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On Alexander Polynomials of Pretzel Links

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Abstract. In this paper, we will find a Seifert matrix for a class of pretzel links with a certain symmetry. Using the symmetry, we find formulae for the Alexander polynomials, determinants and signatures of the pretzel links.

1. Introduction

A pretzel link $P(p_1, p_2, p_3, \cdots, p_n)$ is defined by an n-tuple $(p_1, p_2, p_3, \cdots, p_n)$, $n \geq 3$, such that each p_i is nonzero integer. The absolute value of p_i is the number of half twists and the sign of p_i is either positive or negative as seen in Fig. 1. Pretzel

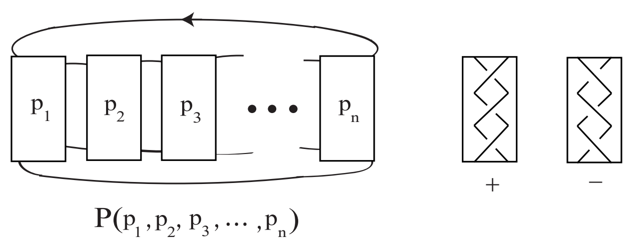

Figure 1:

links are a well-known family of links in knot theory, and they have been studied extensively. J. Ge and L. Zhang [5] used graph theory to study the determinants of pretzel links and Y. Shinohara [9] used the Goreitx matrix to study their signatures. In [8], Y. Nakagawa studied the Alexander polynomials of pretzel links where at

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least two $p_i s$ are even, while D. Kim and J. Lee [7] studied the Conway polynomials of pretzel links. Even though the Alexander polynomial of a link can be obtained from its Conway polynomial, the practical calculation of the Alexander polynomial of a link is very difficult.

Suppose that $P(p_1, p_2, p_3, \cdots, p_n)$ is oriented so that the induced orientation of the tangle p_i is either parallel or opposite, as seen in Fig. 2.

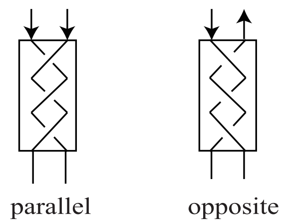

Figure 2:

In this paper, we will use Seifert matrices to find a formula for the Alexander polynomials of pretzel links $P(p_1, p_2, p_3, \cdots, p_n)$ all of whose tangles have opposite orientation. We will also use Seifert matrices to calculate the determinant and the signature of $P(p_1, p_2, p_3, \cdots, p_n)$.

2. Preliminaries

The authors have previously developed techniques for the calculation of the Alexander polynomial. See [1, 2, 3, 4] for details.

A *Seifert surface* for an oriented link L in S^3 is a connected compact oriented surface contained in S^3 which has L as its boundary. The following *Seifert algorithm* is one way to get a Seifert surface from a diagram D of L.

Let D be a diagram of an oriented link L. In a small neighborhood of each crossing, make the following local change to the diagram;

> Delete the crossing and reconnect the loose ends in the only way compatible with the orientation.

When this has been done at every crossing, the diagram becomes a set of disjoint simple loops in the plane. It is a diagram with no crossings. These loops are called *Seifert circles*. By attaching a disc to each Seifert circle and by connecting a half-twisted band at the place of each crossing of D according to the crossing sign, we get a Seifert surface F for L.

The *Seifert graph* Γ of F is constructed as follows.

> Associate a vertex with each Seifert circle and connect two vertices with an edge if their Seifert circles are connected by a twisted band.

Note that the Seifert graph Γ is planar, and that if D is connected, so is Γ. Since Γ is a deformation retract of a Seifert surface F, their homology groups are isomorphic: $H_1(F) \cong H_1(\Gamma)$. Let T be a spanning tree for Γ. For each edge $e \in E(\Gamma) \setminus E(T)$, the graph $T \cup \{e\}$ contains the unique simple closed circuit T_e which represents an 1-cycle in $H_1(F)$. The set $\{T_e \mid e \in E(\Gamma) \setminus E(T)\}$ of these 1-cycles is a homology basis for F. For such a circuit T_e, let T_e^+ denote the circuit in S^3 obtained by lifting slightly along the positive normal direction of F. For $E(\Gamma) \setminus E(T) = \{e_1, \cdots, e_n\}$, the *linking number* between T_{e_i} and $T_{e_j}^+$ is defined by

$$lk(T_{e_i}, T_{e_j}^+) = \frac{1}{2} \sum_{\text{crossing } c \in T_{e_i} \cap T_{e_j}^+} sign(c).$$

A *Seifert matrix* of L associated to F is the $n \times n$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = lk(T_{e_i}, T_{e_j}^+),$$

where $E(\Gamma) \setminus E(T) = \{e_1, \cdots, e_n\}$. A Seifert matrix of L depends on the Seifert surface F and the choice of generators of $H_1(F)$.

