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Divide Knot Presentation of Knots of Berge's Sporadic Lens Space Surgery

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Abstract. Divide knots and links, defined by A'Campo in the singularity theory of complex curves, is a method to present knots or links by real plane curves. The present paper is a sequel of the author's previous result that every knot in the major subfamilies of Berge's lens space surgery (i.e., knots yielding a lens space by Dehn surgery) is presented by an L-shaped curve as a divide knot. In the present paper, L-shaped curves are generalized and it is shown that every knot in the minor subfamilies, called sporadic examples of Berge's lens space surgery, is presented by a generalized L-shaped curve as a divide knot. A formula on the surgery coefficients and the presentation is also considered.

1. Introduction

If r Dehn surgery on a knot K in S^3 yields a lens space, we call the pair (K, r) a *lens space surgery*, and we also say that K admits a lens space surgery, and that r is the *coefficient* of the lens space surgery. The task of classifying lens space surgeries, especially knots that admit lens space surgeries has been a focal point in low-dimensional topology. In 1990, Berge [9] pointed out a "mechanism" of known lens space surgeries, that is, *doubly-primitive knots* in the Heegaard surface of genus 2. Berge also gave a conjecturally complete list of such knots, described them by Osborne–Stevens's "R-R diagrams" in [22], and classified such knots into three families, and into 12 types in detail:

(1) *Knots in a solid torus (Berge–Gabai knots)* : TypeI, II, ... and VI (Berge [10]).

Dehn surgery along a knot in a solid torus whose resulting manifold is also a solid torus. TypeI consists of torus knots. TypeII consists of 2-cable of torus

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knots.

(2) *Knots in a genus-one fiber surface* : TypeVII and VIII (see Baker [6, 8] and also [24, 28]).

(3) *Sporadic examples (a), (b), (c) and (d)* : TypeIX, X, XI and XII, respectively.

Their surgery coefficients are also decided. They are called *Berge's lens space surgeries*. The numbering VII–XII (after VI) are used by Baker [7, 8]. It is conjectured by Gordon [17, 18] that every knot of lens space surgery is a doubly-primitive knot. Recently, Greene [19] proved completeness of Berge's list as a list of doubly-primitive knots.

In the present paper, we are concerned with the minor family (3). It is known that TypeIX and TypeXII (Berge's (a) and (d)) are related, and that TypeX and TypeXI (Berge's (b) and (c)) are related. Thus our targets are TypeIX and TypeX.

Notation 1.1. Throughout the paper, we let Type$\mathfrak{X}$ denote either TypeIX or TypeX, i.e., $\mathfrak{X}$ = IX or X. Knots in Type$\mathfrak{X}$ are parametrized by an integer j with $j \neq 0, -1$ ([9]), thus we call the knots $k_{IX}(j)$ for TypeIX, and $k_X(j)$ for TypeX, respectively. The knot $k_{\mathfrak{X}}(j)$ is obtained from a torus knot by full-twists twice from a torus knot. For the precise construction of them, see Section 3.

Berge's original classification (a)–(d) of sporadic examples in [9] was

$$\begin{array}{ll} \text{(a)}\ \ k_{IX}(j) \text{ with } j > 0, & \text{(b)}\ \ k_X(j) \text{ with } j > 0, \\ \text{(c)}\ \ k_X(j)! \text{ with } j < -1, & \text{(d)}\ \ k_{IX}(j)! \text{ with } j < -1, \end{array}$$

respectively, see Deruelle–Miyaszaki–Motegi's works [13, 14], where $K!$ means the mirror image of a knot K.

The theory of A'Campo's *divide knots and links* came from singularity theory of complex curves. A *divide* is originally a relative, generic immersion of a 1-manifold in the unit disk in $\mathbb{R}^2$, see Section 2. A'Campo [1, 2, 3, 4] formulated the way to associate to each divide P a link $L(P)$ in S^3. In the present paper, we regard a PL (piecewise linear) plane curve as a divide by smoothing the corners and controlling the size. Let X be the $\pi/4$-lattice defined by $\{(x, y) | \cos \pi x = \cos \pi y\}$ in xy-plane. In this paper, we are interested in plane curves constructed as intersection of X and regions. When the region is L-shaped, we call such a curve *L-shaped curve*. See Figure 1, which is the starting examples of our project (Question 1.2). Two L-shaped curves present a same knot, the pretzel knot of type $(-2, 3, 7)$ as a divide knot. Its 18-surgery and 19-surgery are lens spaces (by Fintushel–Stern [15]). Note that the areas of $\mathcal{L}$ are equal to the coefficients of the lens space surgeries. They have different mechanism of lens space surgeries: 18-surgery is in TypeIII, 19-surgery is in TypeVII.

