A TWO-LEVEL FINITE ELEMENT METHOD FOR
THE STEADY-STATE NAVIER–STOKES/DARCY MODEL

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Abstract. A two-level finite element method based on the Newton iterative method is proposed for solving the Navier–Stokes/Darcy model. The algorithm solves a nonlinear system on a coarse mesh $H$ and two linearized problems of different loads on a fine mesh $h = O(H^{4-\epsilon})$. Compared with the common two-grid finite element methods for the considered problem, the presented two-level method allows for larger scaling between the coarse and fine meshes. Moreover, we prove the stability and convergence of the considered two-level method. Finally, we provide numerical experiment to exhibit the effectiveness of the presented method.

1. Introduction

The Navier–Stokes/Darcy model, coupled by certain transmission conditions at the interface, describes the coupling of incompressible fluid flow with porous media flow. This model plays an important role in many current industrial and technological applications, including hydrogeological mechanics, soil pollution simulation, biohydrodynamics, oil drilling and production engineering, industrial filtration and so on. Therefore, much effort has been devoted to the development of efficient numerical approaches for investigating this model. At the time of writing, some efficient numerical methods have been proposed [3,9,10,15,20,22,23]. In addition, Chidyagwai and Riviére [7] have used continuous finite elements in the incompressible flow region and discontinuous finite elements in the porous medium for solving the Navier–Stokes/Darcy model. A non-conforming finite volume element method has designed by Wu and Mei [29]. In [13], Girault and Riviére have proposed a numerical scheme based on discontinuous finite element methods and given the optimal error estimates. A discontinuous Galerkin finite element method for the discretization of this problem is applied by Hadji et al. [14]. Then the authors have developed a posteriori...
error analysis for the resulting discrete problem. Cao et al. [5] have proposed a domain decomposition method to improve the efficiency of the finite element method and applied Newton iteration to deal with the nonlinear systems. In particular, based on two-grid discretization, Du et al. [11,12] have constructed some local and parallel finite element algorithms for the considered problem. Besides, Chidyagwai [6] have designed a multilevel decoupling method for the Navier–Stokes/Darcy model and obtained the optimal error estimates.

When it comes to the two-grid method which is firstly introduced by Xu [30,31] and can save a large amount of CPU time, it is a significant method to deal with the nonlinear problem. Its basic idea is solving one nonlinear system on a coarse mesh as an iterative initial value approximation of a fine mesh and then solving one linear system on the fine mesh. By employing the two-grid strategy, a two-grid finite element method for the Navier–Stokes/Darcy model is given and the efficiency through numerical analysis and experiments is verified [4]. However, it has not theoretically achieved the optimal error estimates. The scaling between the coarse mesh size and fine mesh size is $h = O(H^2)$. Further, Qin and Hou [25] have proved the optimal error estimates for the velocity and the pressure in the fluid flow region, and improved the scaling between the coarse and fine mesh size from $h = O(H^2)$ to $h = O(H^3)$. In addition, according to the work in [4], Jia et al. [19] have proposed and analyzed a modified two-grid decoupling method for the mixed Navier–Stokes/Darcy model, where the scaling between the coarse and fine mesh size is $h = O(H^2)$. This scaling is also obtained in [26,33]. Moreover, based on the two-grid method and a recent local and parallel finite element method, a parallel two-grid linearized method for the coupled Navier–Stokes-Darcy problem is proposed and analyzed [34]. Similarly, it has the same order of accuracy as the standard finite element method when one takes $h = O(H^2)$. However, it is known that the two-level method is considered to be more effective for the case $h \ll H$. Hence, it is important to find an efficient algorithm to increase the ratio between the coarse and fine meshes of the two-level method.

Recently, Dai and Cheng [8] have shown a two-grid method for solving the Navier–Stokes equations based on Newton iteration. This method involves solving one small nonlinear system on a coarse mesh and two large linear problems on the fine mesh, which allows a much higher order scaling between the coarse grid size and fine grid size. Inspired by the idea of [8,16,17,28], a two-level method for the Navier–Stokes/Darcy model based on the Newton iteration is given in this article. This method consists of solving a small nonlinear problem on a coarse mesh and two large linearized problems of different loads on a fine mesh based on the Newton iteration.

The rest of the paper is arranged as follows: In the next section, we introduce some notations, function spaces and some significant results of the steady Navier–Stokes/Darcy model. In Section 3, a two-level finite element method
A TWO-LEVEL FEM FOR NS/DARCY MODEL 917

for the Navier–Stokes/Darcy model is presented. In Section 4, numerical experiment is implemented to verify the effectiveness of this presented method.

2. Notation and preliminaries

In this article, we consider the coupled fluid and the porous media flows on the domain $\Omega \subset \mathbb{R}^2$, which consists of two subdomains $\Omega_f$ and $\Omega_p$ separated by an interface $\Gamma$, i.e., $\overline{\Omega_f} \cap \overline{\Omega_p} = \emptyset$, $\Omega_f \cap \Omega_p = \emptyset$ and $\partial \Omega_f \cap \partial \Omega_p = \Gamma$. Here, we suppose $\Gamma$ is sufficiently smooth as in [4]. Besides, $n_f$ and $n_p$ represent the unit outward normal vectors on $\partial \Omega_f$ and $\partial \Omega_p$, respectively.

In $\Omega_f$, the fluid flow is governed by the stationary incompressible Navier–Stokes equations [27, 33]:

\[
\begin{align*}
\nabla \cdot T(u_f, p_f) + \rho_f (u_f \cdot \nabla) u_f &= f_1 & \text{in } \Omega_f, \\
\nabla \cdot u_f &= 0 & \text{in } \Omega_f,
\end{align*}
\]

where $u_f$ and $p_f$ denote the velocity and the kinetic pressure in $\Omega_f$, respectively. $\rho_f$ is the density of the fluid, $f_1$ is the external force and $T(u_f, p_f) = -p_f I + 2\nu D(u_f)$ is the stress tensor, where $\nu > 0$ represents the viscosity coefficient and $D(u_f) = \frac{1}{2}(\nabla u_f + (\nabla u_f)^T)$ is the deformation tensor.

In $\Omega_p$, the porous media flow is governed by the Darcy equation [24]:

\[
\begin{align*}
q &= -K \cdot \nabla \phi & \text{in } \Omega_p, \\
\nabla \cdot q &= f_2 & \text{in } \Omega_p,
\end{align*}
\]

where $\phi = z + \frac{p_p}{\rho_f g}$ means the piezometric head, $z$ is the elevation from a reference level, $p_p$ is the pressure in $\Omega_p$ and $g$ is the gravity acceleration. The discharge vector $q = \varepsilon u_p$, $\varepsilon$ is the volumetric porosity [21] and $u_p$ is the velocity in $\Omega_p$. In addition, $f_2$ is the source term with a solvability condition $\int_{\Omega_p} f_2 = 0$. $K$ is the hydraulic conductivity tensor of the porous medium. Here, we assume that $K$ is a symmetric positive define matrix uniformly bounded above and below, i.e., there exist $\lambda_{\min} > 0$, $\lambda_{\max} > 0$ such that

\[
a.e. \ x \in \Omega_p, \ \lambda_{\min} x \cdot x \leq K x \cdot x \leq \lambda_{\max} x \cdot x.
\]

Using Darcy’s law, (2) can be rewritten in the elliptic form:

\[
\begin{align*}
-\nabla \cdot (K \cdot \nabla \phi) &= f_2 & \text{in } \Omega_p.
\end{align*}
\]

For boundaries $\partial \Omega_f \setminus \Gamma$ and $\partial \Omega_p \setminus \Gamma$, we impose homogeneous Dirichlet boundary conditions, i.e.,

\[
\begin{align*}
u_f &= 0 & \text{on } \partial \Omega_f \setminus \Gamma, \\
\phi &= 0 & \text{on } \partial \Omega_p \setminus \Gamma.
\end{align*}
\]

About the interface $\Gamma$, we consider the following interface conditions as studied in [2, 18]:

\[
\begin{align*}
u_f \cdot n_f + u_p \cdot n_p &= 0, \\
\tau \cdot [-T(u_f, p_f) \cdot n_f] &= \frac{\alpha}{\sqrt{\tau \cdot \rho_f K \cdot \tau}} u_f \cdot \tau, \\
u_f \cdot [-T(u_f, p_f) \cdot n_f] &= \rho_f g \phi,
\end{align*}
\]
where $\alpha$ is a positive parameter depending on the properties of the porous medium that is experimentally determined and $\tau$ is an unit tangential vector on $\Gamma$. For brevity, we assume that $\varepsilon$ and $\rho_f$ are constants.