Let M be any Seifert matrix for an oriented link L. The *Alexander polynomial* $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$, the *determinant* $\det(L)$ and the *signature* $\sigma(L)$ of L are defined by

$$\begin{aligned} \Delta_L(t) &\doteq \det(t^{\frac{1}{2}} M - t^{-\frac{1}{2}} M^T) \\ \det(L) &= |\det(M + M^T)| \\ \sigma(L) &= \sigma(M + M^T). \end{aligned}$$

See [4, 6] for further details.

For $e, f \in E(\Gamma) \backslash E(T)$, the intersection $T_e \cap T_f$ is either the empty set, a single vertex, or a simple path in the spanning tree T. If $T_e \cap T_f$ is a simple path, and v_0 and v_1 are two ends of $T_e \cap T_f$, we may assume that the neighborhood of v_0 looks like Fig. 3. In other words, the cyclic order of edges incident to v_0 is given by $T_e \cap T_f, T_e, T_f$ with respect to the positive normal direction of the Seifert surface. Also we may assume that the directions of T_e and T_f are given so that v_0 is the starting point of $T_e \cap T_f$. For, if the direction is reversed, one can change the direction to adapt to our setting so that the resulting linking number changes its sign. In [1], the authors showed the following proposition which is the key tool to calculate the linking numbers for Seifert matrix of a link.

Proposition 2.1.([1]) *For $e, f \in E(\Gamma) \backslash E(T)$, let p and q denote the numbers of edges in $T_e \cap T_f$ corresponding to positive crossings and negative crossings, respectively. Suppose that the local shape of $T_e \cap T_f$ in F looks like Fig. 3. Then,*

$$lk(T_e, T_f^+) = \begin{cases} -\frac{1}{2}(p-q), & \text{if } p+q \text{ is even;} \\ -\frac{1}{2}(p-q+1), & \text{if } p+q \text{ is odd, and} \end{cases}$$

$$lk(T_f, T_e^+) = \begin{cases} -\frac{1}{2}(p-q), & \text{if } p+q \text{ is even;} \\ -\frac{1}{2}(p-q-1), & \text{if } p+q \text{ is odd.} \end{cases}$$

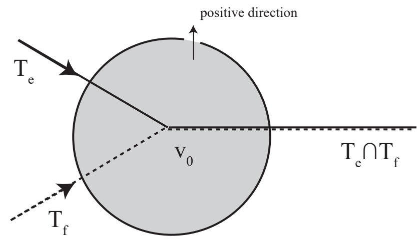

Figure 3:

Let $P(p_1, p_2, p_3, \cdots, p_n)$ denote the pretzel link whose all tangles have opposite orientation. Then the Seifert surface of $P(p_1, p_2, p_3, \cdots, p_n)$ is drawn in Fig. 4. In this case of orientation, we will see that the Seifert matrix has very nice symmetry, in which Viète's formula (from algebra) can be applied.

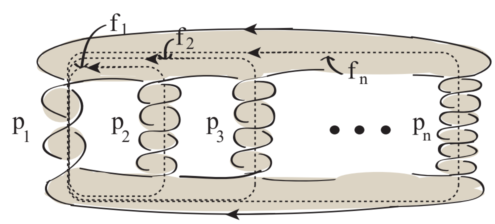

Figure 4:

From now on, we suppose that the Seifert surface of $P(p_1, p_2, p_3, \cdots, p_n)$ is depicted as in Fig. 4. In order for a Seifert surface of $P(p_1, p_2, p_3, \cdots, p_n)$ to be drawn as Fig. 4, the orientations of p_i for all i, $1 \leq i \leq n$ must be opposite. To do this, the p_is must be either all odd or all even. Because if there exist $i \in \{1, 2, \cdots, n-1\}$ such that p_i is odd and p_{i+1} is even, then the Seifert circles of $P(p_1, p_2, p_3, \cdots, p_n)$ are depicted as in Fig. 5.

