REVISIT TO ALEXANDER MODULES OF 2-GENERATOR KNOTS IN THE 3-SPHERE

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ABSTRACT. It is known that a 2-generator knot $K$ has a cyclic Alexander module $\mathbb{Z}[t, t^{-1}]/(\Delta(t))$ where $\Delta(t)$ is the Alexander polynomial of $K$. In this paper we explicitly show how to reduce 2-generator Alexander modules to cyclic ones by using Chiswell, Glass and Wilson’s presentations of 2-generator knot groups

$$\langle x, y \mid (x^{\alpha_1})^{y^{\gamma_1}}, \ldots, (x^{\alpha_k})^{y^{\gamma_k}} \rangle$$

where $a^b = bab^{-1}$.

1. Introduction

A knot $K$ in the 3-sphere $S^3$ whose fundamental group is defined by a presentation with two generators (and hence one relator) is called a 2-generator knot. An arc $\tau$ embedded in $S^3$ so that $K \cap \tau = \partial \tau$ is called an unknotting tunnel of $K$ if the complement of a regular neighbourhood of $K \cup \tau$ in $S^3$ is $H_2$, a handlebody of genus 2. A knot with an unknotting is called a tunnel 1-knot. By attaching to $H_2$ a 2-handle corresponding to $\tau$, one would get the exterior of $K$, the complement of a regular neighbourhood of $K$ in $S^3$. Thus we see that a tunnel 1-knot is a 2-generator knot. The converse statement is one of intriguing conjectures in knot theory. Berge knots admitting lens space Dehn surgeries are well known examples of tunnel 1-knots. In particular, characterization of the Alexander polynomials of Berge knots seems somewhat intriguing subject. Recently Chiswell, Glass and Wilson [2] introduced a handy method of computing the Alexander polynomial of a 2-generator knot via its group presentation

$$\langle x, y \mid (x^{\alpha_1})^{y^{\gamma_1}}, \ldots, (x^{\alpha_k})^{y^{\gamma_k}} \rangle.$$

It is induced by a presentation admitting a generator with zero exponent sum [5, Chapter V, Lemma 11.8].

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Indeed via Nielsen transformations [6, Chapter 3] corresponding to mutual subtractions in the Euclidean algorithm any 2-generator presentation of a knot group can be brought into \( <x, y \mid w> \) so that \( w_y \) and \( w_x \), the sum of exponents of \( y \) and \( x \) in \( w \), are 0 and 1 respectively. Then it is easy to see that the relator \( w \) is cyclically conjugate to that introduced by Chiswell, Glass and Wilson.

Using such special presentations of 2-generator knots, we have:

**Theorem 1.1.** Any 2-generator knot in the 3-sphere has a cyclic Alexander module \( \mathbb{Z}[t, t^{-1}]/(\Delta(t)) \) where \( \Delta(t) \) denotes the Alexander polynomial of a knot.

Milnor [7, Footnote, p. 120] asserted that a 2-generator knot has a cyclic Alexander module. This follows easily from the fact that the Alexander module has deficiency 0. See [4, p. 14] for more details. Hence the Alexander module arising from a 2-generator 1-relator knot group via Fox differential calculus can be always reduced to a cyclic one. The method shown in this paper may be thought of as explicit reducing steps for the desired cyclic Alexander modules. We have in mind a practical application of explicit knowledge of the Alexander polynomial to homology of the cyclic branched covering [9].

A knot \( K \) in \( S^3 \) is said to be a (1,1)-knot if \( K \) is split into a pair of trivial arcs in solid tori determined by a Heegaard torus of \( S^3 \). All torus knots, and all 2-bridge knots are (1,1)-knots. The author [8] showed that any (1,1)-knot in \( S^3 \) admits a cyclic Alexander module by explicitly constructing the infinite cyclic covering space of its exterior.

Finally it is pointed out that in a Chiswell, Glass and Wilson’s presentation, tidiness of a relator word (for the definition see [2, p.2]) would not be necessary to get the desired Alexander polynomial because it is assumed to be in \( \mathbb{Z}[t, t^{-1}] \) instead of \( \mathbb{Z}[t] \).

### 2. Proof of the main theorem

**Lemma 2.1.** A 2-generator knot in \( S^3 \) admits a presentation \( <x, y \mid w> \) such that \( w_y = 0 \), and \( w_x = 1 \).

**Proof.** If necessary replacing a generator to its inverse, we assume that for a knot group presentation \( <a, b \mid r> \) both \( r_a \) and \( r_b \) are relative prime positive integers since the abelianized presentation of a knot group is isomorphic to \( \mathbb{Z} \). Define \( [r] \) to be the largest integer not greater than a real number \( r \). If \( r_a < r_b \), then replacing \( a \) by \( ab^{-[r_a]} \) (and hence \( a^{-1} \) by \( b^{[r_a]}a^{-1} \)) in \( r \), we end up with a presentation with a new pair of sums of exponents \( (r_a, r_b - r_a[r_a]) \).

