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Lightlike Hypersurfaces of an Indefinite Nearly Trans-Sasakian Manifold with an (ℓ, m)-type Connection

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Abstract. We study a lightlike hypersurface M of an indefinite nearly trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that the structure vector field ζ of $\bar{M}$ is tangent to M. In particular, we focus on such lightlike hypersurfaces M for which the structure tensor field F is either recurrent or Lie recurrent, or such that M itself is totally umbilical or screen totally umbilical.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an $(\ell,\, m)$-*type connection* if there exist two smooth functions ℓ and m such that

$$(1.1)\qquad (\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\ \ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\},$$

$$(1.2)\qquad \bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}$$

for any vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ on $\bar{M}$, where $\bar{T}$ is the torsion tensor of $\bar{\nabla}$ and J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Throughout this paper, we set $(\ell, m) \neq (0, 0)$ and denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

The notion of $(\ell,\, m)$-type connection was introduced by Jin [8]. In the case $(\ell,\, m) = (1,\, 0)$, this connection $\bar{\nabla}$ becomes a semi-symmetric non-metric connec-

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tion. The notion of a semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [1]. In the case $(\ell,\, m) = (0,\, 1)$, this connection $\bar{\nabla}$ becomes a non-metric ϕ-symmetric connection such that $\phi(\bar{X}, \bar{Y}) = \bar{g}(J\bar{X}, \bar{Y})$. The notion of the non-metric ϕ-symmetric connection was introduced by Jin [6].

Remark 1.1.([8]) Denote by $\widetilde{\bar{\nabla}}$ a unique Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. Then *a linear connection* $\bar{\nabla}$ *on* $(\bar{M}, \bar{g})$ *is an* $(\ell,\, m)$*-type connection if and only if* $\bar{\nabla}$ *satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\bar{\nabla}}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{\ell\bar{X} + mJ\bar{X}\}. \tag{1.3}$$

The subject of study in this paper is lightlike hypersurfaces of an indefinite nearly trans-Sasakian manifold $\bar{M} = (\bar{M}, \zeta, \theta, J, \bar{g})$ with an $(\ell,\, m)$-type connection subject to the conditions: (1) the tensor field J and the 1-form θ, defined by (1.1) and (1.2 are identical with the indefinite nearly trans-Sasakian structure tensor J and the structure 1-form θ of $\bar{M}$, respectively, and (2) the structure vector field ζ of $\bar{M}$ is tangent to M.

Călin [3] proved that *if the structure vector field* ζ *of* $\bar{M}$ *is tangent to* M*, then it belongs to* $S(TM)$, we assume this in this paper.

2. On (ℓ, m)-type Connections

A hypersurface M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a *lightlike hypersurface* if its normal bundle $TM^{\perp}$ is a vector subbundle of the tangent bundle TM. There exists a screen distribution $S(TM)$ such that

$$TM = TM^{\perp} \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. It is known from [4] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$, of rank 1, in the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $\bar{M}$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in S(TM).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ can be decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* with respect to the screen distribution $S(TM)$, respectively.

In the following, we denote by X, Y and Z smooth vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be an (ℓ, m)-type connection on $\bar{M}$ defined by (1.3) and P the projection morphism of TM on $S(TM)$. As ζ belongs to $S(TM)$, from

(1.1) we have $\bar{g}(\bar{\nabla}_X N, \xi) + \bar{g}(N, \bar{\nabla}_X \xi) = 0$. Thus the local Gauss and Weingarten formulae of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.1}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N\,; \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.3}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.4}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators, and τ is a 1-form.

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite almost contact metric manifold* [5, 6] if there exist a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a $(1,1)$-type tensor field, ζ is a vector field, θ is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ such that

$$\begin{gathered} J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad J\zeta = 0, \quad \theta\circ J = 0, \quad \theta(\zeta) = 1, \\ \theta(\bar{X}) = \bar{g}(\zeta, \bar{X}), \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \theta(\bar{X})\theta(\bar{Y}). \end{gathered} \tag{2.5}$$

It is known [5, 6] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$ such that the structure vector field ζ of $\bar{M}$ is tangent to M, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, such that $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. Thus there exist two non-degenerate almost complex distributions D_o and D with respect to J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$\begin{gathered} S(TM) = \{J(TM^\perp) \oplus J(tr(TM))\} \oplus_{orth} D_o, \\ D = TM^\perp \oplus_{orth} J(TM^\perp) \oplus_{orth} D_o. \end{gathered}$$

In this case, the decomposition form of TM is reformed as follows:

$$TM = D \oplus J(tr(TM)).$$

Consider two lightlike vector fields U and V, and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X,V), \quad v(X) = g(X,U). \tag{2.6}$$

Denote by $\bar{S}$ the projection morphism of TM on D. Any vector field X of M is expressed as $X = \bar{S}X + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.7}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $FX = J\bar{S}X$. Applying J to (2.7) and using (2.5) and (2.6), we have

$$F^2X = -\,X + u(X)U + \theta(X)\zeta. \tag{2.8}$$

Using (1.1), (1.2), (2.1) and (2.7), we see that

$$(2.9)\qquad (\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) - \ell\{\theta(Y)g(X,Z) + \theta(Z)g(X,Y)\} - m\{\theta(Y)\bar{g}(JX,Z) + \theta(Z)\bar{g}(JX,Y)\},$$

$$(2.10)\qquad T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\},$$

$$(2.11)\qquad B(X,Y) - B(Y,X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\},$$

where T is the torsion tensor with respect to the induced connection ∇ on M and η is a 1-form such that $\eta(X) = \bar{g}(X,N)$.