To calculate the Alexander polynomial of a pretzel link $P(p_1, p_2, p_3, \cdots, p_n)$, we introduce that Viète's formula:

Proposition 2.2.([Viète's formula]) *Let* $f(x) = x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0$ *be a polynomial of degree* $n-1$ *and let* $x_1, x_2, \cdots, x_{n-1}$ *be roots of the equation* $f(x) = 0$*. Then the relation between coefficients of* $f(x)$ *and its roots are related to*

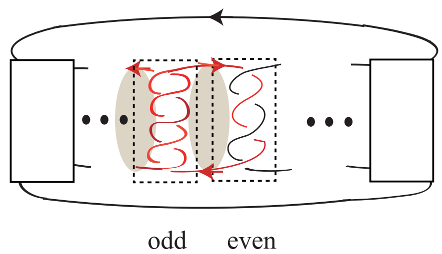

Figure 5:

symmetric polynomial expression:

$$\prod_k (x_1, x_2, \cdots, x_{n-1}) = x_1 x_2 \cdots x_k + \cdots + x_{n-k} x_{n-k+1} \cdots x_{n-1} = (-1)^k C_{n-1-k}.$$

The Alexander polynomial $\Delta_P(t)$ of a pretzel link $P(p_1, p_2, p_3, \cdots, p_n)$ can be expressed as $f(x) = x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0$ where we think of $\Delta_{P(p_1,p_2)}(t)$, $\Delta_{P(p_1,p_3)}(t)$, $\cdots$, $\Delta_{P(p_1,p_n)}(t)$ as roots and think of $\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k})}(t)$ as x^k.

Notice that the signs of the coefficents are always positive, e.g., $\Delta_{P(3,-5,5)}(t) = \Delta_{P(3,-3,-3)}(t) + \{\Delta_{P(3,-5)}(t) + \Delta_{P(3,5)}(t)\}\Delta_{P(3,-3)}(t) + \Delta_{P(3,-5)}(t)\Delta_{P(3,5)}(t)$.

3. Seifert Matrices of Pretzel Links and Related Invariants

Lemma 3.1. *Let $P = P(p_1, p_2, \cdots, p_n)$ be a pretzel link. Suppose that the Seifert surface of the pretzel link P looks like Fig. 4. Then there exist a Seifert matrix M of the pretzel link P such that if p_1 is odd, then a Seifert matrix M of the pretzel link P is given by*

$$M = \frac{1}{2}\begin{pmatrix} p_1+p_2 & p_1-1 & p_1-1 & \cdots & p_1-1 \\ p_1+1 & p_1+p_3 & p_1-1 & \cdots & p_1-1 \\ p_1+1 & p_1+1 & p_1+p_4 & \cdots & p_1-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1+1 & p_1+1 & p_1+1 & \cdots & p_1+p_n \end{pmatrix}_{(n-1)\times(n-1)},$$

and if p_1 is even, then a Seifert matrix M of the pretzel link P is given by

$$\frac{1}{2}\begin{pmatrix} p_1+p_2 & p_1 & p_1 & \cdots & p_1 \\ p_1 & p_1+p_3 & p_1 & \cdots & p_1 \\ p_1 & p_1 & p_1+p_4 & \cdots & p_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & p_1 & \cdots & p_1+p_n \end{pmatrix}_{(n-1)\times(n-1)}.$$

Proof. If we choose the oriented simple closed curves $f_1, f_2, \cdots, f_{n-1}$ shown in Fig. 4 as the basis of $H_1(F, \mathbb{Z})$ where F is the Seifert surface of the pretzel link $P(p_1, p_2, p_3, \cdots, p_n)$, then by using Proposition 2.1 one can calculate the linking numbers to get the result. The proof is completed. □

Theorem 3.2. *Let* $P = P(p_1, \underbrace{-p_1, \cdots, -p_1}_{n-1})$ *be a pretzel link. Suppose that the Seifert surface of the pretzel link* P *looks like Fig. 4. Then the determinant* $\det(P)$ *and the signature* $\sigma(P)$ *of the pretzel link* P *are given by*

$$\det(P) = (n-2)|p_1|^{n-1},$$
$$\sigma(P) = \begin{cases} -n+3, & \text{if } p_1 > 0; \\ n-3, & \text{if } p_1 < 0. \end{cases}$$

Proof. From Lemma 3.2, we know that

$$M + M^T = \begin{pmatrix} 0 & p_1 & p_1 & \cdots & p_1 \\ p_1 & 0 & p_1 & \cdots & p_1 \\ p_1 & p_1 & 0 & \cdots & p_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & p_1 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

Hence $\det(M + M^T) = (-1)^n(n-2)p_1^{n-1}$ by the formula (1) in Appendix A. Since $n - 2 > 0$, $\det(P) = (n-2)|p_1|^{n-1}$. The characteristic equation of $M + M^T$ to be

$$\det((M + M^T) - \lambda I) = \det \begin{pmatrix} -\lambda & p_1 & p_1 & \cdots & p_1 \\ p_1 & -\lambda & p_1 & \cdots & p_1 \\ p_1 & p_1 & -\lambda & \cdots & p_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & p_1 & \cdots & -\lambda \end{pmatrix}_{(n-1)\times(n-1)}$$
$$= (-\lambda - p_1)^{n-2}\{p_1(n-1) + (-\lambda - p_1)\} = 0$$

by the formula (3) in Appendix A. Thus the eigenvalues of $M + M^T$ are $\lambda_1 = p_1(n-2)$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = -p_1$. If p_1 is positive, then $-p_1$ is negative and $p_1(n-2)$ is positive since $n > 3$. Hence the signature of $M + M^T$ is $3 - n$. Similarly, if p_1 is negative, then the signature of $M + M^T$ is $n - 3$. The proof is completed. □

