0 \lim_{n \to \infty} M(x_n, x_0, t) = 1\)
(ii) a sequence \(\{x_n\}\) in \(X\) is a Cauchy sequence [5] if and only if for all \(\varepsilon \in (0, 1)\) and \(t > 0\) there exists \(n_0\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for all \(m, n \geq n_0\)
(iii) \((X, M, \cdot)\) is complete [6] if every Cauchy sequence converges to some \(x \in X\).

Definition 1.5. ([1]). Let \((X, M, \cdot)\) be a fuzzy metric space. The fuzzy metric \(M\) is called triangular when

\[
\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1 \text{ for all } x, y, z \in X \text{ all } t > 0.
\]
2. Common Fixed Point via Simulation Function on Fuzzy Metric Spaces

**Theorem 2.1.** Let $(X, M, \ast)$ be a non-Archimedean fuzzy metric space with $M$ triangular and let $A, B : X \to X$ be two given mappings. Let there exists $\zeta \in \mathbb{Z}$ such that

\begin{equation}
\zeta\left(\frac{1}{M(Ax, Ay, t)} - 1, \frac{1}{M(Bx, By, t)} - 1\right) \geq 0 \text{ for all } x, y \in X
\end{equation}

If $AX \subseteq BX$ and $AX$ or $BX$ is a complete subset of $X$. Then $A$ and $B$ have unique coincidence point in $X$. Moreover if $A$ and $B$ are weakly compatible then $A$ and $B$ have a unique common fixed point in $X$.

**Proof.** First of all we will prove that if coincidence point of $A$ and $B$ exist then it is unique.

Suppose if possible $v_1$ and $v_2$ are two distinct coincidence points of $A$ and $B$ then there exists two points $u_1, u_2 \in X$ such that

$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_2$

then by (2.1) we have

\[
0 \leq \zeta\left(\frac{1}{M(Ax, Ay, t)} - 1, \frac{1}{M(Bx, By, t)} - 1\right) = \zeta\left(\frac{1}{M(v_1, v_2, t)} - 1, \frac{1}{M(v_1, v_2, t)} - 1\right) < 0,
\]

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. Since $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we obtain $Ax_n = Bx_{n+1}$ for all $n \in \mathbb{N}$

Let $Ax_n = Bx_{n+1} = y_n$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus $x_{n+1}$ is the unique coincidence point of $A$ and $B$. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Hence we have

\begin{equation}
0 \leq \zeta\left(\frac{1}{M(Ax_n, Ax_{n+1}, t)} - 1, \frac{1}{M(Bx_n, Bx_{n+1}, t)} - 1\right) = \zeta\left(\frac{1}{M(y_n, y_{n+1}, t)} - 1, \frac{1}{M(y_{n-1}, y_n, t)} - 1\right) < S(y_{n-1}, y_n) - S(y_n, y_{n+1}).
\end{equation}
where \( S(y_{n-1}, y_n, t) = \frac{1}{M(y_{n-1}, y_n, t)} - 1 \).

Therefore \( \{S(y_{n-1}, y_n, t)\} \) is a decreasing sequence of positive real numbers. Thus there exists \( z \geq 0 \) such that

\[
\lim_{n \to \infty} S(y_{n-1}, y_n, t) = z
\]

(Suppose \( z > 0 \) then by (2.2) and (\( \zeta_2 \)) it follows that

\[
0 \leq \lim_{n \to \infty} \sup \zeta(S(y_n, y_{n+1}, t), S(y_{n-1}, y_n, t)) < 0
\]

where \( t_n = S(y_n, y_{n+1}, t) < S(y_{n-1}, y_n, t) = s_n \) and \( t_n, s_n \to z > 0 \).

Clearly this is a contradiction and so \( z = 0 \).

By (2.3) we obtain

\[
\lim_{n \to \infty} M(y_{n-1}, y_n, t) = 1
\]

Now we prove that the sequence \( \{y_n\} \) is Cauchy. Suppose if possible \( \{y_n\} \) is not a Cauchy sequence in \( X \), therefore \( \lim_{n,n \to \infty} \inf M(y_m, y_n, t) < 1 \) for some \( t > 0 \).