Denote $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$, where
\[
H_f = \{ v \in H^1(\Omega_f)^2 : v = 0 \text{ on } \partial \Omega_f \setminus \Gamma \},
\]
\[
H_p = \{ \phi \in H^1(\Omega_p) : \phi = 0 \text{ on } \partial \Omega_p \setminus \Gamma \}.
\]

We equip the space $L^2(\Lambda)$ ($\Lambda = \Omega_f$ or $\Omega_p$) with the usual $L^2$-scalar product $(\cdot, \cdot)$ and $L^2$-norm $\| \cdot \|_{L^2(\Lambda)}$. The space $W$ is equipped with the following norm:
\[
\| u \|_W = \sqrt{2\nu(D(u_f), D(u_f))_{\Omega_f} + K(\nabla \phi, \nabla \phi)_{\Omega_p}}.
\]

Set $f = (f_1, f_2)$, then the weak formulation of the steady Navier–Stokes/Darcy model as follows: Find $u = (u_f, \phi) \in W$, $p_f \in Q$ such that
\[
\begin{align*}
    a(u, v) + d(v, p_f) + b(u_f; u_f, v) &= (f, v) \quad \forall v = (v, \psi) \in W, \\
    d(u, q) &= 0 \quad \forall q \in Q,
\end{align*}
\]
where
\[
a(u, v) = a_{\Omega_f}(u_f, v) + a_{\Omega_p}(\phi, \psi) + a_T(u, v),
\]
\[
a_{\Omega_f}(u_f, v) = \varepsilon \int_{\Omega_f} 2\nu D(u_f) : D(v) + \varepsilon \int_{\Omega_f} \frac{\alpha}{\sqrt{\tau \cdot \nabla \phi}} (u_f \cdot \tau) \cdot (v \cdot \tau),
\]
\[
a_{\Omega_p}(\phi, \psi) = \rho_p g \int_{\Omega_p} K \cdot \nabla \phi \cdot \nabla \psi, \quad a_T(u, v) = \varepsilon \rho_p g \int_{\Gamma} (\phi v \cdot n_f - \psi u_f \cdot n_f),
\]
\[
b(u_f; u_f, v) = \varepsilon \rho_p \left( \int_{\Omega_f} (u_f \cdot \nabla) u_f \cdot v + \frac{1}{2} \int_{\Omega_f} (\nabla \cdot u_f) u_f \cdot v \right),
\]
\[
d(v, p_f) = -\varepsilon \int_{\Omega_f} p_f \nabla \cdot v, \quad (f, v) = \varepsilon \int_{\Omega_f} f_1 \cdot v + \rho_p g \int_{\Omega_p} f_2 \psi.
\]

In addition, for the trilinear form $b(\cdot, \cdot, \cdot)$, we list the following estimates [8]:
\[
\begin{align*}
    &|b(u; v, w)| \leq C_0 \| \nabla u \|_{L^2(\Omega_f)} \| \nabla v \|_{L^2(\Omega_f)} \| \nabla w \|_{L^2(\Omega_f)^2}, \\
    &|b(u; v, w)| \leq C_1 \| u \|_{L^2(\Omega_f)}^{\varepsilon} \| \nabla u \|_{L^2(\Omega_f)}^{\varepsilon} \| \nabla v \|_{L^2(\Omega_f)} \| \nabla w \|_{L^2(\Omega_f)^2},
\end{align*}
\]
where $C_0$ and $C_1$ denote the positive constants and $\varepsilon > 0$ is arbitrarily small.

We also recall the Poincaré and Korn’s inequalities, trace inequalities [32] and Sobolev inequalities that are useful in the analysis. There exist constants $C_2$, $C_3$, $C_4$, $C_5$, $C_6$, $C_7$ and $C_8$ that depend on $\Omega_f$ or $\Omega_p$ such that $\forall v \in H_f$, 

\[
\| v \|_{H^1(\Omega)} \leq C_2 \| v \|_{L^2(\Omega)}, \\
\| v \|_{H^1(\Omega)} \leq C_3 \| v \|_{L^2(\Omega)}^{\varepsilon} \| \nabla v \|_{L^2(\Omega)}^{\varepsilon},
\]

∀ϕ ∈ H_p, we have the following bounds [33]
\[ \|v\|_{L^2(Ω_f)^2} \leq C_2 \|\nabla v\|_{L^2(Ω_f)^2}, \quad \|ϕ\|_{L^2(Ω_p)} \leq C_3 \|\nabla ϕ\|_{L^2(Ω_p)}, \]
\[ \|\nabla v\|_{L^2(Ω_f)^2} \leq C_4 \|D(v)\|_{L^2(Ω_f)^2}, \quad C_5 \|K^{1/2}(v)\|_{L^2(Ω_p)}^2 \leq \alpha(ϕ, ϕ), \]
\[ \|ϕ\|_{L^2(Ω_p)} \leq C_6 \|K^{1/2}(v)\|_{L^2(Ω_p)}, \quad \|v\|_{L^2(Ω_f)^2} \leq C_7 \|\nabla v\|_{L^2(Ω_f)^2}, \]
\[ \|ϕ\|_{L^2(Γ)} \leq C_8 \|\nabla ϕ\|_{L^2(Ω_p)}. \]

Given (3), we obtain
\[ \frac{1}{\sqrt{λ_{\text{max}}}} \|K^{1/2}(v)\|_{L^2(Ω_p)} \leq \|ϕ\|_{L^2(Ω_p)} \leq \frac{1}{\sqrt{λ_{\text{min}}}} \|K^{1/2}(v)\|_{L^2(Ω_p)}. \]

The following well-posedness for the coupled Navier–Nastokes/Darcy model (7) is classical.

**Theorem 2.1** ([1, 33]). Assume that the data satisfies:
\[ \frac{C_2 C_3^2}{4ν^2} \|f\|^2_{L^2(Ω_f)^2} + \frac{ρfC_2}{2νλ_{\text{min}}} \|f\|^2_{L^2(Ω_f)} \leq \frac{4ε^2ν^3}{C_0 C_4}. \]
Then the problem (7) has at most one weak solution satisfying
\[ \|D(u_f)\|^2_{L^2(Ω_f)^2} \leq \frac{C_2 C_3^2}{4ν^2} \|f\|^2_{L^2(Ω_f)^2} + \frac{ρfC_2}{2νλ_{\text{min}}} \|f\|^2_{L^2(Ω_f)} \]