Theorem 3.3. *Let* $P = P(p_1, \underbrace{-p_1, \cdots, -p_1}_{n-1})$ *be a pretzel link. Suppose that the Seifert surface of the pretzel link* P *looks like Fig. 4. Then the Alexander polynomials*

$\Delta_P(t)$ of P are given by

$$\Delta_P(t) = \begin{cases} (-1)^{n-1}\Delta_{P(p_1,-p_1,-p_1)}(t) \\ \quad \times \dfrac{\{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2})^{n-2} - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})^{n-2}\}}{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2}) - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})}, & \text{if } p_1 \text{ is odd;} \\ (-1)^{n-1}(n-2)\Delta_{P(p_1,-p_1,-p_1)}(t)(t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2})^{n-3}, & \text{if } p_1 \text{ is even.} \end{cases}$$

Proof. Suppose that p_1 is odd. From Lemma 3.1, we know that $t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T =$

$$\begin{pmatrix} 0 & t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2} & \cdots & t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2} \\ t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2} & 0 & \cdots & t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2} & t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2} & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

By the formula (1) in Appendix A,

$$\begin{aligned} \det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T) &= (-1)^{n-2}(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2})(t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2}) \\ &\quad \times \frac{\{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2})^{n-2} - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})^{n-2}\}}{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2}) - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})} \\ &= (-1)^{n-1}\Delta_{P(p_1,-p_1,-p_1)}(t) \\ &\quad \times \frac{\{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2})^{n-2} - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})^{n-2}\}}{(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2}) - (t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2})} \end{aligned}$$

since

$$\begin{aligned} \Delta_{P(p_1,-p_1,-p_1)}(t) &= \det\begin{pmatrix} 0 & t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2} \\ t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2} & 0 \end{pmatrix} \\ &= (-1)(t^{\frac{1}{2}}\frac{p_1-1}{2} - t^{-\frac{1}{2}}\frac{p_1+1}{2})(t^{\frac{1}{2}}\frac{p_1+1}{2} - t^{-\frac{1}{2}}\frac{p_1-1}{2}). \end{aligned}$$

Suppose that p_1 is even. From Lemma 3.1, we know that $t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T =$

$$\begin{pmatrix} 0 & t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} & \cdots & t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} \\ t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} & 0 & \cdots & t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} & t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

By the formula (1) in Appendix A,

$$\begin{aligned} \det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T) &= (-1)^{n-2}(n-2)(t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2})^{n-1} \\ &= (-1)^{n-1}(n-2)\Delta_{P(p_1,-p_1,-p_1)}(t)(t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2})^{n-3} \end{aligned}$$

since

$$\Delta_{P(p_1,-p_1,-p_1)}(t) = \det\begin{pmatrix} 0 & t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} \\ t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2} & 0 \end{pmatrix}$$
$$= (-1)(t^{\frac{1}{2}}\frac{p_1}{2} - t^{-\frac{1}{2}}\frac{p_1}{2})^2.$$

The proof is completed. □

Corollary 3.4. *Let $P = P(p_1, p_2, p_3, \cdots, p_n)$ be a pretzel link ($n \geq 3$). If the Seifert surface of pretzel link P is shown in Fig 4, then the determinant* $\det(P)$ *of P is given by*

$$\det(P) = \left| p_2 p_3 \cdots p_n \left\{ \frac{p_1}{p_n}(n-2) + \frac{p_1}{p_2} + 1 \right\} \right|.$$

Proof. From the definition of a link and Lemma 3.1, we can prove it by using the formula (3) in Appendix A. The proof is completed. □

To prove the main theorem, we show the following lemma.

Lemma 3.5. *Suppose that the Seifert surface of the pretzel link P looks like Fig. 4.*

(1) $\Delta_{P(p_1)}(t) = \Delta_O(t) = 1$.

(2) $\Delta_{P(p_1,-p_1)}(t) = \Delta_{OO}(t) = 0$.

(3) $\Delta_{P(p_1,-p_1,-p_k)}(t) = \Delta_{P(p_1,-p_1,-p_1)}(t)$, *for any* $k = 1, 2, \cdots, n$.