Suppose there exists \( 0 < \epsilon < 1 \) and two sub sequences \( \{y_{m_k}\} \) and \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( n_k \) is the smallest index for which \( n_k > m_k \geq k \) and

\[
M(y_{m_k}, y_{n_k}, t) \leq 1 - \epsilon
\]

and

\[
M(y_{m_k}, y_{n_k-1}, t_0) > 1 - \epsilon
\]

Now we have

\[
1 - \epsilon \geq M(y_{m_k}, y_{n_k}, t_0) \\
\geq M(y_{m_k}, y_{n_k-1}, t_0) \cdot M(y_{n_k}, y_{n_0}, t) \\
> 1 - \epsilon \cdot M(y_{n_k}, y_{n_0}, t)
\]

Letting \( k \to \infty \) and using (2.4), we get

\[
\lim_{k \to \infty} M(y_{m_k}, y_{n_k}, t) = 1 - \epsilon
\]

By the same reasoning as above, we obtain

\[
1 - \epsilon \geq M(y_{m_k}, y_{n_k}, t_0) \\
\geq M(y_{m_k}, y_{m_k-1}, t_0) \cdot M(y_{m_k-1}, y_{n_k-1}, t_0) \cdot M(y_{n_k}, y_{n_0}, t)
\]

and

\[
M(y_{m_k-1}, y_{n_k-1}, t_0) \geq M(y_{m_k-1}, y_{m_k}, t_0) \cdot M(y_{m_k}, y_{n_k}, t_0) \cdot M(y_{n_k}, y_{n_k-1}, t_0)
\]
By letting \( k \to \infty \) and using (2.4) and (2.7), we obtain

\[
(2.8) \quad \lim_{k \to \infty} M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = 1 - \epsilon
\]

Using (2.7) and (2.8), we obtain

\[
\lim_{k \to \infty} S(y_{m_k}, y_{n_k}, t_0) = \lim_{k \to \infty} \frac{1}{M(y_{m_k}, y_{n_k}, t_0)} - 1
\]
\[
= \lim_{k \to \infty} \frac{1 - M(y_{m_k}, y_{n_k}, t_0)}{M(y_{m_k}, y_{n_k}, t_0)}
\]
\[
= \frac{1 - (1 - \epsilon)}{1 - \epsilon}
\]
\[
= \frac{\epsilon}{1 - \epsilon}
\]

and

\[
\lim_{k \to \infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = \frac{\epsilon}{1 - \epsilon}
\]

Let

\[
t_k = S(y_{m_k}, y_{n_k}, t_0),
\]
\[
s_k = S(y_{m_{k-1}}, y_{n_{k-1}}, t_0).
\]

Thus by using (2.1) and (\( \zeta_2 \)) we have

\[
0 \leq \lim_{k \to \infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_0), S(y_{m_{k-1}}, y_{n_{k-1}}, t_0)) < 0.
\]

Above inequality is not true and hence \( \{y_n\} \) is a Cauchy sequence in \( X \). Now since \( AX \) or \( BX \) is a complete subset of \((X, M, *)\) therefore there exists \( u \in X \) such that \( y_n \to Bu \) as \( n \to \infty \). If there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( y_{n_k} = Au \) then letting \( k \to \infty \) we get \( Au = Bu \) and hence the claim. So we suppose that \( y_{n_k} \neq Au \) for all \( n \in N \).

Since \( y_{n-1} \neq y_n \) there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( y_{n_k} \neq Bu \) for \( k \in N \). Using (2.1) we have

\[
0 \leq \zeta(\frac{1}{M(Ax_{n_k+1}, Au, t)} - 1, \frac{1}{M(Bx_{n_k+1}, Bu, t)} - 1)
\]
\[
= \zeta(S(y_{n_k+1}, Au, t), S(y_{n_k}, Bu, t))
\]
\[
< S(y_{n_k}, Bu, t) - S(y_{n_k+1}, Au, t).
\]