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$(2.12)\qquad B(X,\xi) = 0, \qquad B(\xi,X) = 0.$$

The local second fundamental forms are related to their shape operators by

$$(2.13)\qquad B(X,Y) = g(A^*_\xi X, Y) + mu(X)\theta(Y),$$

$$(2.14)\qquad C(X,PY) = g(A_N X, PY) + \{\ell\eta(X) + mv(X)\}\theta(PY),$$

$$(2.15)\qquad \bar{g}(A^*_\xi X, N) = 0, \qquad \bar{g}(A_N X, N) = 0.$$

As $S(TM)$ is non-degenerate, taking $X = \xi$ to (2.13), we obtain

$$(2.16)\qquad A^*_\xi \xi = 0, \qquad \bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi.$$

Applying ∇_X to $F\xi = -V$ and $FV = \xi$ by turns and using (2.5), we have

$$(2.17)\qquad (\nabla_X F)\xi = -\nabla_X V + F(A^*_\xi X) - \tau(X)V,$$

$$(2.18)\qquad (\nabla_X F)V = -F\nabla_X V - A^*_\xi X - \tau(X)\xi.$$

Applying ∇_X to $v(Y) = g(Y,U)$ and using (2.9), we obtain

$$(2.19)\qquad (\nabla_X v)(Y) = m\theta(Y)\eta(X) - \ell\theta(Y)v(X) + B(X,U)\eta(Y) + g(Y,\nabla_X U).$$

Applying ∇_X to $g(U,U) = 0$ and $g(V,V) = 0$ and using (2.9), we get

$$(2.20)\qquad v(\nabla_X U) = 0, \qquad u(\nabla_X V) = 0.$$

3. Recurrents and Lie Recurrents

Definition 3.1.([7]) The structure tensor field F of M is said to be *recurrent* if there exists a 1-form ω on M such that

$$(3.1)\qquad (\nabla_X F)Y = \omega(X)FY.$$

Theorem 3.2. *Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$ with an (ℓ, m)-type connection $\bar{\nabla}$ such that ζ is tangent to M. If F is recurrent, then F is parallel with respect to the induced connection ∇ from $\bar{\nabla}$.*

Proof. Comparing (2.18) with (3.1) in which we replace Y with V, we obtain

$$F\nabla_X V + A^*_\xi X + \{\omega(X) + \tau(X)\}\xi = 0. \tag{3.2}$$

Also, comparing (2.17) with (3.1), taking $Y = \xi$, we obtain

$$\nabla_X V - F(A^*_\xi X) - \{\omega(X) - \tau(X)\}V = 0. \tag{3.3}$$

Taking the scalar product with V and ζ to (3.3), we have

$$u(\nabla_X V) = 0, \qquad \theta(\nabla_X V) = 0. \tag{3.4}$$

Applying F to (3.2) and using (2.8) and (3.4) and then, comparing this result with (3.3), we get $\omega = 0$. Thus F is parallel with respect to ∇. □

Definition 3.3.([7]) The structure tensor field F of M is called *Lie recurrent* if there exists a 1-form θ on M such that

$$(\mathcal{L}_X F)Y = \sigma(X)FY, \tag{3.5}$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{3.6}$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$.

Theorem 3.4. *Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$ with an (ℓ, m)-type connection $\bar{\nabla}$ such that ζ is tangent to M. If F is Lie recurrent, then F is Lie parallel.*

Proof. As the induced connection ∇ from $\bar{\nabla}$ is torsion-free, from (3.5) and (3.6) we have

$$(\nabla_X F)Y = \nabla_{FY} X - F\nabla_Y X + \sigma(X)FY. \tag{3.7}$$

Comparing (2.18) with (3.7), taking $Y = V$, we obtain

$$\nabla_\xi X = -F(\nabla_X V - \nabla_V X) - A^*_\xi X - \{\sigma(X) + \tau(X)\}\xi. \tag{3.8}$$

Also, comparing (2.17) with (3.7), taking $Y = \xi$, we obtain

$$F\nabla_\xi X = \nabla_X V - \nabla_V X - F(A^*_\xi X) - \{\sigma(X) - \tau(X)\}V. \tag{3.9}$$

Taking the scalar product with V and ζ to (3.9), we obtain

$$u(\nabla_X V - \nabla_V X) = 0, \qquad \theta(\nabla_X V - \nabla_V X) = 0. \tag{3.10}$$

Applying F to (3.8) and using (2.8) and (3.10) and then, comparing this result with (3.9), we have $\sigma = 0$. Thus F is Lie parallel. □

4. Indefinite Nearly Trans-Sasakian Manifolds

Definition 4.1.([9]) An indefinite almost contact metric manifold $\bar{M}$ is called an *indefinite nearly trans-Sasakian manifold* if $\{J, \zeta, \theta, \bar{g}\}$ satisfies

$$(\widetilde{\nabla}_{\bar{X}} J)\bar{Y} + (\widetilde{\nabla}_{\bar{Y}} J)\bar{X} = \alpha\{2\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} - \beta\{\theta(\bar{Y})J\bar{X} + \theta(\bar{X})J\bar{Y}\}. \tag{4.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of $\bar{M}$. We say that the set $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite nearly trans-Sasakian structure of type* (α, β).

