CORRIGENDUM TO “THE IDEAL OF WEAKLY $p$-NUCLEAR OPERATORS AND ITS INJECTIVE AND SURJECTIVE HULLS” [J. KOREAN MATH. SOC. 56 (2019), NO. 1, PP. 225–237]

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Abstract. We indicate that some results in [2] are wrong, and obtain some new results on them.

1. Weakly 1-nuclear operators

We use all notations, terminologies and definitions in [2]. Let us recall the concept of a weakly 1-nuclear operator from a Banach space $X$ to a Banach space $Y$ as any operator which can be represented as

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \in \mathcal{N}_{w1}(X,Y),$$

where $(x_n^*)_n \in \ell^w_1(X^*)$ and $(y_n)_n \in c^w_0(Y)$. Every weakly 1-nuclear operator $T : X \to Y$ is weakly compact because $T(B_X)$ is contained in the convex hull of a weakly null sequence in $Y$.

Proposition 1.1 ([2, Proposition 2.2]). Let $1 \leq p \leq \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{N}_{wp}(X,Y)$ if and only if there exist $R \in \mathcal{L}(X,\ell_p)$ and $S \in \mathcal{L}(\ell_p,Y)$ ($\ell_p$ is replaced by $c_0$ if $p = \infty$) such that $T = SR$. In this case, $\|T\|_{\mathcal{N}_{wp}} = \inf \|S\|\|R\|$, where the infimum is taken over all such factorizations.

The case $p = 1$ in Proposition 1.1 is wrong. Indeed, if that statement would be true, then the identity map $id_{\ell_1} : \ell_1 \to \ell_1$ should be a weakly compact operator. This is a contradiction because $\ell_1$ has the Schur property.

The following lemma is well known but we provide a proof for the sake of completeness of our presentation.

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Lemma 1.2. Let $X$ and $Y$ be Banach spaces. An operator $T : X^* \to Y$ is weak$^*$ to weak continuous if and only if $T^*(Y^*) \subset i_X(X)$, where $i_X : X \to X^{**}$ is the canonical isometry.

Proof. Assume that $T$ is weak$^*$ to weak continuous and let $y^* \in Y^*$. To show that $T^*y^*$ is a weak$^*$ continuous functional, let $(x_n^*)_n$ be a net in $X^*$ and let $x^* \in X^*$ be such that $\lim_n x_n^* = x^*$ in the weak$^*$ topology on $X^*$. Since $T$ is weak$^*$ to weak continuous,

$$\lim_n T^*y^*(x_n^*) = \lim_n y^*(Tx_n^*) = y^*(Tx^*) = T^*y^*(x^*).$$

To show the converse, let $(x_n^*)_n$ be a net in $X^*$ and let $x^* \in X^*$ be such that $\lim_n x_n^* = x^*$ in the weak$^*$ topology on $X^*$. By assumption, for every $y^* \in Y^*$,

$$\lim_n y^*(Tx_n^*) = \lim_n T^*y^*(x_n^*) = T^*y^*(x^*) = y^*(Tx^*).$$

Hence $T$ is weak$^*$ to weak continuous. \[\square\]

We now obtain some factorizations of weakly 1-nuclear operators.

Theorem 1.3. Let $X$ and $Y$ be Banach spaces and let $T : X \to Y$ be a linear map. Then the following statements are equivalent.

(a) $T \in \mathcal{N}_{w1}(X,Y)$.
(b) There exist an operator $R : X \to \ell_1$ and a weak$^*$ to weak continuous operator $S : \ell_1 \to Y$ such that $T = SR$.
(c) There exist operators $R : X \to \ell_1$ and $S \in \mathcal{N}_{w1}(\ell_1,Y)$ such that $T = SR$.

In this case, $\|T\|_{\mathcal{N}_{w1}} = \inf \|S\| \|R\| = \inf \|S\|_{\mathcal{N}_{w1}} \|R\|$, where the infimums are taken over all such factorizations.

Proof. (c)$\Rightarrow$(a) is clear and $\|T\|_{\mathcal{N}_{w1}} \leq \inf \| \cdot \|_{\mathcal{N}_{w1}} \| \cdot \|$.

(a)$\Rightarrow$(b): Let $T \in \mathcal{N}_{w1}(X,Y)$ and let

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$$

be an arbitrary weakly 1-nuclear representation. Consider the maps

$$R : X \to \ell_1, \, x \mapsto (x_n^*(x))_n \quad \text{and} \quad S : \ell_1 \to Y, \, (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n y_n.$$

Then we see that $\|R\| = \|(x_n^*)_n\|_1$ and $\|S\| = \|(y_n)_n\|_\infty$.

Also, for every $y^* \in Y^*$ and $(\alpha_n)_n \in \ell_1$,

$$(S^*y^*)((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n y^*(y_n) = ((\alpha_n)_n, (y^*(y_n))_n).$$
Since \((y_n) \in c_0^w(Y)\), \(S^*(y^*) \in i_{c_0}(c_0)\). Thus by Lemma 1.2, \(S\) is weak* to weak continuous and the following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow & & \downarrow S \\
\ell_1 & & R
\end{array}
\]

Since the weakly 1-nuclear representation of \(T\) was arbitrary, \(\inf \| \cdot \|_{\mathcal{N}_w} \leq \|T\|_{\mathcal{N}_w}^w\).

(b)⇒(c): Let \(T\) have the following factorization in (b).