This shows that \( y_{n_k+1} \to Au \) and hence \( Au = Bu \) is a unique coincidence point of \( A \) and \( B \). If \( A \) and \( B \) are weakly compatible then by using well known result due to Jungck, we can prove the existence of unique common fixed point of \( A \) and \( B \).
Theorem 2.2. Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $M$ triangular and $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in \mathbb{Z}$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that

\begin{equation}
\zeta \left( \frac{1}{M(Ax, Ay, t)} - 1 \right) - 1, \phi \left( \frac{1}{M(Bx, By, t)} - 1 \right) \geq 0 \text{ for all } x, y \in X
\end{equation}

\begin{equation}
0 < \phi(t) \leq t \text{ for all } t \in (0, +\infty) \text{ and } \phi(0) = 0
\end{equation}

If $AX \subseteq BX$ and $AX$ or $BX$ is a complete subset of $X$. Then $A$ and $B$ have unique coincidence point in $X$. Moreover if $A$ and $B$ are weakly compatible then $A$ and $B$ have a unique common fixed point in $X$.

Proof. First of all we will prove that if coincidence point of $A$ and $B$ exist then it is unique.

suppose if possible $v_1$ and $v_2$ are two distinct coincidence points of $A$ and $B$ then there exists two points $u_1, u_2 \in X$ such that

$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_2$

then by (2.9) we have

$0 \leq \zeta \left( \frac{1}{M(Au_1, Au_2, t)} - 1 \right) - 1, \phi \left( \frac{1}{M(Bu_1, Bu_2, t)} - 1 \right) \leq 0$

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. Since $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we obtain $Ax_n = Bx_{n+1}$ for all $n \in \mathbb{N}$

Let $Ax_n = Bx_{n+1} = y_n$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus $x_{n+1}$ is the unique coincidence point of $A$ and $B$. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Hence we have

$0 \leq \zeta \left( \frac{1}{M(Ax_n, Ax_{n+1}, t)} - 1 \right) - 1, \phi \left( \frac{1}{M(Bx_n, Bx_{n+1}, t)} - 1 \right) = \zeta \left( \frac{1}{M(y_n, y_{n+1}, t)} - 1 \right) - 1, \phi \left( \frac{1}{M(y_{n-1}, y_n, t)} - 1 \right)$
\begin{equation}
< \phi\left( \frac{1}{M(y_{n-1}, y_n, t)} - 1 \right), \left( \frac{1}{M(y_n, y_{n+1}, t)} - 1 \right)
= S(y_{n-1}, y_n, t) - S(y_n, y_{n+1}, t) \quad \text{for all } n \in N
\end{equation}

where \( S(y_{n-1}, y_n, t) = \frac{1}{M(y_{n-1}, y_n, t)} - 1. \)

Therefore \( \{ S(y_{n-1}, y_n, t) \} \) is a decreasing sequence of positive real numbers. Thus there exists \( z \geq 0 \) such that

\begin{equation}
\lim_{n \to \infty} S(y_{n-1}, y_n, t) = z
\end{equation}

Suppose \( z > 0 \) then

\[ 0 \leq \lim_{n \to \infty} \sup \zeta(S(y_n, y_{n+1}, t), \phi(S(y_{n-1}, y_n, t))) < 0 \]

where \( t_n = S(y_n, y_{n+1}, t), s_n = \phi(S(y_{n-1}, y_n, t)) < S(y_{n-1}, y_n, t), \) and \( t_n < s_n, t_n, s_n \to z > 0. \)

This is a contradiction. Thus we have

\[ \lim_{n \to \infty} S(y_{n-1}, y_n, t) = 0 \]

By (2.12) we obtain

\begin{equation}
\lim_{n \to \infty} M(y_n, y_{n+1}, t) = 1
\end{equation}

Now we claim that the sequence \( \{ y_n \} \) is Cauchy sequence in \( (X, d) \). Suppose if possible \( \{ y_n \} \) is not a Cauchy sequence in \( X \), therefore \( \lim_{m,n \to \infty} \inf M(y_m, y_n, t_0) < 1 \) for some \( t_0 > 0. \)