Note that the indefinite nearly Sasakian manifolds, indefinite nearly Kanmotsu manifolds and indefinite nearly cosymplectic manifolds are important examples of indefinite nearly trans-Sasakian manifold such that

$$\alpha = 1,\ \beta = 0; \qquad \alpha = 0,\ \beta = 1; \qquad \alpha = \beta = 0, \text{ respectively.}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the (ℓ, m)-type connection $\bar{\nabla}$ given by (1.3), the equation (4.1) is reduced to

$$\begin{aligned}(\bar{\nabla}_{\bar{X}} J)\bar{Y} + (\bar{\nabla}_{\bar{Y}} J)\bar{X} = {} & (m - \alpha)\{\theta(\bar{Y})\bar{X} + \theta(\bar{X})\bar{Y}\} \\ & - (\ell + \beta)\{\theta(\bar{Y})J\bar{X} + \theta(\bar{X})J\bar{Y}\} \\ & + 2\{\alpha\bar{g}(\bar{X}, \bar{Y}) - m\theta(\bar{X})\theta(\bar{Y})\}\zeta.\end{aligned} \tag{4.2}$$

Applying $\bar{\nabla}_\zeta$ to $\bar{g}(\zeta, \zeta) = 1$ and using (1.1), we have $\theta(\bar{\nabla}_\zeta \zeta) = \ell$. Taking $\bar{X} = \bar{Y} = \zeta$ to (4.2), we obtain $(\bar{\nabla}_\zeta J)\zeta = 0$. It follows that $J(\bar{\nabla}_\zeta \zeta) = 0$. Applying J to this equation and using (2.5) and the fact that $\theta(\bar{\nabla}_\zeta \zeta) = \ell$, we have $\bar{\nabla}_\zeta \zeta = \ell\zeta$. From this equation, (2.1) and (2.3), we obtain

$$\nabla_\zeta \zeta = \ell\zeta, \qquad B(\zeta, \zeta) = 0, \qquad C(\zeta, \zeta) = 0. \tag{4.3}$$

Definition 4.2.([4]) A lightlike hypersurface M of $(\bar{M}, \bar{g})$ is said to be

(1) *totally umbilical* if there is a smooth function ρ on a coordinate neighborhood $\mathcal{U}$ in M such that $A^*_\xi X = \rho PX$ or equivalently

$$B(X, Y) = \rho\, g(X, Y). \tag{4.4}$$

In case $\rho = 0$ on $\mathcal{U}$, we say that M is *totally geodesic.*

(2) *screen totally umbilical* if there exist a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that $A_N X = \gamma PX$ or equivalently

$$C(X, PY) = \gamma\, g(X, PY). \tag{4.5}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that M is *screen totally geodesic.*

Theorem 4.3. *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that the structure vector field ζ of $\bar{M}$ is tangent to M.*

(1) *If M is totally umbilical, then M is totally geodesic and $m = 0$.*

(2) *If M is screen totally umbilical, then M is screen totally geodesic.*

Proof. (1) If M is totally umbilical, then, taking $X = Y = \zeta$ to (4.4) and using (4.3), we have $\rho = 0$. Thus M is totally geodesic. On the other hand, since $B = 0$, taking $X = U$ and $Y = \zeta$ to (2.11), we see that $m = 0$.

(2) If M is screen totally umbilical, then, taking $X = PY = \zeta$ to (4.5) and using (4.3), we have $\gamma = 0$. Thus M is screen totally geodesic. □

Applying $\bar{\nabla}_X$ to $JY = FY + u(Y)N$ and using (2.3), we have

$$(\bar{\nabla}_X J)Y = (\nabla_X F)Y - u(Y)A_N X + B(X,Y)U + \{(\nabla_X u)(Y) + u(Y)\tau(X) + B(X, FY)\}N. \tag{4.6}$$

Substituting (4.6) into (4.2) and using (2.7) and (2.11), we obtain

$$\begin{aligned}(\nabla_X F)Y + (\nabla_Y F)X = {} & (m - \alpha)\{\theta(Y)X + \theta(X)Y\} \\ & - (\ell + \beta)\{\theta(Y)FX + \theta(X)FY\} \\ & + 2\{\alpha g(X,Y) - m\theta(X)\theta(Y)\}\zeta \\ & + u(X)A_N Y + u(Y)A_N X - 2B(X,Y)U \\ & + m\{\theta(Y)u(X) - \theta(X)u(Y)\}U.\end{aligned} \tag{4.7}$$

Lemma 4.4. *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection $\bar{\nabla}$ such that the structure vector field ζ of $\bar{M}$ is tangent to M. Then we have*

$$\begin{cases} B(U,V) = C(V,V), & B(U,\zeta) + C(V,\zeta) = 2(m - \alpha), \\ B(U,U) = C(U,V), & v(\nabla_U V) = -\tau(U), \\ C(U,\zeta) = 0, & 2C(V,\zeta) + C(\zeta,V) = 2m - 3\alpha, \\ B(U,\zeta) = C(V,\zeta) + C(\zeta,V) + \alpha, & \\ B(U,\zeta) = m + \theta(A^*_\xi U), & \theta(\nabla_\xi U) = \theta(A^*_\xi U), \end{cases} \tag{4.8}$$

where ∇ is the induced connection from $\bar{\nabla}$.