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow & & \downarrow S \\
\ell_1 & & R
\end{array}
\]

It follows that

\[
S = \sum_{n=1}^{\infty} e_n^* \otimes Se_n
\]

and \(\|(e_n^*)\|_w^w = 1\), where \(e_n\) and \(e_n^*\) are the standard unit vectors in \(\ell_1\) and \(c_0\), respectively. Since \(S\) is weak* to weak continuous and \(\lim_{n \to \infty} e_n = 0\) in the weak* topology on \(\ell_1\), \((Se_n)_{n} \in c_0^w(Y)\) and \(\|(Se_n)_{n}\|_\infty \leq \|S\|\).

Consequently, \(S \in \mathcal{N}_w(\ell_1, Y)\) and

\[
\inf \| \cdot \|_{\mathcal{N}_w} \| \cdot \| \leq \|S\| \|R\|.
\]

It was shown in [2, Lemma 2.3] that if \(1 < p \leq \infty\), then for every Banach space \(X\), \(\mathcal{N}_{wp}(X, \ell_p)\) (respectively, \(\mathcal{N}_{wp}(\ell_p, X)\)) is isometrically equal to \(\mathcal{L}(X, \ell_p)\) (respectively, \(\mathcal{L}(\ell_p, X)\) \((\ell_p = c_0\) when \(p = \infty\)). For the case \(p = 1\), we have:

**Proposition 1.4.** For every Banach space \(X\),

\[
\mathcal{N}_w(X, \ell_1) = \mathcal{K}(X, \ell_1)
\]

holds isometrically.

**Proof.** Note that

\[
\mathcal{N}_w(X, \ell_1) \subset W(X, \ell_1) = \mathcal{K}(X, \ell_1).
\]

To show the reverse inclusion, let \(T = \sum_{n=1}^{\infty} e_n^* T \otimes e_n \in \mathcal{K}(X, \ell_1)\) and let \(\varepsilon > 0\). Since \(T(B_X)\) is a relatively compact subset of \(\ell_1\),

\[
\lim_{l \to \infty} \sup_{x \in B_X} \sum_{n \geq l} |e_n^* Tx| = 0.
\]

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Then there exists a sequence \((\beta_n)\) with \(\beta_n > 1\) and \(\lim_{n \to \infty} \beta_n = \infty\) such that
\[
\lim_{l \to \infty} \sup_{x \in B_X} \sum_{n \geq l} |\beta_n e_n^* Tx| = 0 \quad \text{and} \quad \sup_{x \in B_X} \sum_{n = 1}^{\infty} |\beta_n e_n^* Tx| \leq (1 + \varepsilon) \sup_{x \in B_X} \sum_{n = 1}^{\infty} |e_n^* Tx|
\]
(cf. [3, Lemma 3.1]). Now, we see that
\[
T = \sum_{n=1}^{\infty} \beta_n e_n^* T \otimes (e_n / \beta_n) \in N_{w1}(X, \ell_1)
\]
and
\[
\|T\|_{N_{w1}} \leq (1 + \varepsilon) \sup_{x \in B_X} \sum_{n = 1}^{\infty} |e_n^* Tx| = (1 + \varepsilon) \|T\|.
\]

2. Weakly 1-compact sets

A subset \(K\) of a Banach space \(X\) is called weakly 1-compact if there exists \((x_n) \in \ell^w_p(X)\) such that
\[
K \subset 1-\text{co}(x_n) := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{c_0} \right\}.
\]

**Proposition 2.1** ([2, Lemma 3.5(a)]). Let \(X\) be a Banach space. For \(1 \leq p < \infty\), if \((x_n) \in \ell^w_p(X)\), then the set \(p-co(x_n)\) is balanced, convex and weakly compact.

The case \(p = 1\) in Proposition 2.1 is wrong. Indeed, let \((e_n)\) be the sequence of standard unit vectors in \(c_0\). Then we see that \((e_n) \in \ell^w_1(c_0)\) and \(1-co(e_n) = B_{c_0}\). Consequently, \(B_{c_0}\) is a weakly 1-compact subset of \(c_0\). But it is not weakly compact. Generally, we have:

**Proposition 2.2.** The following statements are equivalent for a Banach space \(X\).

(a) \(X\) does not have an isomorphic copy of \(c_0\).

(b) Every weakly 1-compact set in \(X\) is relatively compact.

(c) Every weakly 1-compact set in \(X\) is relatively weakly compact.

(d) For every \((x_n) \in \ell^w_1(X)\), the set \(1-co(x_n)\) is relatively weakly compact.

**Proof.** (b)⇒(c) and (c)⇒(d) are trivial.

It is well known that a Banach space \(X\) does not have an isomorphic copy of \(c_0\) if and only if every weakly \(1\)-summable sequence in \(X\) is unconditionally summable (cf. [4, Theorem 4.3.12]). Also a sequence \((x_n)\) in \(X\) is unconditionally summable if and only if
\[
\lim_{l \to \infty} \sup_{x^* \in B_{X^*}} \sum_{n \geq l} |x^*(x_n)| = 0
\]
(cf. [1, Theorem 1.9]).
(a)⇒(b): Let \((x_n)_n \in \ell^p_w(X)\). By (a), \((x_n)_n\) is unconditionally summable. Hence by [1, Theorem 1.9], \(1\)-co\((x_n)_n\) is relatively compact.

(d)⇒(a): Let \((x_n)_n \in \ell^p_w(X)\). Define the map

\[ S : c_0 \to X \text{ by } S(\alpha_n)_n = \sum_{n=1}^{\infty} \alpha_n x_n. \]

By (d), \(S\) is a weakly compact operator. We see that the adjoint operator \(S^* : X^* \to \ell_1\) is defined by

\[ S^*x^* = (x^*(x_n))_n. \]

Since \(S^*\) is weakly compact, by the Schur property \(S^*\) is compact. Consequently, \((x_n)_n\) is unconditionally summable. \(\square\)

References


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