Suppose there exists \( 0 < \epsilon < 1 \) and two sub sequences \( \{ y_{m_k} \} \) and \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that \( n_k \) is the smallest index for which \( n_k > m_k \geq k \) and

\begin{equation}
M(y_{m_k}, y_{n_k}, t_0) \leq 1 - \epsilon
\end{equation}

and

\begin{equation}
M(y_{m_k}, y_{n_{k-1}}, t_0) > 1 - \epsilon
\end{equation}

Now we have

\[ 1 - \epsilon \geq M(y_{m_k}, y_{n_k}, t_0) \]
\[ \geq M(y_{m_k}, y_{n_{k-1}}, t_0) * M(y_{n_{k-1}}, y_{n_k}, t_0) \]
\[ \geq 1 - \epsilon * M(y_{n_{k-1}}, y_{n_k}, t_0) \]

Letting \( k \to \infty \) and using (2.13), we get

\begin{equation}
\lim_{k \to \infty} M(y_{m_k}, y_{n_k}, t_0) = 1 - \epsilon
\end{equation}
By the same reasoning as above, we obtain

\[ 1 - \epsilon \geq M(y_{m_k}, y_{n_k}, t_o) \]
\[ \geq M(y_{m_k}, y_{m_{k-1}}, t_o) \times M(y_{m_{k-1}}, y_{n_{k-1}}, t_o) \times M(y_{n_{k-1}}, y_{n_k}, t_o) \]

and

\[ M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) \geq M(y_{m_{k-1}}, y_{m_k}, t_0) \times M(y_{m_k}, y_{n_k}, t_0) \times M(y_{n_k}, y_{n_{k-1}}, t_0) \]

From the last inequality, by letting \( k \to \infty \) and using (2.13), (2.16) we get

(2.17) \[ \lim_{k \to \infty} M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = 1 - \epsilon \]

By letting \( k \to \infty \) and using (2.16) and (2.17) we obtain

\[ \lim_{k \to \infty} S(y_{m_k}, y_{n_k}, t_o) = \frac{1}{1 - \epsilon} \]

and

\[ \lim_{k \to \infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = \frac{\epsilon}{1 - \epsilon} \]

Let

\[ t_k = S(y_{m_k}, y_{n_k}, t_o) \]
\[ s_k = \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_o)) < S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) \]

By (2.9), we have

(2.18) \[ 0 \leq \zeta\left(\frac{1}{M(Ax_{m_k}, Ay_{n_k}, t_0)} - 1, \phi\left(\frac{1}{M(Bx_{m_k}, Bx_{n_k}, t_0)} - 1\right)\right) \]
\[ = \zeta\left(\frac{1}{M(y_{m_k}, y_{n_k}, t_0)} - 1, \phi\left(\frac{1}{M(y_{m_{k-1}}, y_{n_{k-1}}, t_0)} - 1\right)\right) \]
\[ = \zeta(s_k, t_k, t_o) \times \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0)) \]
\[ < \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) - S(y_{m_k}, y_{n_k}, t_0) \to 0 \text{ as } k \to \infty \]

From (2.18) we deduce that

\[ \lim_{k \to \infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_o), \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_o))) = 0. \]
Clearly this is a contradiction to (2.1) and hence we conclude that \( \{y_n\} \) is a Cauchy sequence in \( X \). Now since \( AX \) or \( BX \) is a complete subset of \( (X, M, \ast) \) therefore there exists \( u \in X \) such that \( y_n \rightarrow Bu \) as \( n \rightarrow \infty \). If there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( y_{n_k} = Au \) then letting \( k \rightarrow \infty \) we get \( Au = Bu \) and hence the claim. So we suppose that \( y_{n_k} \neq Au \) for all \( n \in N \).