Proof. Applying ∇_X to $FU = 0$ and $FV = \xi$ by turns, we obtain

$$(\nabla_X F)U = -F\nabla_X U, \quad (\nabla_X F)V = -F\nabla_X V - A^*_\xi X - \tau(X)\xi.$$

From these two equations, we obtain

$$(\nabla_U F)V + (\nabla_V F)U = -F(\nabla_U V + \nabla_V U) - A^*_\xi U - \tau(U)\xi.$$

Comparing this result with (4.7), taking $X = U$ and $Y = V$, we have

$$F(\nabla_U V + \nabla_V U) + A^*_\xi U + \tau(U)\xi = -2\alpha\zeta - A_N V + 2B(U,V)U.$$

Taking the scalar product with V, ζ, U and N to this and using (2.13), (2.14), (2.20) and $\eta(\nabla_X PY) = C(X, PY)$, we get (4.8).

By direct calculation from $FU = 0$, $F\zeta = 0$ and (4.7), we obtain

$$F(\nabla_U \zeta + \nabla_\zeta U) = -A_N \zeta + \{\alpha - 2m + 2B(U,\zeta)\}U.$$

Taking the scalar product with U and V to this by turns and using (2.5), (2.7), (2.14) and $\eta(\nabla_U \zeta + \nabla_\zeta U) = C(U,\zeta) + C(\zeta, U)$, we get (4.8) and

$$2B(U,\zeta) - C(\zeta, V) = 2m - \alpha. \tag{4.9}$$

Substituting (4.8) into (4.9), we have (4.8).

By directed calculation from $FV = \xi$, $F\zeta = 0$ and (4.7), we obtain

$$\begin{aligned} F(\nabla_V \zeta + \nabla_\zeta V) \;&= -A^*_\xi \zeta + 2B(V,\zeta)U \\ &- \;(m - \alpha)V + \{\ell + \beta - \tau(\zeta)\}\xi. \end{aligned}$$

Taking the scalar product with U and using (2.3), (2.11) and (2.13), we get (4.8): $B(U,\zeta) = C(V,\zeta) + C(\zeta, V) + \alpha$.

Taking $X = U$ and $Y = \zeta$ to (2.13), we have (4.8). On the other hand, applying $\bar{\nabla}_X$ to $v(Y) = g(FY, N)$ and using (1.1), (2.1) and (2.2), we get

$$g((\nabla_X F)Y, N) = (\nabla_X v)(Y) - v(Y)\tau(X) + g(A_N X, FY).$$

Taking the scalar product with N to (4.7), we obtain

$$\begin{aligned} (\nabla_X v)Y + (\nabla_Y v)X \;&= \;(m - \alpha)\{\theta(Y)\eta(X) + \theta(X)\eta(Y)\} \\ &- (\ell + \beta)\{\theta(Y)v(X) + \theta(X)v(Y)\} \\ &+ v(Y)\tau(X) + v(X)\tau(Y) \\ &- g(A_N X, FY) - g(A_N Y, FX). \end{aligned}$$

Substituting (2.19) into the last equation, we have

$$\begin{aligned} &B(X,U)\eta(Y) + B(Y,U)\eta(X) + g(Y, \nabla_X U) + g(X, \nabla_Y U) \\ &= -\,\alpha\{\theta(Y)\eta(X) + \theta(X)\eta(Y)\} - \beta\{\theta(Y)v(X) + \theta(X)v(Y)\} \\ &\quad + v(Y)\tau(X) + v(X)\tau(Y) - g(A_N X, FY) - g(A_N Y, FX). \end{aligned}$$

Taking $X = \zeta$ and $Y = \xi$ to this and using (2.11) and (2.12), we have

$$B(U,\zeta) - C(\zeta, V) = m - \alpha - \theta(\nabla_\xi U),$$

due to (2.14). Substituting this equation into (4.9), we obtain

$$B(U,\zeta) = m + \theta(\nabla_\xi U).$$

Comparing this equation with (4.8), we have (4.8). □

Lemma 4.5. *Let M be a lightlike hypersurfac of an indefinite nearly trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection $\bar{\nabla}$ such that ζ is tangent to M. If one of the following three conditions is satisfied,*

(1) $(\nabla_X F)Y + (\nabla_Y F)X = 0$,

(2) *F is parallel with respect to the induced connection ∇ on M, that is,* $\nabla_X F = 0$,