Otherwise exchanging roles of \( a \) and \( b \), we end up with a presentation with a new pair of sums of exponents \( (r_a - r_b[r_b], r_b) \). Inductively executing Nielsen transformations corresponding to mutual subtractions, we eventually end up with \( <x, y \mid w> \) such that \( w_y = 0 \), and \( w_x = 1 \). \( \square \)

A presentation \( <x, y \mid w> \) of a knot group with \( w_y = 0 \) and \( w_x = 1 \) is said to be **normalized**.
Example 2.2. The fundamental group of a torus knot $t(5, 7)$ has a presentation $<x, y \mid x^5y^{-7}>$. Put $w_0 = x^5y^7$. Replacing $x$ by $xy^{-\frac{5}{7}} = xy^{-1}$ (and hence $x^{-1}$ by $yx^{-1}$) in $w_0$, we have

$$w_1 = xy^{-1}xy^{-1}xy^{-1}xy^{-1}xy^6$$

where $(w_1)_x = 5$, and $(w_1)_y = 2$. Replacing $y$ by $yx^{-\frac{5}{7}} = yx^{-2}$ in $w_1$, we have

$$w_2 = xy^{-1}x^3y^{-1}x^3y^{-1}xyx^{-2}yx^{-2}yx^{-2}yx^{-2}y$$

where $(w_2)_x = 1$, and $(w_2)_y = 2$. Finally replacing $x$ by $yx^{-2}$ in $w_2$, we have the desired normalized relator.

$$w(x, y) = yxy^{-3}xy^{-2}xy^{-3}xy^{-2}xy^{-3}x$$

Remark 2.3. A normal presentation of a 2-generator knot group is not unique. For a normalized presentation $<x, y \mid w(x, y) >$, we may get another normalized presentation $<x, y \mid w(x, yk)>$ for any integer $k \in \mathbb{Z}$.

Lemma 2.4. Assume that a presentation $<x, y \mid w = y^{\beta_1}x^{\alpha_1}, \ldots, y^{\beta_k}x^{\alpha_k}>$ is normalized so that $w_y = 0$. Then $w$ is cyclically conjugate to a word $(x_{\alpha_1}^\gamma y_1, \ldots, (x_{\alpha_k}^\gamma y_k)^{y_k}$. 

Proof. For each $1 \leq j \leq k$, take $\gamma_j = \sum_{i=1}^{j} \beta_i$. Then the last term $(x_{\alpha_k}^\gamma y_k)^{y_k}$ is always equal to $x_{\alpha_k}^\gamma$ since $w_y = 0$. 

Example 2.5. 

For $w$ in Example 2.2, we have the following product of conjugates :

$$x^y x^y x^{-2}x^y x^{-4}x^y x^{-6}x^y x^{-9}x^y x^{-11}x^y x^{-13}$$

$$x^y x^y x^{-16}x^y x^{-18}x^y x^{-20}x^y x^{-23}$$

$$(x^{-1})^y x^{-22}(x^{-1})^y x^{-20}(x^{-1})^y x^{-17}(x^{-1})^y x^{-15}(x^{-1})^y x^{-12}$$

$$(x^{-1})^y x^{-10}(x^{-1})^y x^{-7}(x^{-1})^y x^{-5}(x^{-1})^y x^{-2}x^{-1}$$

Let $X$ be a standard 2-complex associated with a presentation of $<x, y \mid w>$ of a knot group $G$ with a single 0-cell $v$, two 1-cells $x, y$ and one 2-cell $w$ such that $\pi_1(X, v) = G$. And let $\tilde{X}$ be an infinite cyclic covering space of $X$ such that $\pi_1(\tilde{X}, \tilde{v}) = G$, the commutator subgroup of $G$ where $\tilde{v}$ is 0-cell chosen in the 0-skeleton $\tilde{X}^0$ of $\tilde{X}$. Under action of the covering transformation group $G/G = < t^n | n \in \mathbb{Z} >$, $H_1(\tilde{X}) = G/G$ admits a $\mathbb{Z}[t, t^{-1}]$ module structure so called the Alexander module of a knot. For the canonical homomorphism $\phi : G = < x, y \mid w > \rightarrow G_{ab} \cong < t^n | n \in \mathbb{Z} > \cong G/G$. The linear extension to the group ring is also denoted by $\phi : \mathbb{Z}G \rightarrow \mathbb{Z}[t, t^{-1}]$, and $\phi(w) = w^\phi$ is denoted by
[w] for \( w \in \mathbb{Z}G \). Fox derivatives of \( w \in \mathbb{Z}G \) with respect to \( x, y \) are denoted by \( \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \) respectively.

Lemma 2.6 follows immediately from Fox differential calculus, lemma 2.4 and the fact that the canonical homomorphism \( \phi \) carries \( x, y \) to 1, \( t \) respectively.