Since \( y_{n-1} \neq y_n \) there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( y_{n_k} \neq Bu \) for \( k \in N \). Using (2.9) we have

\[
0 \leq \zeta\left(\frac{1}{M(Ax_{n_k+1}, Au, t)} - 1, \phi\left(\frac{1}{M(Bx_{n_k+1}, Bu, t)} - 1\right)\right) = \zeta(S(y_{n_k+1}, Au, t), \phi(S(y_{n_k}, Bu, t)))
\]

\[
< \phi(S(y_{n_k}, Bu, t) - S(y_{n_k+1}, Au, t)) = S(y_{n_k}, Bu, t) - S(y_{n_k+1}, Au, t) \text{ for all } n \in N
\]

This shows that \( y_{n_k+1} \rightarrow Au \) and hence \( Au = Bu \) is a unique coincidence point of \( A \) and \( B \). If \( A \) and \( B \) are weakly compatible then by using well known result due to Jungck, we can prove the existence of unique common fixed point of \( A \) and \( B \).

**Theorem 2.3.** Let \( (X, M, \ast) \) be a non-Archimedean fuzzy metric space and \( A, B : X \rightarrow X \) be two given mappings. Suppose there exists \( \zeta \in Z \) and a function \( k \in (0, \frac{1}{2}) \) such that for all \( x, y \in X \)

\[
(2.19) \quad \zeta\left(\frac{1}{M(Ax, Ay, t)} - 1, k \max\left\{ \frac{1}{M(Bx, By, t)} - 1, \frac{1}{M(Bx, Ay, t)} - 1, \frac{1}{M(By, Ay, t)} - 1\right\} \right) \geq 0
\]

If \( AX \subseteq BX \) and \( AX \) or \( BX \) is a complete subset of \( X \). Then \( A \) and \( B \) have unique coincidence point in \( X \). Moreover if \( A \) and \( B \) are weakly compatible then \( A \) and \( B \) have a unique common fixed point in \( X \).

**Corollary 2.4.** If in (2.19) we put \( Bx = x \) for all \( x \in X \), then \( A : X \rightarrow X \) has a unique fixed point in \( (X, d) \).

### 3. Extended Approach to a Modular Metric

**Definition 3.1** ([2, 3]). Let \( \omega : (0, \infty) \times X \times X \rightarrow [0, \infty) \) be a function satisfying the following conditions for all \( \lambda, \mu > 0 \) and \( x, y, z \in X \)

(i) \( x = y \) iff \( \omega(\lambda, x, y) = 0 \) for all \( \lambda > 0 \).

(ii) \( \omega(\lambda, x, y) = \omega(\lambda, y, x) \)
Then \( \omega \) is called a modular metric on \( X \). If we replace (i) by
(iv) \( \omega(\lambda, x, x) = 0 \) for all \( \lambda > 0 \),
then \( \omega \) is called pseudo modular metric on \( X \). If we replace (iii) by
(v) \( \omega(\lambda, x, y) \leq \omega(\lambda, x, z) + \omega(\lambda, z, y) \) for all \( \lambda > 0 \) and \( x, y, z \in X \).
Then \( \omega \) is called non-Archimedean. Moreover \( \omega \) is called convex if the following
inequality is satisfied for all \( \lambda, \mu > 0 \) and \( x, y, z \in X \)
(vi) \( \omega(\lambda + \mu, x, z) \leq \frac{\lambda}{\lambda + \mu} \omega(\lambda, x, z) + \frac{\mu}{\lambda + \mu} \omega(\mu, z, y) \).

**Remark 3.2.** (i) A metric on a set \( X \) is a finite distance between any two points of \( X \) while a modular on a same set \( X \) is a way to consider a nonnegative “field of velocities” precisely an average velocity \( \omega(\lambda, x, y) \) is associated to each \( \lambda > 0 \); \( \omega(\lambda, x, y) \) that is one takes time \( \lambda \) to move from \( x \) to \( y \).
(ii) ([6]). Let \((X, M, \ast)\) be a triangular fuzzy metric space. Define a function \( \omega : (0, \infty) \times X \times X \to [0, \infty) \) as

\[
\omega(\lambda, x, y) = \frac{1}{M(x, y, \lambda)} - 1
\]

for all \( x, y \in X \) and \( \lambda > 0 \). Then \( \omega_\lambda \) is a modular metric on \( X \).