(4) $\Delta_{P(p_1,p_2,\cdots,p_n)}(t) = \Delta_{P(p_1,p_i)}(t)\Delta_{P(p_1,p_2,\cdots,p_{i-1},p_{i+1},\cdots,p_n)}(t)$
$+ \Delta_{P(p_1,p_2,\cdots,p_{i-1},-p_i,p_{i+1},\cdots,p_n)}(t)$.

Proof. (1) and (2) are trivial.
(3) Suppose that p_1 is odd. Then a Seifert matrix M_L of $P(p_1, -p_1, -p_k)$ and a Seifert matrix M_R of $P(p_1, -p_1, -p_1)$ are given by $M_L = \begin{pmatrix} 0 & \frac{p_1-1}{2} \\ \frac{p_1+1}{2} & \frac{p_1+p_k}{2} \end{pmatrix}$ and $M_R = \begin{pmatrix} 0 & \frac{p_1-1}{2} \\ \frac{p_1+1}{2} & 0 \end{pmatrix}$ if p_1 is odd. For p_1 is even, it is similar for p_1 is odd.

(4) The basic idea of determining the determinant of matrix is as

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & a_{1i} & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & a_{2i} & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1i-1} & a_{i-1i} & a_{i-1i+1} & \cdots & a_{i-1n} \\ a_{i1} & a_{i2} & \cdots & a_{ii-1} & a_{ii} & a_{ii+1} & \cdots & a_{i-1n} \\ a_{i+11} & a_{i+22} & \cdots & a_{i+1i-1} & a_{i+1i} & a_{i+1i+1} & \cdots & a_{i+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & a_{ni} & a_{ni+1} & \cdots & a_{nn} \end{pmatrix}$$

$$= a_{ii}\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1i-1} & a_{i-1i+1} & \cdots & a_{i-1n} \\ a_{i+11} & a_{i+22} & \cdots & a_{i+1i-1} & a_{i+1i+1} & \cdots & a_{i+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & a_{ni+1} & \cdots & a_{nn} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & a_{1i} & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & a_{2i} & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1i-1} & a_{i-1i} & a_{i-1i+1} & \cdots & a_{i-1n} \\ a_{i1} & a_{i2} & \cdots & a_{ii-1} & \mathbf{0} & a_{ii+1} & \cdots & a_{i-1n} \\ a_{i+11} & a_{i+22} & \cdots & a_{i+1i-1} & a_{i+1i} & a_{i+1i+1} & \cdots & a_{i+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & a_{ni} & a_{ni+1} & \cdots & a_{nn} \end{pmatrix}.$$

The proof is completed. □

The following is the main theorem of the paper.

Theorem 3.6. *Let $P = P(p_1, p_2, p_3, \cdots, p_n)$ be a pretzel link ($n \geq 3$). If the Seifert surface of the pretzel link P is shown in Fig. 4, then the Alexander polynomial $\Delta_P(t)$ of P is given by*

$$\sum_{k=1}^{n}\left\{\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k-1})}(t) \times \prod_{n-k}(\Delta_{P(p_1,p_2)}(t), \Delta_{P(p_1,p_3)}(t), \cdots, \Delta_{P(p_1,p_n)}(t))\right\}.$$

Proof. We divide our proof into two cases (Case 1) All of p_i are odd, and (Case 2) All of p_i are even.

(Case 1) All of p_i are odd. By the definition of the Alexander polynomial of a link and by Lemma 3.1, we have the Alexander polynomial of P is given by

$$\det\begin{pmatrix} (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\frac{p_1+p_2}{2} & t^{\frac{1}{2}}\frac{p_1-1}{2}-t^{-\frac{1}{2}}\frac{p_1+1}{2} & \cdots & t^{\frac{1}{2}}\frac{p_1-1}{2}-t^{-\frac{1}{2}}\frac{p_1+1}{2} \\ t^{\frac{1}{2}}\frac{p_1+1}{2}-t^{-\frac{1}{2}}\frac{p_1-1}{2} & (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\frac{p_1+p_3}{2} & \cdots & t^{\frac{1}{2}}\frac{p_1-1}{2}-t^{-\frac{1}{2}}\frac{p_1+1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ t^{\frac{1}{2}}\frac{p_1+1}{2}-t^{-\frac{1}{2}}\frac{p_1-1}{2} & t^{\frac{1}{2}}\frac{p_1+1}{2}-t^{-\frac{1}{2}}\frac{p_1-1}{2} & \cdots & (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\frac{p_1+p_n}{2} \end{pmatrix}.$$