**Lemma 2.6.** For a presentation \(< x, y \mid w = (x^{\alpha_1})y^{\gamma_1}, \ldots, (x^{\alpha_k})y^{\gamma_k} >\) with \( w_x = 1 \), we have:

1. \( \frac{\partial w}{\partial x} = \sum_{i=1}^{k} \alpha_i t^{\gamma_i} \), and
2. \( \frac{\partial w}{\partial y} = 0 \)

For any positive integer \( n \), a tamed embedding of the \( n \)-sphere \( S^n \) in the \( n+2 \)-sphere \( S^{n+2} \) is said to be \( n \)-knot. From lemma 2.6, we have:

**Corollary 2.7.** If a \( n \)-knot has a presentation

\(< x, y \mid w = (x^{\alpha_1})y^{\gamma_1}, \ldots, (x^{\alpha_k})y^{\gamma_k} >,\)

then it has the Alexander polynomial \( \Delta(t) = \sum_{i=1}^{k} \alpha_i t^{\gamma_i} \).

**Example 2.8.** The Alexander polynomial corresponding to the normal presentation in Example 2.5 is

\[
t + t^{-2} + t^{-4} + t^{-6} + t^{-9} + t^{-11} + t^{-13} + t^{-16} + t^{-18} + t^{-20} + t^{-23} \\
- t^{-22} - t^{-20} - t^{-17} - t^{-15} - t^{-12} - t^{-10} - t^{-7} - t^{-5} - t^{-2} - 1
\]

The following example is prepared to show that we may get the desired Alexander polynomial from a normalized presentation \(< x, y \mid w >\) without the tidy condition of \( w \) in \([2]\).

**Example 2.9.** Kanenobu and Sumi \([3, \text{Example 2.1}]\) showed that a ribbon 2-knot \( K^2 = R(1,2,-3,1) \) admits a knot group presentation

\(< x, y \mid x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}xy^2x^{-3}y >,\)

which is normalized to a presentation

\(< x, y \mid y^{-1}x^2y^{-1}x^{-1}yx^{-1}y^{-2}xyx^{-1}y^2xy^{-1}x > .\)

The relator word can be brought into the product of conjugates;

\((x^2)^{y^{-1}}(x^{-1})^{y^{-2}}(x^{-1})^y(x^{-1})(x)^{y^{-1}}(x^{-1})^{y^{-2}}(x^{-1})^y(x)^y x.\)
Finally we end up with the desired Alexander polynomial

\[ \Delta(t) = 2t^{-1} - t^{-2} - t^{-1} + t^{-2} - t^{-1} + t + 1 \]

From the homology long exact sequence of of a pair \((\tilde{X}, \tilde{X}_0)\), we have a short exact sequence

\[ 0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} \ker i_* \to 0 \]

where the boundary homomorphism \(\partial\) has the right inverse \(\sigma\), and hence the short exact sequence is split in such a way that \(H_1(\tilde{X}, \tilde{X}_0) \cong H_1(\tilde{X}) \oplus \mathbb{Z}[t, t^{-1}]\) where \(\mathbb{Z}[t, t^{-1}]\) stands for a free \(\mathbb{Z}[t, t^{-1}]\)-module of rank 1 generated by \((t-1)\tilde{v}\)

From \([1, \text{Proposition 9.2}]\) we have:

**Lemma 2.10.** Let \(< x, y | w >\) be a knot group presentation, \(\tilde{x}, \tilde{y}\) lifted 1-cells of \(x, y\) respectively, and \(\tilde{w}\) a lifted 2-cell of \(w\). Then \(H_1(\tilde{X}, \tilde{X}_0)\) admits a \(\mathbb{Z}[t, t^{-1}]\)-module presentation

\[ < \tilde{x}, \tilde{y} | \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} + [\frac{\partial w}{\partial y}]\tilde{y} > \]

where \(\partial \tilde{x} = ([x] - 1)\tilde{v}\), and \(\partial \tilde{y} = ([y] - 1)\tilde{v}\) for the connecting homomorphism \(\partial : H_1(\tilde{X}, \tilde{X}_0) \to \ker i_*\)

From lemma 2.10, we have:

**Proposition 2.11.** If a knot group presentation \(< x, y | w >\) is normalized, then \(H_1(\tilde{X})\) admits a \(\mathbb{Z}[t, t^{-1}]\)-module presentation

\[ < \tilde{x} | \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} > \cong \mathbb{Z}[t, t^{-1}]/(\Delta(t)) \]

**Proof.** Since \([\frac{\partial w}{\partial y}] = 0\), \(H_1(\tilde{X}, \tilde{X}_0)\) admits a \(\mathbb{Z}[t, t^{-1}]\)-module presentation

\[ < \tilde{x}, \tilde{y} | \tilde{w} = [\frac{\partial w}{\partial x}]\tilde{x} > . \]

Furthermore since \(\partial \tilde{y} = (t-1)\tilde{v}\), removing \(\tilde{y}\) corresponding to the free \(\mathbb{Z}[t, t^{-1}]\)-module generator from the presentation of \(H_1(\tilde{X}, \tilde{X}_0)\) we get the desired cyclic module presentation of \(H_1(\tilde{X})\).

\[ \square \]

Theorem 1.1 follows from Lemma 2.1 and Proposition 2.11.

**References**


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