**Definition 3.3.** Let \( X_\omega \) be a modular metric space. Then
(i) \( \{x_n\} \) in \( X_\omega \) is called \( \omega \)-convergent to \( x \in X_\omega \), if \( \omega(\lambda, x_n, x) \to 0 \) as \( n \to \infty \) for all \( \lambda > 0 \). In this case we say \( x \) is the \( \omega \)-limit of \( \{x_n\} \).
(ii) \( \{x_n\} \) in \( X_\omega \) is called \( \omega \)-Cauchy if \( \omega(\lambda, x_n, x_m) \to 0 \) as \( m, m \to \infty \) for all \( \lambda > 0 \).
(iii) A subset \( Y \) of \( X_\omega \) is called \( \omega \)-complete if any \( \omega \)-Cauchy sequence in \( Y \) is a \( \omega \)-convergent sequence and it’s \( \omega \)-limit is in \( Y \).

Now we state two existence results for unique fixed point in the setting of modular space. Clearly these results are modular counterparts of Theorem 3.1 and Theorem 3.2.

**Theorem 3.4.** Let \( X_\omega \) be a non-Archimedean modular metric space and let \( A, B : X \to X \) be two given mappings. Let there exists \( \zeta \in Z \) such that

\[
\zeta(\omega(\lambda, Ax, Ay), \omega(\lambda, Bx, By)) \geq 0
\]

for all \( x, y \in X \) and for all \( \lambda > 0 \).
If \( AX \subseteq BX \) and \( AX \) or \( BX \) is a complete subset of \( X \). Then \( A \) and \( B \) have unique coincidence point in \( X \). Moreover if \( A \) and \( B \) are weakly compatible then \( A \) and \( B \) have a unique common fixed point in \( X \).

**Theorem 3.5.** Let \( X_\omega \) be a non-Archimedean modular metric space and let \( A, B : X \to X \) be two given mappings. Suppose there exists \( \zeta \in \mathbb{Z} \) and a function \( \phi : [0, \infty) \to [0, \infty) \) such that

\[
\zeta(\omega(\lambda, Ax, Ay), \phi(\omega(\lambda, Bx, By))) \geq 0
\]

for all \( x, y \in X \) and for all \( \lambda > 0 \).

\[
0 < \phi(t) \leq t \text{ for all } t \in (0, \infty) \text{ and } \phi(0) = 0
\]

If \( AX \subseteq BX \) and \( AX \) or \( BX \) is a complete subset of \( X \). Then \( A \) and \( B \) have unique coincidence point in \( X \). Moreover if \( A \) and \( B \) are weakly compatible then \( A \) and \( B \) have a unique common fixed point in \( X \).

The proof of Theorem 3.4 and Theorem 3.5 are established by applying Theorem 2.1 and Theorem 2.2. We give outline of the proof of Theorem 3.4.

**Proof.** Let \( M \) be a fuzzy metric induced by \( \omega \) and defined by (3.1). It follows that the triple \((X, M, \ast)\) is non-Archimedean fuzzy metric space. Then by (3.2) we have

\[
\zeta\left(\frac{1}{M(Ax, Ay, \lambda)} - 1, \frac{1}{M(Bx, By, \lambda)} - 1\right) \geq 0
\]

for all \( x, y \in X_\omega \) and for all \( \lambda > 0 \). Therefore, we apply Theorem 3.1 to conclude that \( A \) and \( B \) have a unique common fixed point in \( X \).

**References**

6. N. Hussain & P. Salimi: Implicit contractive mappings in modular metric and fuzzy
7. F. Khojasteh, S. Shukla & S. Radenovic: A new approach to the study of fixed point
11 (1975), 336-344.
9. A. Nastasi & P. Vetro: Fixed point results on metric and partial metric spaces via
10. A. Nastasi & P. Vetro: Existence and uniqueness for a first order periodic differential
problem via fixed point results. Results Mth. 71 (2017), 889-909.
Martinez-Moreno: Coincidence point theorems on metric spaces via simulation func-
13. S. Radenovic, F. Vetro & J. Vujakovic: An alternative and easy approach to fixed point
15. F. Tchier, C. Vetro & F. Vetro: Best approximation and variational inequality problems

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