We proceed by the mathematical induction on $n (n \geq 3)$. For $n = 3$,

$$\begin{aligned}
\Delta_{P(p_1,p_2,p_3)}(t) &= \det\begin{pmatrix} \frac{t^{\frac{1}{2}}(p_1+p_2)-t^{-\frac{1}{2}}(p_1+p_2)}{2} & \frac{t^{\frac{1}{2}}(p_1-1)-t^{-\frac{1}{2}}(p_1+1)}{2} \\ \frac{t^{\frac{1}{2}}(p_1+1)-t^{-\frac{1}{2}}(p_1-1)}{2} & \frac{t^{\frac{1}{2}}(p_1+p_3)-t^{-\frac{1}{2}}(p_1+p_3)}{2} \end{pmatrix} \\
&= \det\left(\frac{t^{\frac{1}{2}}(p_1+p_2)-t^{-\frac{1}{2}}(p_1+p_2)}{2}\right)\det\left(\frac{t^{\frac{1}{2}}(p_1+p_3)-t^{-\frac{1}{2}}(p_1+p_3)}{2}\right) \\
&\quad + \det\begin{pmatrix} \frac{t^{\frac{1}{2}}(p_1+p_2)-t^{-\frac{1}{2}}(p_1+p_2)}{2} & \frac{t^{\frac{1}{2}}(p_1-1)-t^{-\frac{1}{2}}(p_1+1)}{2} \\ \frac{t^{\frac{1}{2}}(p_1+1)-t^{-\frac{1}{2}}(p_1-1)}{2} & 0 \end{pmatrix}, \\
&\quad \text{by Lemma 3.5(4)} \\
&= \Delta_{P(p_1,p_2)}(t)\Delta_{P(p_1,p_3)}(t) + \Delta_{P(p_1,p_2,-p_1)}(t) \\
&= \Delta_{P(p_1,p_2)}(t)\Delta_{P(p_1,p_3)}(t) + \Delta_{P(p_1,-p_1,-p_1)}(t) \text{ by Lemma 3.5(3).} \\
&= \Delta_{P(p_1,p_2)}(t)\Delta_{P(p_1,p_3)}(t) + \{\Delta_{P(p_1,p_2)}(t) + \Delta_{P(p_1,p_3)}(t)\}\Delta_{P(p_1,-p_1)}(t) \\
&\quad + \Delta_{P(p_1,-p_1,-p_1)}(t), \text{ by Lemma 3.5(2).}
\end{aligned}$$

Assume that the formula is true for $n-1$.

$$\begin{aligned}
\Delta_{P(p_1,p_2,\cdots,p_n)}(t) &= \Delta_{P(p_1,p_n)}(t)\Delta_{P(p_1,p_2,\cdots,p_{n-1})}(t) + \Delta_{P(p_1,p_2,\cdots,p_{n-1},-p_1)}(t) \\
&= \Delta_{P(p_1,p_n)}(t)\Delta_{P(p_1,p_2,\cdots,p_{n-1})}(t) + \Delta_{P(p_1,p_{n-1})}(t) \\
&\quad \Delta_{P(p_1,p_2,\cdots,p_{n-2},-p_1)}(t)\Delta_{P(p_1,p_2,\cdots,p_{n-2},-p_1,-p_1)}(t), \\
&\quad \text{by Lemma 3.5(4).}
\end{aligned}$$

By applying the identity Lemma 3.5(4) to the last polynomial $\Delta_{P(p_1,p_2,\cdots,p_{n-2},-p_1,-p_1)}(t)$ repeatedly, we get the following result.

$$\begin{aligned}
&\Delta_{P(p_1,p_2,\cdots,p_n)}(t) \\
&= \sum_{i=2}^{n} \Delta_{P(p_1,p_i)}(t)\Delta_{P(p_1,p_2,\cdots,p_{i-1},\underbrace{-p_1,\cdots,-p_1}_{n-i})}(t) + \Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{n-1})}(t).
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{n} \Delta_{P(p_1,p_i)}(t) \sum_{k=1}^{n-1} \{\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k-1})}(t) \\
&\quad \times \prod_{n-1-k} (\Delta_{P(p_1,p_2)}(t),\cdots,\Delta_{P(p_1,p_{i-1})}(t),\underbrace{\Delta_{P(p_1,-p_1)}(t),\cdots,\Delta_{P(p_1,-p_1)}(t)}_{n-i})\} \\
&\quad + \Delta_{P(p_1,\underbrace{-p_1,\cdots,,-p_1}_{n-1})}(t) \\
&= \sum_{k=1}^{n-1} \{\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k-1})}(t) \sum_{i=2}^{n} \Delta_{P(p_1,p_i)}(t) \\
&\quad \times \prod_{n-1-k} (\Delta_{P(p_1,p_2)}(t),\cdots,\Delta_{P(p_1,p_{i-1})}(t),\underbrace{\Delta_{P(p_1,-p_1)}(t),\cdots,\Delta_{P(p_1,-p_1)}(t)}_{n-i})\} \\
&\quad + \Delta_{P(p_1,\underbrace{-p_1,\cdots,,-p_1}_{n-1})}(t) \\
&= \sum_{k=1}^{n-1} \{\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k-1})}(t) \times \prod_{n-k} (\Delta_{P(p_1,p_2)}(t),\cdots,\Delta_{P(p_1,p_n)}(t))\} \\
&\quad + \Delta_{P(p_1,\underbrace{-p_1,\cdots,,-p_1}_{n-1})}(t) \\
&= \sum_{k=1}^{n} \{\Delta_{P(p_1,\underbrace{-p_1,\cdots,-p_1}_{k-1})}(t) \prod_{n-k} (\Delta_{P(p_1,p_2)}(t),\Delta_{P(p_1,p_3)}(t),\cdots,\Delta_{P(p_1,p_n)}(t))\}.
\end{aligned}
$$