We partition Ω_f and Ω_p by quasi-uniform triangulations K_{f, μ} and K_{p, μ} with a positive parameter μ (μ = h or H with h ≪ H). For given K_{p, μ} and K_{f, μ}, we consider the following finite element spaces W_μ = H_{f, μ} × H_{p, μ} ⊂ W and Q_μ ⊂ Q:
\[ H_{f, μ} = (P^0_1)_μ^d \cap H_f, \quad H_{p, μ} = \{ψ_μ ∈ C^0(Ω_p) : ψ_μ|_K ∈ P_1(K), ∀K ∈ K_{p, μ}\}, \]
\[ Q_μ = \{q_μ ∈ C^0(Ω_f) : q_μ|_K ∈ P_1(K), ∀K ∈ K_{f, μ}\}. \]

where
\[ P^0_1 = \{v_μ ∈ C^0(Ω_f) : v_μ|_K ∈ P_1(K) \oplus \text{span}\{b\}, ∀K ∈ K_{f, μ}\}, \]
\[ b \] is a bubble function, and P_1(K) is a space of linear polynomials on element K. Furthermore, we need the subspace H_{f, μ} of H_{f, μ} which is defined as
\[ H_{f, μ} = \{v_μ ∈ H_{f, μ} : (∇ · v_μ, q_μ) = 0, ∀q_μ ∈ Q_μ\}. \]

Note that the inf-sup condition holds, i.e., there is a positive constant β independent of μ such that
\[ d(v_μ, q_μ) ≥ β\|v_μ\|_{W_μ} \|q_μ\|_{Q_μ}, \quad ∀v_μ ∈ W_μ, q_μ ∈ Q_μ. \]

Further, the finite element scheme of (7) is defined as the following coupled system: Find u_μ = (u_{f, μ}, ϕ_μ) ∈ W_μ, and p_{f, μ} ∈ Q_μ, such that
\[ \begin{cases} \alpha(u_μ, v_μ) + d(v_μ, p_{f, μ}) + b(u_{f, μ}, v_μ) + (f, v_μ) & ∀v_μ ∈ W_μ, q_μ ∈ Q_μ, \\ d(u_μ, q_μ) = 0 & ∀q_μ ∈ Q_μ. \end{cases} \]
The following theorems establish the stability and error estimate results for the finite element discretization (12) of the considered problem.

**Theorem 2.2** ([4, 7, 33]). Let

\[ R = \left( \max \left\{ \frac{1}{\epsilon}, \frac{1}{C_0} \right\} \right)^{1/2} \left( \frac{\varepsilon C_2^2 C_3^2}{\nu} ||f||^2_{L^2(\Omega_f)} + \frac{2\nu_0^2 \nu C_5^2}{C_5^2} \|f_f\|^2_{L^2(\Omega_f)} \right)^{1/2}. \]

Then, under the assumption \( R^2 \leq \frac{8\varepsilon^2 C_2^2}{C_3^2 C_5^2} \), the problem (12) admits a unique solution satisfying

\[ 2\nu\|D(u_{f,h})\|^2_{L^2(\Omega_f)} + \|K^{1/2}\nabla \phi\|^2_{L^2(\Omega_h)} \leq R^2. \]

Besides, let \( (u_f, \phi_f, p_f) \in H^2(\Omega_f)^2 \times H^1(\Omega_f) \times H^1(\Omega_f) \) be the solution of (7), we have the following error estimate

\[ \|D(u_f - u_{f,h})\|^2_{L^2(\Omega_f)} + \|\nabla (\phi - \phi_h)\|^2_{L^2(\Omega)} + \|p_f - p_{f,h}\|^2_{L^2(\Omega_f)} \leq c \mu (\|u_f\|_{L^2(\Omega_f)} + \|\phi\|_{H^2(\Omega)} + \|p_f\|_{H^1(\Omega_f)}). \]

**Theorem 2.3** ([4]). Under the assumption of Theorem 2.2, let \( (u_{f,h}, p_{f,h}, h) \) be the finite element solution of (12) \( (\mu = h) \), we have the \( L^2 \)-error estimate

\[ \|u_f - u_{f,h}\|^2_{L^2(\Omega_f)} + \|\phi - \phi_h\|^2_{L^2(\Omega)} \leq c h^2 (\|u_f\|^2_{L^2(\Omega_f)} + \|\phi\|^2_{H^2(\Omega)} + \|p_f\|^2_{H^1(\Omega_f)}) \]

and for \( h \ll H \), we then have

\[ \|u_{f,h} - u_{f,H}\|^2_{H^1(\Omega_f)} \leq c H (\|u_f\|^2_{L^2(\Omega_f)} + \|\phi\|^2_{H^2(\Omega)} + \|p_f\|^2_{H^1(\Omega_f)}). \]

3. A two-level finite element method

In this section, we show an effective two-level finite element method for the Navier-Stokes/Darcy model. The algorithm is shown as follows:

**Algorithm 3.1.**

**Step I.** Solve the nonlinear problem on a coarse grid: Find \( u_H = (u_{f,H}, \phi H) \in W_H, p_{f,H} \in Q_H \) such that for all \( v = (v, \psi) \in W_H, q \in Q_H \),

\[ \begin{cases} a(u_H, v) + d(v, p_{f,H}) + b(u_f, u_f; u_{f,H}, v) = (f, v), \\ d(u_H, q) = 0. \end{cases} \]

**Step II.** Solve a linear problem on a fine grid based on Newton iteration: Find \( u_h = (u_{f,h}, \phi_h) \in W_h, p_{f,h} \in Q_h \) such that for all \( v = (v, \psi) \in W_h, q \in Q_h \),

\[ \begin{cases} a(u_{h}, v) + d(v, p_{f,h}) + b(u_f, u_f; u_{f,h}, v) + b(u_f, u_f; u_{f,h}, v) \\ = (f, v) + b(u_f, u_f; u_{f,h}, v), \\ d(u_{h}, q) = 0. \end{cases} \]
Step III. Update on the same fine mesh: Find $u^h = (u_h^f, \phi_h^s) \in W_h, p_h^f \in Q_h$ such that for all $v = (v, \psi), q \in Q_h$,

$$
\begin{cases}
(a(u^h, v) + d(v, p_h^f) + b(u_{f,H}; u_f^h; v) + b(u_f^h; u_f,H; v) = (f, v) + b(u_{f,H}; u^*_{f,h}, v) + b(u^*_{f,h}; u_{f,H} - u^*_{f,h}, v), \\
d(u^h, q) = 0.
\end{cases}
$$

(19)

Next, we derive stability and error estimates of the presented method for the Navier–Stokes/Darcy equations.

**Theorem 3.1.** Suppose $0 < \delta < 1$ with $\delta = 1 - \frac{C_6 R C_1^2}{\sqrt{2} \nu^{3/2} \varepsilon}$. Under the assumption of Theorem 2.2, $(u_{f,h}^*, \phi_{h}^s)$ defined by Step II of Algorithm 3.1 satisfies

$$
2\nu \|D(u_{f,h}^*)\|_{L^2(\Omega_f)^2} + \sigma^{-1} \|K^{1/2}\nabla \phi_{h}^s\|_{L^2(\Omega_p)} \leq \sigma^{-1} R_2.
$$

Moreover, $(u_f^*, \phi^s)$ defined by Step III of Algorithm 3.1 satisfies

$$
2\nu \|D(u_f^*)\|_{L^2(\Omega_f)^2} + \sigma^{-1} \|K^{1/2}\nabla \phi^s\|_{L^2(\Omega_p)} \leq \sigma^{-1} R_2,
$$

where

$$
\sigma = \frac{\varepsilon}{C_5}, \quad R_1 = \frac{\varepsilon C^2 C_2}{\nu C_5^2} \|f\|_{L^2(\Omega_f)}^2 + \frac{\rho_1^2 g^2 C_3^2}{\lambda_{min} C_5^2} \|f\|_{L^2(\Omega_p)}^2 + \frac{C_6^2 R^4 C_5^6}{4 \varepsilon \nu^3 C_5},
$$

and

$$
R_2^2 = \frac{3\varepsilon C^2 C_2^2}{2\nu C_5^2} \|f\|_{L^2(\Omega_f)}^2 + \frac{\rho_1^2 g^2 C_3^2}{\lambda_{min} C_5^2} \|f\|_{L^2(\Omega_p)}^2 + \frac{3C_6^2 R^2 C_5^4}{8\varepsilon \nu^3 C_5} + \frac{3C_6^2 R^4 C_5^6}{8\varepsilon \nu^3 C_5}.
$$