(3) *F is recurrent,*

then $\alpha = m$ and $\beta = -\ell$. The shape operators A^*_ξ and A_N satisfy

$$A^*_\xi V = 0, \quad A_N V = -2\alpha\zeta, \quad A_N \xi = 0, \quad \theta(A^*_\xi U) = 0, \\ \theta(\nabla_\xi U) = 0, \qquad A_N X = C(X, V)U - 2\alpha v(X)\zeta. \tag{4.10}$$

Proof. (1) Assume that $(\nabla_X F)Y + (\nabla_Y F)X = 0$. Taking the scalar product with N to (4.7) and using (2.15), we have

$$(m-\alpha)\{\theta(Y)\eta(X) + \theta(X)\eta(Y)\} = \ell + \beta)\{\theta(Y)v(X) + \theta(X)v(Y)\}.$$

Taking $X = \xi$, $Y = \zeta$ and $X = V$, $Y = \zeta$ in this equation, we obtain $\alpha = m$ and $\beta = -\ell$. As $\alpha = m$ and $\beta = -\ell$, (4.7) is reduced to

$$2\alpha\{g(X,Y) - \theta(X)\theta(Y)\}\zeta + u(X)A_N Y + u(Y)A_N X \\ - 2B(X,Y)U + m\{\theta(Y)u(X) - \theta(X)u(Y)\}U = 0. \tag{4.11}$$

Taking the scalar product with V to (4.11), we have

$$2B(X,Y) = u(Y)u(A_N X) + u(X)u(A_N Y) \\ + m\{\theta(Y)u(X) - \theta(X)u(Y)\}. \tag{4.12}$$

Taking $Y = V$ in this equation and using (2.14), we obtain

$$2B(X,V) = u(X)C(V,V).$$

Replacing X by U to this equation, we have $2B(U,V) = C(V,V)$. Comparing this result with (4.8), we have $C(V,V) = 0$. Thus we obtain

$$B(U,V) = C(V,V) = 0, \qquad B(X,V) = 0. \tag{4.13}$$

Using (2.11) and (4.13), we see that $B(V,X) = 0$. From this, (2.13) and the fact that $S(TM)$ is non-degenerate, we have (4.10): $A^*_\xi V = 0$. Taking $X = U$ and $Y = V$ to (4.11) and using (4.13), we get (4.10): $A_N V = -2\alpha\zeta$. Also, taking $X = U$ and $Y = \xi$ to (4.11) and using (2.12), we get (4.10): $A_N\xi = 0$. Taking $X = V$ and $Y = \zeta$ to (2.14) and using (4.10) and the fact that $m = \alpha$, we obtain

$C(V,\zeta)=-m$. From this result and (4.8), we have $B(U,\zeta)=m$. Thus, from (4.8) we get (4.10): $\theta(A^*_\xi U)=\theta(\nabla_\xi U)=0$.

Taking $Y=U$ to (4.12), we obtain

$$2B(X,U)+m\theta(X)=u(A_N X)+u(X)u(A_N U).$$

Replacing Y by U to (4.11) and using the last equation, we get

$$A_N X-u(A_N X)U+u(X)\{A_N U-u(A_N U)U\}+2\alpha v(X)\zeta=0.$$

Taking $X=U$ to this, we have $A_N U=u(A_N U)U$. Thus we have

$$A_N X=u(A_N X)U-2\alpha v(X)\zeta.$$

(2) If F is parallel with respect to ∇, then $(\nabla_X F)Y+(\nabla_Y F)X=0$. By item (1), we see that $\alpha=m$ and $\beta=-\ell$. A^*_ξ and A_N satisfy (4.10).

(3) If F is recurrent, then F is parallel with respect to ∇ by Theorem 3.2. By item (2), we see that $\alpha=m$ and $\beta=-\ell$. A^*_ξ and A_N satisfy (4.10). □

Theorem 4.6. *Let M be a lightlike hypersurface of an indefinite nearly trans-Sasakian manifold $\bar M$ with an (ℓ,m)-type connection $\bar\nabla$ such that ζ is tangent to M. If F is Lie recurrent, then $\bar M$ is an indefinite nearly β-Kenmotsu manifold with an (ℓ,m)-type connection $\bar\nabla$.*

Proof. If F is Lie recurrent, then F is Lie parallel, *i.e.*, $\sigma=0$, by Theorem 3.4. Replacing Y by U to (3.7), we have $(\nabla_X F)U=-F\nabla_U X$. Applying ∇_X to $FU=0$, we get $(\nabla_X F)U=-F\nabla_X U$. Therefore we have

$$F(\nabla_X U-\nabla_U X)=0. \tag{4.14}$$

Taking the scalar product with N to this and using (2.20), we obtain

$$v(\nabla_U X)=0, \qquad \tau(U)=0, \tag{4.15}$$

due to (4.8). Taking $X=U$ to (3.8) and using (4.14) and (4.15), we get

$$\nabla_\xi U=-A^*_\xi U. \tag{4.16}$$

Taking the scalar product with ζ to this equation, we have

$$\theta(\nabla_\xi U)=-\theta(A^*_\xi U).$$

Comparing this with (4.8) and using (2.11) and (4.8), we have

$$\theta(\nabla_\xi U)=\theta(A^*_\xi U)=0, \qquad B(U,\zeta)=m, \qquad B(\zeta,U)=0. \tag{4.17}$$