(Case 2) All of p_i are even, is similar to the proof of (Case 1). The proof is completed. □

Example 3.7 Let $P(3,-5,5)$ be a pretzel link. Then a Seifert matrix of $P(3,-5,5)$ is given by $M = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix}$. By direct calculation, one can see that $\Delta_{P(3,-5,5)}(t) = -6t^2 + 13t - 6$.

And by using the main theorem, one can get the same result.

$$
\begin{aligned}
\Delta_{P(3,-5,5)}(t) &= \Delta_{P(3,-5)}(t)\Delta_{P(3,5)}(t) + \Delta_{P(3,-3)}(t)\{\Delta_{P(3,-5)}(t) + \Delta_{P(3,5)}(t)\} \\
&\quad + \Delta_{P(3,-3,-3)}(t) \\
&= (-t+1)(4t-4) + \frac{(-1)^2}{t+1}\left(\frac{(3-1)t-(3+1)}{2}\right)\left(\frac{(3+1)t-(3-1)}{2}\right) \\
&\quad \times \left\{\left(\frac{(3-1)t-(3+1)}{2}\right) - \left(\frac{(3+1)t-(3-1)}{2}\right)\right\} \\
&= -6t^2 + 13t - 6.
\end{aligned}
$$

Appendix A

The following formulae for the determinants of matrices are the key tools for the calculation of the determinant, the signature and the Alexander polynomial of pretzel links. We leave a proof in Appendix A. It may be proven somewhere in the linear algebra because is can be proved using mathematical induction.

(1) For an integer $n (n \geq 2)$,

$$\det \begin{pmatrix} 0 & a & a & \cdots & a & a \\ b & 0 & a & \cdots & a & a \\ b & b & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & 0 & a \\ b & b & b & \cdots & b & 0 \end{pmatrix}_{n\times n} = \begin{cases} \dfrac{(-1)^{n-1}ab(a^{n-1}-b^{n-1})}{a-b}, & \text{if } a \neq b; \\ (-1)^{n-1}(n-1)a^n, & \text{if } a = b. \end{cases}$$

Proof. We can prove inductively the lemma by the following recurrence formula. Let

$$f(n) = \det \begin{pmatrix} 0 & a & a & \cdots & a & a \\ b & 0 & a & \cdots & a & a \\ b & b & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & 0 & a \\ b & b & b & \cdots & b & 0 \end{pmatrix}_{n\times n.}$$

Add the (-1)(the 2nd column) to the 1st column and then, add the (-1)(the 2nd row) to the 1st row. Hence

$$f(n) = \det \begin{pmatrix} -a-b & a & 0 & \cdots & 0 & 0 \\ b & 0 & a & \cdots & a & a \\ 0 & b & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & b & \cdots & 0 & a \\ 0 & b & b & \cdots & b & 0 \end{pmatrix}_{n\times n.}$$

$$f(n+2) = (-a-b)\det \begin{pmatrix} 0 & a & \cdots & a & a \\ b & 0 & \cdots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & 0 & a \\ b & b & \cdots & b & 0 \end{pmatrix} - a\det \begin{pmatrix} b & a & \cdots & a & a \\ 0 & 0 & \cdots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \cdots & 0 & a \\ 0 & b & \cdots & b & 0 \end{pmatrix}$$

$$= (-a-b)\det\begin{pmatrix} 0 & a & \cdots & a & a \\ b & 0 & \cdots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & 0 & a \\ b & b & \cdots & b & 0 \end{pmatrix} - ab\det\begin{pmatrix} 0 & \cdots & a & a \\ \vdots & \ddots & \vdots & \vdots \\ b & \cdots & 0 & a \\ b & \cdots & b & 0 \end{pmatrix}$$
$$= (-a-b)\cdot f(n+1) - ab\cdot f(n).$$