**Proof.** Firstly, we consider the stability of $(u_{f,h}^*, \phi_{h}^s)$. Taking $(v, q) = (u_{f,h}^*, p_{f,h}^h)$, i.e., $v = (v, \psi) = (u_{f,h}^*, \phi_{h}^s) = u_{f,h}^*$ and $q = p_{f,h}^h$ in (18), we have

$$
\begin{cases}
(a(u_{h}^*, u_{h}^*) + d(u_{h}^*, p_{f,h}^h) + b(u_{f,H}; u_f^h, u_{f,h}^*) + b(u_f^h; u_{f,H}; u_f^h) = (f, u_{h}^*) + b(u_{f,H}; u_{f,h}^*, u_{f,h}^*) , \\
d(u_{h}^*, p_{f,h}^h) = 0.
\end{cases}
$$

(22)

Noting that $a_{f}(u_{h}^*, u_{h}^*) = 0$ and $b(u_{f,H}; u_f^h, u_{f,h}^*) = 0$ yields

$$
\begin{align*}
a_{f}(u_{f,h}^*, u_{f,h}^*) &= \alpha_{f}(u_{f,h}^*, u_{f,h}^*) + \alpha_{f}(\phi_{h}^s, \phi_{h}^s) \\
&= (f, u_{h}^*) + b(u_{f,H}; u_{f,h}^*, u_{f,h}^*) - b(u_{f,h}^*, u_{f,H}; u_{f,h}^*).
\end{align*}
$$

(23)

By using (9) and the triangle inequality, we get

$$
2\nu \varepsilon \|D(u_{f,h}^*)\|_{L^2(\Omega_f)^2} + C_5 \|K^{1/2}\nabla \phi_{h}^s\|_{L^2(\Omega_p)}^2 \\
\leq \varepsilon \|(f, u_{f,h}^*)_{\Omega_f} + \rho_1 g(\|f\|_{L^2(\Omega_f)}^2 + \|f\|_{L^2(\Omega_p)}^2) + \|b(u_{f,H}; u_{f,h}^*, u_{f,h}^*)\| + \|b(u_{f,h}^*; u_{f,H}; u_{f,h}^*)\|).
$$

Then, applying the Hölder, Poincaré, Young inequalities and Theorem 2.2, it follows that

$$
\frac{2\nu \varepsilon}{C_5} \|D(u_{f,h}^*)\|_{L^2(\Omega_f)^2} + \|K^{1/2}\nabla \phi_{h}^s\|_{L^2(\Omega_p)}^2
$$
\[
\frac{2}{C_5} \|D(u_{f,h})\|^2_{L^2(\Omega_2)^2} \leq \frac{\varepsilon C_5^2 C_2^2}{C_5} \|f_1\|^2_{L^2(\Omega_2)^2} + \frac{\rho_2^2 C_3^2}{2 \lambda_{\min} C_2^2} \|f_2\|^2_{L^2(\Omega_2)^2} + \frac{\rho_2^2 C_3^2}{2 \lambda_{\min} C_2^2} \|D(u_{f,H})\|^2_{L^2(\Omega_2)^2} + \frac{\nu \varepsilon}{2 C_5} \|\nabla \phi_h\|^2_{L^2(\Omega_2)} + \frac{\nu \varepsilon}{2 C_5} \|\nabla \phi_h\|^2_{L^2(\Omega_2)}.
\]

Let \( \delta = 1 - \frac{C_4 R C_4^2}{\sqrt{2} \varepsilon \nu}, \) and assume \( 0 < \delta < 1. \) Then we have

\[
\frac{\varepsilon}{C_5} \delta^2 \nu \|D(u_{f,h})\|^2_{L^2(\Omega_2)^2} \leq \frac{\nu \varepsilon}{2 C_5} \|\nabla \phi_h\|^2_{L^2(\Omega_2)} + \frac{\rho_2^2 C_3^2}{2 \lambda_{\min} C_2^2} \|f_2\|^2_{L^2(\Omega_2)^2} + \frac{C_4 R C_4^2}{4 \varepsilon \nu C_5}. \]

Further, set \( \sigma = \frac{C_4 R C_4^2}{\sqrt{2} \varepsilon \nu}, \) we arrive at

\[
2 \nu \|D(u_{f,h})\|^2_{L^2(\Omega_2)^2} \leq \sigma^{-1} \left( \frac{\varepsilon C_5^2 C_2^2}{\nu C_5} \|f_1\|^2_{L^2(\Omega_2)^2} + \frac{\rho_2^2 C_3^2}{\lambda_{\min} C_2^2} \|f_2\|^2_{L^2(\Omega_2)^2} + \frac{C_4 R C_4^2}{4 \varepsilon \nu C_5} \right). \]

Next, taking \((v, q) = (u^h, p^f_j), \) i.e., \((v, \psi) = (u^h_p, \phi^h)\) and \(q = p^f_j\) in (19), we know that

\[
\begin{aligned}
&\begin{cases}
a(h^h, u^h) + d(u^h, p^f_j) + b(u_{f,H}; u^h, u^h) + b(u^h; u_{f,H}, u^h) \\
= (f, u^h) + b(u_{f,H}; u^h, u^h) + b(u^h; u_{f,H}, u^h),
\end{cases}
\end{aligned}
\]
By \(a_F(u^h, u^h) = 0\) and \(b(u_{f,H}; u_{f}^h, u_{f}^h) = 0\), we obtain
\[
\begin{align*}
& a_{\lambda f}(u_{f}^h, u_{f}^h) + a_{\lambda r}(\phi^h, \phi^h) \\
& = (f, u^h) + b(u_{f,H}; u_{f,h}, u_{f}^h) + b(u_{f,H} - u_{f,h}, u_{f}^h) - b(u_{f}^h, u_{f,H}, u_{f}^h)
\end{align*}
\]
(28)

Analogously,
\[
\begin{align*}
2\nu\varepsilon\|D(u_{f}^h)\|_{L^2(\Omega_f)}^2 &+ \|K^{1/2}\nabla \phi^h\|_{L^2(\Omega_p)}^2 \\
&\leq \varepsilon|\langle f_1, u_{f}^h \rangle_{\Omega_f}| + \rho f g (f_2, \phi^h)_{\Omega_p} + |b(u_{f,H}; u_{f,h}, u_{f}^h)| \\
&+ |b(u_{f,H} - u_{f,h}, u_{f}^h)| + |b(u_{f}^h, u_{f,H}, u_{f}^h)|
\end{align*}
\]
(29)

Hence,
\[
\begin{align*}
\frac{2\nu\varepsilon}{C_5} &\|D(u_{f}^h)\|_{L^2(\Omega_f)}^2 + \|K^{1/2}\nabla \phi^h\|_{L^2(\Omega_p)}^2 \\
&\leq \varepsilon |\langle f_1, u_{f}^h \rangle_{\Omega_f}| + \frac{\rho f g}{C_5} (f_2, \phi^h)_{\Omega_p} \\
&+ \frac{1}{C_5} |b(u_{f,H}; u_{f,h}, u_{f}^h)| + \frac{1}{C_5} |b(u_{f,H} - u_{f,h}, u_{f}^h)| + \frac{1}{C_5} |b(u_{f}^h, u_{f,H}, u_{f}^h)|
\end{align*}
\]
(30)
By simplifying, we can get
\[
\frac{\varepsilon}{C_5} (1 - \frac{C_0 R C_4^2}{\sqrt{2} \nu^3 \varepsilon^2}) 2 \nu \| D(u_f^*) \|_{L^2(\Omega_f)^2} + \| K^{1/2} \nabla \phi^h \|_{L^2(\Omega_f)}^2
\]
(31)
\[
\leq \frac{3 \nu^3 C_4^2}{2 \nu C_5} \| \xi \|_{L^2(\Omega_f)^1}^2 + \frac{\mu^2 \nu C_4^2}{\lambda_{\min} C_5^2} \| f_2 \|_{L^2(\Omega_f)}^2 + \frac{3 \nu^3 R^2 R_5^2 C_6^2}{2 \varepsilon_8 \nu^3 C_5^2} + \frac{3 \nu^3 R_4^2 C_6^6}{8 \varepsilon_8 \nu^3 C_5^2}.
\]