Applying ∇_ξ to $g(U,\zeta)=0$ and using (2.9) and (4.17), we obtain

$$v(\nabla_\xi\zeta)=-m. \tag{4.18}$$

Taking the scalar product with U to (3.8), we have

$$v(\nabla_\xi X) = \eta(\nabla_X V - \nabla_V X) - B(X, U).$$

Replacing X by ζ to this and using (2.4), (4.17) and (4.18), we have

$$C(\zeta, V) = C(V, \zeta) - m.$$

As $B(U, \zeta) = m$, from (4.9), we obtain

$$C(\zeta, V) = \alpha, \qquad C(V, \zeta) = m + \alpha.$$

Substituting the last two results into (4.8), we get $\alpha = 0$. Thus $\bar{M}$ is an indefinite nearly β-Kenmotsu manifold with an (ℓ, m)-type connection. □

5. Indefinite Nearly Generalized Sasakian Space Forms

Denote by $\bar{R}$, R and R^* the curvature tensors of the (ℓ, m)-type connection $\bar{\nabla}$ of $\bar{M}$ and the induced connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae for M and $S(TM)$, we obtain two Gauss equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\quad - \ell[\theta(X)B(Y,Z) - \theta(Y)B(X,Z)] \\ &\quad - m[\theta(X)B(FY,Z) - \theta(Y)B(FX,Z)]\}N, \end{aligned} \tag{5.1}$$

$$\begin{aligned} R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ &\quad - \ell[\theta(X)C(Y,PZ) - \theta(Y)C(X,PZ)] \\ &\quad - m[\theta(X)C(FY,PZ) - \theta(Y)C(FX,PZ)]\}\xi. \end{aligned} \tag{5.2}$$

Definition 5.1. An indefinite nearly trans-Sasakian manifold $\bar{M}$ is said to be a *indefinite nearly generalized Sasakian space form*, denoted by $\bar{M}(f_1, f_2, f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \widetilde{R}(\bar{X},\bar{Y})\bar{Z} = \; & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ &+ f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ &+ f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ &\qquad + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\}, \end{aligned} \tag{5.3}$$

where $\widetilde{R}$ is the curvature tensors of the Levi-Civita connection $\widetilde{\nabla}$ of $\bar{M}$.

The notion of (Riemannian) generalized Sasakian space form was introduced by Alegre *et. al.* [2]. Sasakian, Kenmotsu and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

By direct calculations from (1.2), (1.3) and (2.5), we have

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} \;=\;& \widetilde{R}(\bar{X},\bar{Y})\bar{Z} \\ &+ \{\ell(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) + [\bar{X}\ell + m^2\theta(\bar{X})]\theta(\bar{Z})\}\bar{Y} \\ &- \{\ell(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) + [\bar{Y}\ell + m^2\theta(\bar{Y})]\theta(\bar{Z})\}\bar{X} \\ &+ \{m(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) + [\bar{X}m - \ell m\theta(\bar{X})]\theta(\bar{Z})\}J\bar{Y} \\ &- \{m(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) + [\bar{Y}m - \ell m\theta(\bar{Y})]\theta(\bar{Z})\}J\bar{X} \\ &+ m\theta(\bar{Z})\{(\bar{\nabla}_{\bar{X}}J)\bar{Y} - (\bar{\nabla}_{\bar{Y}}J)\bar{X}\}.\end{aligned} \tag{5.4}$$

Comparing the tangential, transversal and radical components of the left-right terms of (5.4) such that $\bar{X} = X, \bar{Y} = Y$ and $\bar{Z} = Z$ and using (2.11), (2.15), (4.6), (5.1), (5.2), (5.3) and the last two equations, we obtain

$$\begin{aligned}R(X,Y)Z =\;& B(Y,Z)A_{N}X - B(X,Z)A_{N}Y \\ &+ \{\ell(\bar{\nabla}_X\theta)(Z) + [X\ell + m^2\theta(X)]\theta(Z)\}Y \\ &- \{\ell(\bar{\nabla}_Y\theta)(Z) + [Y\ell + m^2\theta(Y)]\theta(Z)\}X \\ &+ \{m(\bar{\nabla}_X\theta)(Z) + [Xm - \ell m\theta(X)]\theta(Z)\}FY \\ &- \{m(\bar{\nabla}_Y\theta)(Z) + [Ym - \ell m\theta(Y)]\theta(Z)\}FX \\ &+ m\theta(Z)\{(\nabla_X F)Y - (\nabla_Y F)X \\ &\qquad + u(X)A_{N}Y - u(Y)A_{N}X \\ &\qquad + m[\theta(Y)u(X) - \theta(X)u(Y)]U\} \\ &+ f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} \\ &+ f_3\{[\theta(X)Y - \theta(Y)X]\theta(Z) \\ &\qquad + [g(X,Z)\theta(Y) - g(Y,Z)\theta(X)]\zeta\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \{\tau(X) - \ell\theta(X)\}B(Y,Z) - \{\tau(Y) - \ell\theta(Y)\}B(X,Z) \\ &- m\{\theta(X)B(FY,Z) - \theta(Y)B(FX,Z)\} \\ &= \{m(\bar{\nabla}_X\theta)(Z) + [Xm - \ell m\theta(X)]\theta(Z)\}u(Y) \\ &- \{m(\bar{\nabla}_Y\theta)(Z) + [Ym - \ell m\theta(Y)]\theta(Z)\}u(X) \\ &+ m\theta(Z)\{(\nabla_X u)Y - (\nabla_Y u)X + u(Y)\tau(X) \\ &\qquad - u(X)\tau(Y) + B(X,FY) - B(Y,FX)\} \\ &+ f_2\{\bar{g}(X,JZ)u(Y) - \bar{g}(Y,JZ)u(X) + 2\bar{g}(X,JY)u(Z)\},\end{aligned} \tag{5.6}$$