Suppose that $a \neq b$. Then $f(n+2)-(a+b)f(n+1)-abf(n)=0$. Since $f(2)=-ab$, $f(3)=ab(a+b)$ and $x^2+(a+b)x+ab=0$ has two roots $-a$ and $-b$,

$$f(n) = \frac{-b}{a-b}(-a)^n + \frac{a}{a-b}(-b)^n = \frac{(-1)^{n-1}ab(a^{n-1}-b^{n-1})}{a-b}.$$

If $a=b$, then $f(n+2)-(2a)\cdot f(n+1)-a^2 f(n)=0$. Since $f(2)=-a^2$, $f(3)=2a^3$ and $x^2+(2a)x+a^2=0$ has multiple root $-a$,

$$f(n) = (-a)^n - n(-a)^n = (-1)^{n+1}(n-1)a^n. \qquad \square$$

(2) Let n be an integer ($n \geq 2$). If $a_i \neq b$ for all i, then

$$\det\begin{pmatrix} a_1 & b & b & \cdots & b & b \\ b & a_2 & b & \cdots & b & b \\ b & b & a_3 & \cdots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & a_{n-1} & b \\ b & b & b & \cdots & b & a_n \end{pmatrix} = \left\{\prod_{i=1}^{n}(a_i-b)\right\}\left(\frac{bn}{a_n-b}+\frac{a_1}{a_1-b}-\frac{b}{a_n-b}\right).$$

Proof. We can prove the following recurrence formula. Add the (-1)(the first column) to the kth column for any $k=2,3\cdots,n$. Then,

$$\det\begin{pmatrix} a_1 & b & b & \cdots & b & b \\ b & a_2 & b & \cdots & b & b \\ b & b & a_3 & \cdots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & a_{n-1} & b \\ b & b & b & \cdots & b & a_n \end{pmatrix} = \det\begin{pmatrix} a_1 & b-a_1 & b-a_1 & \cdots & b-a_1 \\ b & a_2-b & 0 & \cdots & 0 \\ b & 0 & a_3-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & a_n-b \end{pmatrix}.$$

$$\text{Let } f(n) = \det\begin{pmatrix} a_1 & b-a_1 & b-a_1 & \cdots & b-a_1 & b-a_1 \\ b & a_2-b & 0 & \cdots & 0 & 0 \\ b & 0 & a_3-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & 0 & \cdots & a_{n-1}-b & 0 \\ b & 0 & 0 & \cdots & 0 & a_n-b \end{pmatrix}_{n\times n.}$$

$$f(n) = (a_n - b)\det\begin{pmatrix} a_1 & b-a_1 & \cdots & b-a_1 & b-a_1 \\ b & a_2-b & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & \cdots & a_{n-2}-b & 0 \\ b & 0 & \cdots & 0 & a_{n-1}-b \end{pmatrix}$$
$$+ (-1)^{n+1} b \det\begin{pmatrix} b-a_1 & b-a_1 & \cdots & b-a_1 & b-a_1 \\ a_2-b & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}$$
$$= (a_n - b) f(n-1) + b(a_1-b)(a_2-b)\cdots(a_{n-1}-b).$$

Then $\dfrac{f(n)}{(a_1-b)(a_2-b)\cdots(a_n-b)} = \dfrac{f(n-1)}{(a_1-b)(a_2-b)\cdots(a_{n-1}-b)} + \dfrac{b}{a_n-b}$.
Let $g(n) = \dfrac{f(n)}{(a_1-b)(a_2-b)\cdots(a_n-b)}$. Then $g(n) = g(n-1) + \dfrac{b}{a_n-b}$. Since $g(1) = \dfrac{f(1)}{a_1-b} = \dfrac{a_1}{a_1-b}$, $g(n) = \dfrac{bn}{a_n-b} + \dfrac{a_1}{a_1-b} - \dfrac{b}{a_n-b}$. Hence $f(n) = (a_1 - b)(a_2-b)\cdots(a_n-b)\left(\dfrac{bn}{a_n-b} + \dfrac{a_1}{a_1-b} - \dfrac{b}{a_n-b}\right)$. □

(3) If all a_i are the same, then we have the following result. For an integer $n (n \geq 2)$,

$$\det\begin{pmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix}_{n\times n} = \begin{cases} (a-b)^{n-1}\{bn + (a-b)\}, & \text{if } a \neq b; \\ 0, & \text{if } a = b,. \end{cases}$$

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