Set \( \sigma = \frac{\varepsilon}{C_5} \delta = \frac{\varepsilon}{C_5} \left(1 - \frac{C_0 R C_4^2}{\sqrt{2} \nu^3 \varepsilon^2}\right) \) and use \( 0 < \delta < 1 \) to yield
\[
2 \nu \| D(u_f^*) \|_{L^2(\Omega_f)^2}^2 + \sigma^{-1} \| K^{1/2} \nabla \phi^h \|_{L^2(\Omega_f)}^2
\]
\[
\leq \sigma^{-1} \left( \frac{3 \nu^3 C_4^2}{2 \nu C_5} \| \xi \|_{L^2(\Omega_f)^1}^2 + \frac{\mu^2 \nu C_4^2}{\lambda_{\min} C_5^2} \| f_2 \|_{L^2(\Omega_f)}^2 + \frac{3 \nu^3 R^2 R_5^2 C_6^2}{2 \varepsilon_8 \nu^3 C_5^2} + \frac{3 \nu^3 R_4^2 C_6^6}{8 \varepsilon_8 \nu^3 C_5^2} \right).
\]

**Theorem 3.2.** Let \( (u_f, \phi, p_f) \in H^2(\Omega_f)^2 \times H^2(\Omega_p) \times H^1(\Omega_f) \) be the solution of (7). Then, under the assumption of Theorem 3.1, we have the following estimate:
\[
\| D(u_f - u_f^*) \|_{L^2(\Omega_f)^2} + \| \nabla (\phi - \phi^h) \|_{L^2(\Omega_f)} + \| p_f - p_f^h \|_{L^2(\Omega_f)}
\]
\[
\leq C(\eta + H^{4-\nu}) (\| u_f \|_{H^2(\Omega_f)^2} + \| \phi \|_{H^2(\Omega_f)} + \| p_f \|_{H^1(\Omega_f)}),
\]
where \( \varepsilon \) is arbitrarily small.

**Proof.** Firstly, we consider the error of Step II of Algorithm 3.1. Subtracting (18) from (7), we obtain
\[
\begin{aligned}
& a_{\Omega_f}(u_f - u_f^*, v) + a_{\Omega_p}(\phi - \phi^h, \psi) + a_T(u - u_h^*, v) + d(v, p_f - p_f^h)
\end{aligned}
\]
\[
+ b(u_f; u_f, v) + b(u_f; u_f; u_f, v) - b(u_f; u_f^*; u_f, v) - b(u_f^*; u_f, v) = 0,
\]
\[
d(u - u_h^*, q) = 0.
\]

For the trilinear terms, it is easy to verify that
\[
b(u_f; u_f, v) + b(u_f; u_f; u_f, v) - b(u_f; u_f^*; u_f, v) - b(u_f^*; u_f, v)
\]
\[
= b(u_f; u_f; u_f - u_f^*, v) + b(u_f - u_f^*, u_f; v)
\]
\[
- b(u_f - u_f^*, u_f; v).
\]

Let us denote
\[
\begin{align*}
\zeta_1 - \chi_1 &= (u_f - \Pi_h u_f) - (u_f^* - \Pi_h u_f), \\
\eta_1 - \xi_1 &= (\phi - \Pi_h \phi) - (\phi^* - \Pi_h \phi), \\
\eta_2 - \xi_2 &= (p_f - \Pi_h p_f) - (p_f^* - \Pi_h p_f),
\end{align*}
\]
where \( \Pi_h s \) denotes interpolation of \( s \) in its finite element space, and \( s = u_f, \phi \) and \( p_f \).
Then we choose \( v = \chi_1 \in H_{f, h, 0} \), \( \psi = \xi_1 \in H_{f, h} \) and \( q = \xi_2 \in Q_h \) in (33) and combine (34) to yield
\[
\begin{align*}
\alpha_{\Omega_1}(\chi_1, \chi_1) + \alpha_{\Omega_2}(\xi_1, \xi_1) + b(\chi_1; u_{f, h}; \chi_1) \\
= \alpha_{\Omega_1}(\xi_1, \chi_1) + \alpha_{\Omega_2}(\xi_1, \xi_1) + a_f(u - u_h^*, u_h^* - \Pi_h u) \\
+ d(\xi_1, p_f - \Pi_h p_f) + b(u_{f, h}; \xi_1, \chi_1) \\
+ b(u_{f, h}, \chi_1) - b(u_f - u_{f, h}; u_{f, h} - u_f, \chi_1) =: \sum_{i=1}^{7} R_i.
\end{align*}
\] (35)

Using (9), the left-hand side of (35) is bounded from below by
\[
(6) \quad l.h.s \geq \nu \varepsilon \left( 2 - \frac{C_0 R C_1^2}{\sqrt{2} \nu \varepsilon^{3/2}} \right) \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 + \nu \varepsilon \| K^{1/2} \nabla \xi_1 \|_{L^2(\Omega_h)}^2.
\]

Next, applying Poincaré, Young inequalities, Theorem 2.2 and Theorem 2.3, we estimate the terms of the right-hand side of (35) as follows:

\[ |R_1| \leq 2 \nu \varepsilon \| D(\xi_1) \|_{L^2(\Omega_1)^2} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 + C \| \xi_1 \|_{L^2(\Gamma)^2} \| \chi_1 \|_{L^2(\Gamma)^2}^2 \]
\[
\leq 6 \nu \varepsilon \| D(\xi_1) \|_{L^2(\Omega_1)^2}^2 + \frac{1}{6} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 \\
+ C C^2_1^2 C_1^4 \| D(\xi_1) \|_{L^2(\Omega_1)^2} \| D(\chi_1) \|_{L^2(\Omega_1)^2} \\
\leq 6 \nu \varepsilon \| D(\xi_1) \|_{L^2(\Omega_1)^2}^2 + \frac{3 C^2 C_1^4 C_1^4}{2 \nu \varepsilon} \| D(\xi_1) \|_{L^2(\Omega_1)^2}^2 + \frac{\nu \varepsilon}{3} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 \\
\leq C h^2 \| u_f \|_{H^2(\Omega_h)}^2 + \| \phi \|_{H^2(\Omega_h)} + \| p_f \|_{H^1(\Omega_1)}^2 + \frac{\nu \varepsilon}{3} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2,
\]

\[ |R_2| \leq \frac{\nu \varepsilon}{2} \| K^{1/2} \nabla \xi_1 \|_{L^2(\Omega_h)}^2 \]

\[ \leq \frac{\nu \varepsilon}{2} \| K^{1/2} \nabla \xi_1 \|_{L^2(\Omega_h)}^2 + \frac{\nu \varepsilon}{2} \| K^{1/2} \nabla \xi_1 \|_{L^2(\Omega_h)}^2 \\
\leq C h^2 \| u_f \|_{H^2(\Omega_h)}^2 + \| \phi \|_{H^2(\Omega_h)} + \| p_f \|_{H^1(\Omega_1)}^2 + \frac{\nu \varepsilon}{2} \| K^{1/2} \nabla \xi_1 \|_{L^2(\Omega_h)}^2.
\]