$$\begin{aligned}
(5.7)\qquad & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
& - \{\tau(X) + \ell\theta(X)\}C(Y, PZ) + \{\tau(Y) + \ell\theta(Y)\}C(X, PZ) \\
& - m\{\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)\} \\
& = \{\ell(\bar{\nabla}_X\theta)(PZ) + [X\ell + m^2\theta(X)]\theta(PZ)\}\eta(Y) \\
& - \{\ell(\bar{\nabla}_Y\theta)(PZ) + [Y\ell + m^2\theta(Y)]\theta(PZ)\}\eta(X) \\
& + \{m(\bar{\nabla}_X\theta)(PZ) + [Xm - \ell m\theta(X)]\theta(PZ)\}v(Y) \\
& - \{m(\bar{\nabla}_Y\theta)(PZ) + [Ym - \ell m\theta(Y)]\theta(PZ)\}v(X) \\
& + m\theta(PZ)\{(\nabla_X v)Y - (\nabla_Y v)X \\
& \qquad - v(Y)\tau(X) + v(X)\tau(Y) \\
& \qquad + g(A_N X, FY) - g(A_N Y, FX)\} \\
& + f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\
& + f_2\{\bar{g}(X, JPZ)v(Y) - \bar{g}(Y, JPZ)v(X) + 2\bar{g}(X, JY)v(PZ) \\
& + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ),
\end{aligned}$$

due to the following equations:

$$\bar{g}((\bar{\nabla}_X J)Y, \xi) = (\nabla_X u)(Y) + u(Y)\tau(X) + B(X, FY),$$

$$\bar{g}((\bar{\nabla}_X J)Y, N) = (\nabla_X v)(Y) - v(Y)\tau(X) + g(A_N X, FY).$$

Using the Gauss-Weingarten formulae for $S(TM)$, we obtain the following Codazzi equations for $S(TM)$ such that

$$\begin{aligned}
R(X, Y)\xi = & -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X, Y] \\
& - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\
& + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi.
\end{aligned}$$

Replacing Z by ξ to (5.5) and using (2.12) and (5.9), we have

$$\begin{aligned}
R(X, Y)\xi = & \theta(A^*_\xi X)\{\ell Y + mFY\} - \theta(A^*_\xi Y)\{\ell X + mFX\} \\
& + f_2\{u(Y)FX - u(X)FY - 2\bar{g}(X, JY)V\}.
\end{aligned}$$

Comparing the radical components of the last two equations, we obtain

$$\begin{aligned}
(5.8)\qquad & f_2\{u(Y)v(X) - u(X)v(Y)\} \\
& = g(A_N Y, A^*_\xi X) - g(A_N X, A^*_\xi Y) - 2d\tau(X, Y).
\end{aligned}$$

Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and $\theta(\xi) = 0$ and using (2.16), we obtain

$$(5.9)\qquad (\bar{\nabla}_X\theta)(U) = -\theta(\nabla_X U), \qquad (\bar{\nabla}_X\theta)(\xi) = \theta(A^*_\xi X).$$

Theorem 5.2. *Let M be a lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type connection $\bar{\nabla}$ such that ζ is tangent to M. If one of the following conditions is satisfied;*

(1) $(\nabla_X F)Y + (\nabla_Y F)X = 0$,

(2) *F is parallel with respect to the induced connection* ∇, *that is,* $\nabla_X F = 0$,

(3) *F is recurrent,*

then $f_1 + f_2 = 0$ *and* $f_2 = 2d\tau(U, V)$.

Proof. If one of the items (1) ∼ (3) is satisfied, then A^*_ξ and A_N satisfy (4.10). Taking the scalar product with U to (4.10) and using (2.14), we have

$$C(X, U) = 0.$$

Applying ∇_X to $C(Y, U) = 0$ and using the last equation, we have

$$(\nabla_X C)(Y, U) = -C(Y, \nabla_X U).$$

Substituting the last two equations into (5.7) with $PZ = U$, we obtain

$$\begin{aligned} C(X, \nabla_Y U) - C(Y, \nabla_X U) = (\bar{\nabla}_X \theta)(U)\{\ell\eta(Y) + mv(Y)\} \\ - (\bar{\nabla}_Y \theta)(U)\{\ell\eta(X) + mv(X)\} \\ + (f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} \end{aligned}$$