\[ |R_5 + R_6| \leq 2 C_0 C_1^3 \| D(u_f, h) \|_{L^2(\Omega_1)^2} \| D(\xi_1) \|_{L^2(\Omega_1)^2} \| D(\chi_1) \|_{L^2(\Omega_1)^2} \\
\leq 6 C_0 C_1^3 \| D(u_f, h) \|_{L^2(\Omega_1)^2}^2 \| D(\xi_1) \|_{L^2(\Omega_1)^2}^2 + \frac{\nu \varepsilon}{6} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 \\
\leq C h^2 \| u_f \|_{H^2(\Omega_h)}^2 + \| \phi \|_{H^2(\Omega_h)} + \| p_f \|_{H^1(\Omega_1)}^2 + \frac{\nu \varepsilon}{6} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2,
\]

\[ |R_7| \leq C_1 \| u_f - u_{f, h} \|_{L^2(\Omega_1)^2} \| \nabla (u_f, h - u_f) \|_{L^2(\Omega_1)^2}^2 \| \nabla \chi_1 \|_{L^2(\Omega_1)^2} \\
\leq C_1 C_1^2 \| u_f - u_{f, h} \|_{L^2(\Omega_1)^2} \| \nabla (u_f, h - u_f) \|_{L^2(\Omega_1)^2}^2 \| \nabla \chi_1 \|_{L^2(\Omega_1)^2} \\
\leq \frac{3 C_1^2 C_1^4}{2 \nu \varepsilon} \| u_f - u_{f, h} \|_{L^2(\Omega_1)^2}^4 \| D(u_f, h - u_f) \|_{L^2(\Omega_1)^2}^4 + \frac{\nu \varepsilon}{6} \| D(\chi_1) \|_{L^2(\Omega_1)^2}^2 \\
\leq C [H^{3-\varepsilon} \| u_f \|_{H^2(\Omega_1)^2}^2 + \| \phi \|_{H^2(\Omega_h)} + \| p_f \|_{H^1(\Omega_1)}^2] \]
\[ + \frac{\nu \varepsilon}{6} \|D(\chi_1)\|_{L^2(\Omega_f)}^2. \]

Similarly, we have
\[
|R_4| \leq |d(\chi_1, \eta_2)| \leq \frac{\nu \varepsilon}{6} \|D(\chi_1)\|_{L^2(\Omega_f)}^2 + \frac{3C_4^2}{2\nu \varepsilon} \|\eta_2\|_{L^2(\Omega_f)}^2 \\
\leq C h^2 (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}^2) \\
+ \frac{\nu \varepsilon}{6} \|D(\chi_1)\|_{L^2(\Omega_f)}^2.
\]

Besides, for \(R_3\), we obtain
\[
|R_3| = \left| \rho_f g \int_{\Gamma} [\eta_1 \chi_1 \cdot n_f - \xi_1 \zeta_1 \cdot n_f] \right| \\
\leq \rho_f g \|\eta_1\|_{L^2(\Gamma)} \|\chi_1\|_{L^2(\Gamma)} \|n_f\|_{L^2(\Gamma)} + \rho_f g \|\xi_1\|_{L^2(\Gamma)} \|\zeta_1\|_{L^2(\Gamma)} \\
\leq \rho_f g C_4 C_7 C_9 (\|\nabla \eta_1\|_{L^2(\Omega_p)} \|D(\chi_1)\|_{L^2(\Omega_f)} + \|D\xi_1\|_{L^2(\Omega_f)} \|\nabla \xi_1\|_{L^2(\Omega_p)}) \\
\leq \frac{\nu \varepsilon}{6} \|D(\chi_1)\|_{L^2(\Omega_f)}^2 + \frac{3\rho_f g C_4 C_7 C_9}{\lambda_{\min}} \|\nabla \eta_1\|_{L^2(\Omega_p)}^2 + \frac{\lambda_{\min} \rho_f g}{4} \|\nabla \xi_1\|_{L^2(\Omega_p)}^2 \\
+ \frac{\rho_f g}{4} \|K^{1/2} \nabla \xi_1\|_{L^2(\Omega_p)}^2.
\]

Further, by using the results of the above estimates, the right-hand side of (35) is bounded by
\[
r.h.s \leq C h^2 (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}^2 + \nu \varepsilon \|D(\chi_1)\|_{L^2(\Omega_f)}^2 \\
+ C [h^{3-\epsilon} (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}^2)]^2 \\
+ \frac{3\rho_f g}{4} \|K^{1/2} \nabla \xi_1\|_{L^2(\Omega_p)}^2.
\]

Then, combining the above two inequalities (36) and (38), we get
\[
\nu \varepsilon \delta \|D(\chi_1)\|_{L^2(\Omega_f)}^2 + \frac{\rho_f g}{4} \|K^{1/2} \nabla \xi_1\|_{L^2(\Omega_p)}^2 \\
\leq \frac{C \rho_f}{4} (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}^2) \\
+ C [h^{3-\epsilon} (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}^2)]^2,
\]

where \(\delta = 1 - \frac{C_5 R C_3}{\sqrt{2\pi \varepsilon \varepsilon}}\). In addition, we have
\[
\|D(u_f - u_f^h)\|_{L^2(\Omega_f)}^2 + \|\nabla (\phi - \phi^h)\|_{L^2(\Omega_p)}^2 \\
\leq C (h + H^{3-\epsilon}) (\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_p)} + \|p_f\|_{H^1(\Omega_f)}).
\]
Secondly, we give the error of Step III of Algorithm 3.1. Subtracting (19) from (7), we have

\[
\begin{aligned}
&\left\{ \begin{array}{l}
a_{\Omega}(u_f - u_f^h, v) + a_{\Omega}(\phi - \phi^h, \psi) + a_{\Omega}(u - u^h, v) + d(v, p_f - p_f^h) \\
+b(u_f; u_f, v) - b(u_{f,H}; u_f^h, v) - b(u_f^h; u_f, v) + b(u_{f,H}; u_{f,H}^h, v) \\
+b(u_{f,H}^h; u_f, v - u_{f,H}^h, v) = 0,
\end{array} \right.
\end{aligned}
\]

(41)

Similarly,

\[
\begin{aligned}
b(u_f; u_f, v) &+ b(u_{f,H}; u_f^h, v) + b(u_f^h; u_f, v) \\
-b(u_f^h; u_f, v) &- b(u_f - u_f^h; u_{f,H}^h, v) - b(u_f - u_f^h; u_{f,H} - u_f, v) \\
-b(u_{f,H}^h - u_f; u_f^h - u_f, v).
\end{aligned}
\]

(42)

Moreover, we need some analogous definitions for the later derivation:

\[
\begin{aligned}
\tau_1 - \bar{\tau}_1 &= (u_f - \Pi_h u_f) - (u_f^h - \Pi_h u_f), \\
\eta_1 - \bar{\eta}_1 &= (\phi - \Pi_h \phi) - (\phi^h - \Pi_h \phi), \\
\eta_2 - \bar{\eta}_2 &= (p_f - \Pi_h p_f) - (p_f^h - \Pi_h p_f).
\end{aligned}
\]

Taking \( v = \bar{\tau}_1 \in H_{f,h_0} \), \( \psi = \bar{\eta}_1 \in H_{p,h} \) and \( q = \bar{\eta}_2 \in Q_h \) in (41) and combining (42) yield

\[
\begin{aligned}
a_{\Omega}(\bar{\tau}_1, \bar{\tau}_1) + a_{\Omega}(\bar{\eta}_1, \bar{\eta}_1) + b(\bar{\tau}_1; u_f, \bar{\tau}_1) \\
= a_{\Omega}(\bar{\tau}_1, \bar{\tau}_1) + a_{\Omega}(\bar{\eta}_1, \bar{\eta}_1) + a_{\Omega}(u - u^h, u^h - \Pi_h u) + d(\bar{\tau}_1, p_f - \Pi_h p_f) \\
+b(\bar{\eta}_1; u_f, \bar{\tau}_1) + b(u_{f,H}; \bar{\tau}_1, \bar{\tau}_1) - b(u_f - u_f^h; u_{f,H}^h - u_f, \bar{\tau}_1) \\
-b(u_f - u_f^h; u_{f,H} - u_f, \bar{\tau}_1) - b(u_f^h - u_f; u_{f,H}^h - u_f, \bar{\tau}_1) := \sum_{i=1}^{q} R_i.
\end{aligned}
\]

(43)

Do the same as (36), the left-hand side of (43) can be bounded by

\[
\begin{aligned}
l.h.s. &\geq \nu \varepsilon \left( 2 - \frac{C_B R C_A}{\sqrt{2}\varepsilon^2} \right) \| D(\bar{\tau}_1) \|^2_{L^2(\Omega_f^T)} + \rho_f g \| K^{1/2} \nabla \bar{\eta}_1 \|^2_{L^2(\Omega_p)}.
\end{aligned}
\]

(44)

Since \( R_i \) are similar as \( R_i, i = 1, 2, 3, 4, 5, 6 \), it is easy to get

\[
\begin{aligned}
|\bar{R}_1| &\leq Ch^2(\| u_f \|_{H^2(\Omega_f)} + \| \phi \|_{H^2(\Omega_p)} + \| p_f \|_{H^1(\Omega_f)}^2 \\
&+ \frac{\nu g}{4} \| D(\bar{\tau}_1) \|^2_{L^2(\Omega_f^T)},
\end{aligned}
\]

\[
\begin{aligned}
|\bar{R}_2| &\leq Ch^2(\| u_f \|_{H^2(\Omega_f)} + \| \phi \|_{H^2(\Omega_p)} + \| p_f \|_{H^1(\Omega_f)}^2 \\
&+ \frac{\rho_f g}{2} \| K^{1/2} \nabla \bar{\eta}_1 \|^2_{L^2(\Omega_p)},
\end{aligned}
\]
we can attain

\begin{align*}
|\mathcal{R}_3| & \leq Ch^2(\|u_f\|_{H^2(\Omega_f)}^2 + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2 \\
& \quad + \frac{\nu\varepsilon}{8} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 + \frac{\rho f_g}{4} \|K^{1/2}\nabla_1\|_{L^2(\Omega_f)}^2,
|\mathcal{R}_4| & \leq Ch^2(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2 \\
& \quad + \frac{\nu\varepsilon}{8} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2,
|\mathcal{R}_5 + \mathcal{R}_6| & \leq Ch^2(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2 \\
& \quad + \frac{\nu\varepsilon}{8} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2.
\end{align*}

Now, we bound the last three \(R_i, i = 7, 8, 9\). By using (9) and Theorem 2.2, we can attain

\begin{align*}
|\mathcal{R}_7 + \mathcal{R}_8| & \leq 2C_0C_1^2 \|D(u_f - u_f, H)\|_{L^2(\Omega_f)}^2 \|D(u_{f, h}^* - u_f)\|_{L^2(\Omega_f)}^2 \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 \\
& \leq \frac{4C_0C_1^2}{\nu\varepsilon} \|D(u_f - u_f, H)\|_{L^2(\Omega_f)}^2 \|D(u_{f, h}^* - u_f)\|_{L^2(\Omega_f)}^2 \\
& \quad + \frac{\nu\varepsilon}{4} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 \\
& \leq CH^2(h^2 + H^{6-\varepsilon})(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^4 \\
& \quad + \frac{\nu\varepsilon}{4} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2. \tag{45}
\end{align*}

For \(\mathcal{R}_9\), we use the second inequality in (8) to get

\begin{align*}
|\mathcal{R}_9| & \leq C_1 \|u_{f, h}^* - u_f\|_{L^2(\Omega_f)}^2 \|\nabla(u_{f, h}^* - u_f)\|_{L^2(\Omega_f)}^2 \|\nabla \nabla_1\|_{L^2(\Omega_f)}^2 \\
& \leq C_1 C_1^2 \|u_{f, h}^* - u_f\|_{L^2(\Omega_f)}^2 \|D(u_{f, h}^* - u_f)\|_{L^2(\Omega_f)}^2 \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 \\
& \leq \frac{2C_0^2 C_1^2}{\nu\varepsilon} \|u_{f, h}^* - u_f\|_{L^2(\Omega_f)}^2 \|D(u_{f, h}^* - u_f)\|_{L^2(\Omega_f)}^2 \\
& \quad + \frac{\nu\varepsilon}{8} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 \\
& \leq C(h + H^{3-\varepsilon})^4(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^4 \\
& \quad + \frac{\nu\varepsilon}{8} \|D(\nabla_1)\|_{L^2(\Omega_f)}^2. \tag{46}
\end{align*}

By using the above estimates, the right-hand side of (43) is bounded by

\begin{align*}
\text{r.h.s} & \leq \nu\varepsilon \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 + \frac{3\rho f_g}{4} \|K^{1/2}\nabla_1\|_{L^2(\Omega_f)}^2 \\
& \quad + CH^2(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2 \\
& \quad + CH^2(h^2 + H^{4-\varepsilon})^2(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2 \\
& \leq \nu\varepsilon \|D(\nabla_1)\|_{L^2(\Omega_f)}^2 + \frac{3\rho f_g}{4} \|K^{1/2}\nabla_1\|_{L^2(\Omega_f)}^2 \\
& \quad + C(h^2 + H^{4-\varepsilon})^2(\|u_f\|_{H^2(\Omega_f)} + \|\phi\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)})^2.
\end{align*}
Then we can obtain
\begin{equation}
\nu \delta \| \mathbf{D}(\mathbf{u}_f) \|_{L^2(\Omega_f)}^2 + \frac{\rho_f g}{4} \| \mathbf{K}^{1/2} \nabla \mathbf{u}_f \|_{L^2(\Omega_p)}^2
\leq C \left( h^2 + (H^4)^2 \right) \left( \| \mathbf{u}_f \|_{H^2(\Omega_f)} + \| \phi \|_{H^2(\Omega_p)} + \| p_f \|_{H^1(\Omega_f)} \right)^2.
\end{equation}

Then we can obtain
\begin{equation}
\| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)} + \| \nabla (\phi - \phi^h) \|_{L^2(\Omega_p)}
\leq C (h + H^4) \left( \| \mathbf{u}_f \|_{H^2(\Omega_f)} + \| \phi \|_{H^2(\Omega_p)} + \| p_f \|_{H^1(\Omega_f)} \right).
\end{equation}

Furthermore, thanks to the discrete inf-sup condition (11) one finds
\begin{equation}
\beta \| p_f - p_f^h \|_{L^2(\Omega_f)}
\leq \frac{d(v, p_f - p_f^h)}{\| v \| W}
\leq 2 \nu \varepsilon \| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)} + \varepsilon \frac{\alpha}{\sqrt{\tau \cdot \nu K \cdot \tau}} \| (\mathbf{u}_f - \mathbf{u}_f^h) \cdot \tau \|_{L^2(\Gamma)}
+ \rho_f g C_4 C_7 C_8 \| \nabla (\phi - \phi^h) \|_{L^2(\Omega_p)} + \| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)}
+ \rho_f g \| \mathbf{K}^{1/2} \nabla (\phi - \phi^h) \|_{L^2(\Omega_p)} + C_6 C_8 \| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)}
+ 2 C_6 C_8 \| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)} + \| \mathbf{D}(\mathbf{u}_f - \mathbf{u}_f^h) \|_{L^2(\Omega_f)}
\leq C (h + H^4) \left( \| \mathbf{u}_f \|_{H^2(\Omega_f)} + \| \phi \|_{H^2(\Omega_p)} + \| p_f \|_{H^1(\Omega_f)} \right)
\end{equation}

Finally, gathering the (48) and (49) leads to (32). \qed

4. Numerical experiment

In this section, numerical experiment is to verify the numerical theory of the two-level method for the Navier–Stokes/Darcy model developed in the previous section.