Taking $Y = V$ and $X = \xi$ to this and using (4.10) and (5.9), we get

$$C(\xi, \nabla_V U) - C(V, \nabla_\xi U) = \ell\theta(\nabla_V U) + f_1 + f_2.$$

By using (2.14), (4.10) and the fact that $m = \alpha$, we see that

$$\begin{aligned} C(\xi, \nabla_V U) &= g(A_N \xi, \nabla_V U) + \ell\theta(\nabla_V U) = \ell\theta(\nabla_V U), \\ C(V, \nabla_\xi U) &= g(A_N V, \nabla_\xi U) + m\theta(\nabla_\xi U) = -m\theta(\nabla_\xi U) = 0. \end{aligned}$$

From the last three equations, we get $f_1 + f_2 = 0$. Taking $Y = V$ and $X = U$ to (5.8) and using (4.10), we have $f_2 = 2d\tau(U, V)$ □

Definition 5.3. A lightlike hypersurface M is said to be a *Hopf lightlike hypersurface* if the structure vector field U is an eigenvector of A^*_ξ.

Theorem 5.4. *Let M be a lightlike hypersurface of an indefinite nearly generalized Sasakian space form* $\bar{M}(f_1, f_2, f_3)$ *with an* (ℓ, m)*-type connection such that* ζ *is tangent to M and F is Lie recurrent. Then*

$$g(A^*_\xi U, A^*_\xi U) = 3f_2.$$

If M is a Hopf lightlike hypersurface of $\bar{M}(c)$, *then* $f_2 = 0$.

Proof. Taking the scalar product with U to (4.16) and using (2.20), we get

$$B(U, U) = 0. \qquad (5.10)$$

Applying ∇_ξ to (5.10) and using (2.11), (2.13), (4.16) and (4.17), we have

$$(\nabla_\xi B)(U,U) = 2g(A^*_\xi U, A^*_\xi U).$$

Applying ∇_U to $B(\xi,U) = 0$ and using (2.4) and (2.11) $\sim$ (2.13), we have

$$(\nabla_U B)(\xi,U) = g(A^*_\xi U, A^*_\xi U),$$

due to (4.17). Taking $X = \xi$, $Y = U$ and $Z = U$ to (5.6) and using (2.12), (4.17), (5.9), (5.10) and the last two equations, we obtain

$$g(A^*_\xi U, A^*_\xi U) = 3f_2.$$

If M is a Hopf lightlike hypersurface of $\bar{M}(c)$, that is, $A^*_\xi U = \lambda U$ for some smooth function λ, then $g(A^*_\xi U, A^*_\xi U) = 0$. Thus $f_2 = 0$. □

Theorem 5.5. *Let M be a totally umbilical lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type connection such that ζ is tangent to M. Then*

$$f_2 = 0, \qquad d\tau(U,V) = 0.$$

Proof. If M is totally umbilical, then $B = 0$ and $m = 0$ by (1) of Theorem 4.3. As $B = m = 0$ and $S(TM)$ is non-degenerate, (2.13) is reduced

$$A^*_\xi X = 0. \tag{5.11}$$

Taking $X = \xi$ and $Y = Z = U$ to (5.6) and using (4.8), (5.9) and (5.11), we get $f_2 = 0$. Taking $X = U$ and $Y = V$ to (5.8) and using (5.11), we have $d\tau(U,V) = 0$. Thus we have our theorem. □

Theorem 5.6. *Let M be a screen totally umbilical lightlike hypersurface of an indefinite nearly generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type connection such that ζ is tangent to M. Then*

$$\begin{aligned} f_1 &= \ell\theta(\nabla_U V - \nabla_V U) - 2m(m-\alpha),\\ f_2 &= \ell\theta(\nabla_V U - \nabla_U V) + m(m-\alpha),\\ f_3 &= \ell\theta(\nabla_U V - \nabla_V U) - 2m(m-\alpha) - \zeta\ell + \ell^2. \end{aligned}$$

Proof. If M is screen totally umbilical, then $C = 0$ by (2) of Theorem 4.3. As $C = 0$, from (2.11) and (4.8), we have

$$2m = 3\alpha, \quad B(U,\zeta) = \alpha, \quad B(\zeta,U) = \alpha - m, \quad \theta(\nabla_\xi U) = \alpha - m. \tag{5.12}$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and $\theta(V) = 0$, we have

$$(\bar{\nabla}_X\theta)(\zeta) = -\ell\theta(X), \qquad (\bar{\nabla}_X\theta)(V) = -\theta(\nabla_X V), \tag{5.13}$$

due to $\theta(\bar{\nabla}_X\zeta) = \ell\theta(X)$. Taking (1) $X = \xi$, $Y = PZ = \zeta$; (2) $X = \xi$, $Y = U$, $PZ = V$; (3) $X = \xi$, $Y = V$, $PZ = U$ to (5.7) and using (2.19), (5.9), (5.13) and (5.12), we have

$$f_1 - f_3 = \zeta\ell - \ell^2, \qquad f_1 + 2f_2 = -\ell\theta(\nabla_U V),$$

$$f_1 + f_2 = -m(m - \alpha) - \ell\theta(\nabla_V U).$$

From these equations, we have our theorem. □

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