We present some numerical results of the Navier–Stokes/Darcy problem under a known analytical solution with the computational domain \( \Omega_p := [0, 1] \times [0, 1] \) and \( \Omega_f := [0, 1] \times [1, 2] \) with the interface \( \Gamma = (0, 1) \times \{1\} \). In this paper, the exact solution is given by
\[
\mathbf{u}_f = \left( \frac{x^2(y - 1)^2 + y - \frac{2}{3} x(y - 1)^3 + 2 - \pi \sin(\pi x)}{2} \right),
\]  
\[
p_f = \left[ 2 - \pi \sin(\pi x) \right] \sin \left( \frac{\pi}{2} y \right), \quad \phi = \left[ 2 - \pi \sin(\pi x) \right] \left( 1 - y - \cos(\pi y) \right).
\]

The force terms \( f_1 \) and \( f_2 \) are determined by (1) and (4), respectively. For simplicity, the model parameters \( \rho_f, g, \varepsilon, \alpha \) equal to 1 and \( \mathbf{K} = \mathbf{I} \).

For a given \( h \), we consider that the \( H \) satisfy \( h = O(H^4) \) for Algorithm 3.1 and \( h = O(H^2) \) for the common two-level finite element method. We list the numerical results of Algorithm 3.1, the one-level method and common two-level finite element method in Tables 1-3. From these tables, we can see that three
methods work well and keep the convergence rates just like the theoretical analysis. Besides, Algorithm 3.1 is competitive with the common two-level method in accuracy, especially for the pressure. However, as expected, the coarse mesh of Algorithm 3.1 can be chosen as a coarser one.

In addition, we compare the computing time of Algorithm 3.1 with the one-level method and the common two-level method in Tables 4-5. As expected, our algorithm spends less computing time than the other two methods under nearly the same accuracy. In conclusion, Algorithm 3.1 is more efficient than the other two methods.

Table 1. The one-level method for the steady Navier–Stokes/Darcy model.

<table>
<thead>
<tr>
<th>H</th>
<th>h</th>
<th>$|D(\bf{u}_f - \bf{u}_h)f|_0$ Rate</th>
<th>$|p_f - p_hf|_0$ Rate</th>
<th>$|\nabla(\phi - \phi_h)|_0$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1/7</td>
<td>9.503E-2 - 4.431E-1 - 1.398E-1</td>
<td>1.04 - 2.38E-1 - 1.080</td>
<td>8.125E-2 - 0.944</td>
</tr>
<tr>
<td>1/3</td>
<td>1/4</td>
<td>5.223E-2 - 1.016 - 8.645E-2</td>
<td>1.249</td>
<td>3.479E-2 - 1.046</td>
</tr>
<tr>
<td>1/3</td>
<td>1/2</td>
<td>1.259E-2 - 1.040 - 5.009E-2</td>
<td>1.048</td>
<td>1.930E-3 - 1.024</td>
</tr>
</tbody>
</table>

Table 2. The common two-level finite element method for the steady Navier–Stokes/Darcy model.

<table>
<thead>
<tr>
<th>H</th>
<th>h</th>
<th>$|D(\bf{u}_f - \bf{u}_h)f|_0$ Rate</th>
<th>$|p_f - p_hf|_0$ Rate</th>
<th>$|\nabla(\phi - \phi_h)|_0$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1/7</td>
<td>9.505E-2 - 4.588E-1 - 1.418E-1</td>
<td>1.041 - 2.452E-1 - 1.089</td>
<td>8.245E-2 - 0.943</td>
</tr>
<tr>
<td>1/3</td>
<td>1/4</td>
<td>5.223E-2 - 1.017 - 8.685E-2</td>
<td>1.280</td>
<td>3.554E-2 - 1.038</td>
</tr>
<tr>
<td>1/3</td>
<td>1/2</td>
<td>1.258E-2 - 1.040 - 5.729E-2</td>
<td>0.723</td>
<td>1.968E-2 - 1.028</td>
</tr>
</tbody>
</table>

Table 3. Algorithm 3.1 for the steady Navier–Stokes/Darcy model.

<table>
<thead>
<tr>
<th>H</th>
<th>h</th>
<th>$|D(\bf{u}_f - \bf{u}_h)f|_0$ Rate</th>
<th>$|p_f - p_hf|_0$ Rate</th>
<th>$|\nabla(\phi - \phi_h)|_0$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1/7</td>
<td>1.072E-1 - 5.061E-1 - 1.562E-1</td>
<td>1.073 - 2.040E-1 - 1.580</td>
<td>8.360E-2 - 1.086</td>
</tr>
<tr>
<td>1/3</td>
<td>1/4</td>
<td>5.784E-2 - 1.035 - 7.941E-2</td>
<td>1.163</td>
<td>3.495E-2 - 1.075</td>
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<tr>
<td>1/3</td>
<td>1/2</td>
<td>1.310E-2 - 1.123 - 4.502E-2</td>
<td>1.090</td>
<td>1.792E-2 - 1.111</td>
</tr>
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</table>

Table 4. Comparisons of the one-level method and Algorithm 3.1.

<table>
<thead>
<tr>
<th>Methods</th>
<th>H</th>
<th>h</th>
<th>CPU-time</th>
<th>$|D(\bf{u}_f - \bf{u}_h)f|_0$</th>
<th>$|p_f - p_hf|_0$</th>
<th>$|\nabla(\phi - \phi_h)|_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-level</td>
<td>1/3</td>
<td>1/7</td>
<td>8.320</td>
<td>3.738E-1 - 1.627E-1</td>
<td>3.363E-2 - 1.086</td>
<td></td>
</tr>
<tr>
<td>Algorithm 3.1</td>
<td>1/3</td>
<td>1/4</td>
<td>6.741</td>
<td>3.762E-1 - 1.760E-1</td>
<td>3.868E-1 - 1.086</td>
<td></td>
</tr>
<tr>
<td>One-level</td>
<td>1/3</td>
<td>1/2</td>
<td>2903.15</td>
<td>7.261E-2 - 2.872E-2</td>
<td>7.377E-2</td>
<td></td>
</tr>
<tr>
<td>Algorithm 3.1</td>
<td>1/3</td>
<td>1/2</td>
<td>1295.54</td>
<td>8.082E-2 - 7.115E-2</td>
<td>7.379E-2</td>
<td></td>
</tr>
</tbody>
</table>
Table 5. Comparisons of the common two-level method and Algorithm 3.1.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$H$</th>
<th>$h$</th>
<th>CPU-time</th>
<th>$|D(u_f^h-u_f^H)|_0$</th>
<th>$|p_f^H-p_f^h|_0$</th>
<th>$|\nabla(\phi-\phi^h)|_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common two-level</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>9.020</td>
<td>$2.309E-1$</td>
<td>$8.198E-2$</td>
<td>$2.413E-1$</td>
</tr>
<tr>
<td>Algorithm 3.1</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>6.741</td>
<td>$3.762E-1$</td>
<td>$1.760E-1$</td>
<td>$3.868E-1$</td>
</tr>
<tr>
<td>Common two-level</td>
<td>$1$</td>
<td>$\frac{1}{4}$</td>
<td>3701.55</td>
<td>$5.198E-2$</td>
<td>$4.927E-2$</td>
<td>$5.023E-2$</td>
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<tr>
<td>Algorithm 3.1</td>
<td>$1$</td>
<td>$\frac{1}{4}$</td>
<td>1295.54</td>
<td>$8.082E-2$</td>
<td>$7.115E-2$</td>
<td>$7.379E-2$</td>
</tr>
</tbody>
</table>

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References


A TWO-LEVEL FEM FOR NS/DARCY MODEL


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