We are concerned with the following question:

Question 1.2. *Is every knot of (Berge's) lens space surgeries a divide knot?*

Our main theorem is:

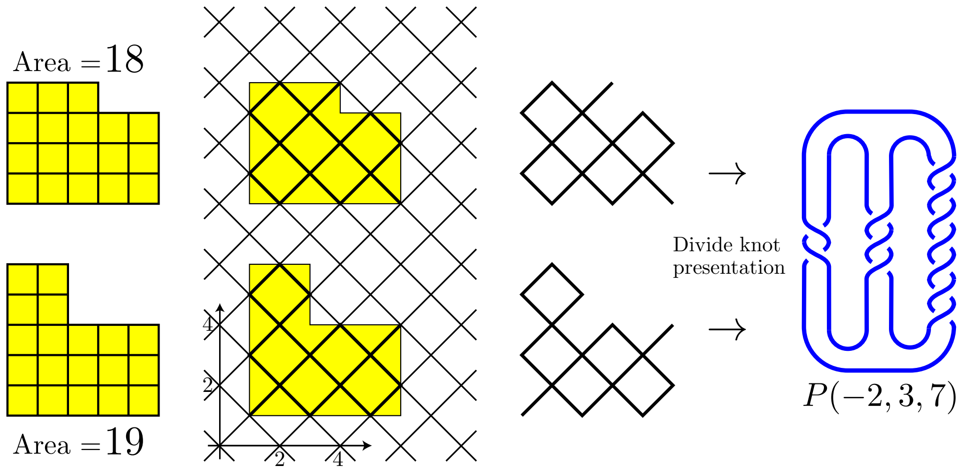


Figure 1: Divide knot presentation of $P(-2,3,7)$ (see [24, 27])

Theorem 1.3. *Up to mirror image, every knot in TypeIX and TypeX in Berge's list of lens space surgery is a divide knot.*

In fact, we will construct plane curves $P_{\mathcal{X}}(j)$ that present the knots $k_{\mathcal{X}}(j)$ for $\mathcal{X} = \mathrm{IX}$ and X, $j \in \mathbb{Z}$ with $j \neq 0, -1$. They are a little complicated (generalized) L-shaped curves. In the next subsection, we roughly review the knots $k_{\mathcal{X}}(j)$ of Berge's TypeIX and TypeX and explain how to construct $P_{\mathcal{X}}(j)$. Theorem 1.5 is a more precise version of Theorem 13. We will also study the relation between the coefficients of lens space surgeries and the divide presentations.

We postpone the strict definition of $P_{\mathcal{X}}(j)$ (and $P(j)$) until Definition 3.1, after the definition of generalized L-shaped curves and the notion "type" of L-shaped curves in Section 2.

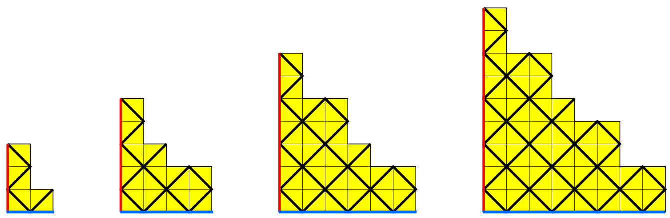
1.1. Presentation of the Sporadic Knots

First, we review the knots $k_{\mathcal{X}}(j)$ of Berge's TypeIX and TypeX ($\mathcal{X} = \mathrm{IX}$ or X, $j \in \mathbb{Z}$ and $j \neq 0, -1$). They are called knots of sporadic lens space surgeries. Table 1 is a list of some data on $k_{\mathcal{X}}(j)$: the coefficients p of their lens space surgeries of $k_{\mathcal{X}}(j)$ as Type$\mathcal{X}$, the second parameter q of the resulting lens space $L(p,q)$ and the genus of $k_{\mathcal{X}}(j)$, which depends on the sign of j. Our convention about orientations of lens spaces is "p/q Dehn surgery along an unknot is $-L(p,q)$".

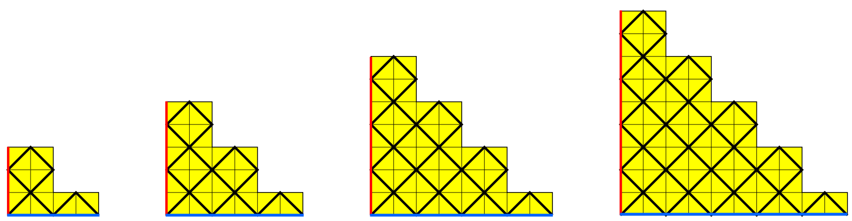
Before a more precise version of the main theorem, we show some plane curves, see $P(j)$ in Figure 2. Each curve $P(j)$ is constructed as intersection of X and a region that consists of some rectangles sharing the left bottom corner. We call such a curve *a generalized L-shaped curve*, or *an L-shaped curve* for short. The precise

knot	p (= coefficient)	q of $L(p,q)$	genus ($j>0$)	genus ($j<-1$)
$k_{IX}(j)$	$22j^2+9j+1$	$-(11j+2)^2$	$11j^2$	$11j^2+9j+2$
$k_X(j)$	$22j^2+13j+2$	$-(11j+3)^2$	$11j^2+2j$	$11j^2+11j+3$

Table 1: Data on Sporadic knots



Case $j>0$: $P(1), P(2), P(3), P(4)$



Case $j<-1$: $P(-2), P(-3), P(-4), P(-5)$

Figure 2: Plane curves $P(j)$, see Definition 3.1

definition and some notations will be given in Section 2. See Definition 3.1 for the strict definition of $P(j)$ as L-shaped curves. For example, the curve $P(3)$ will be defined as an L-shaped curve of type $[(1,7),(3,5),(4,3),(6,2)]$. We define plane curves $P_{\mathcal{X}}(j)$ from $P(j)$ by *adding a square* twice in the sense [27, Lemma 4.2] as follows.

Definition 1.4.(Plane curves $P_{\mathcal{X}}(j)$) We let E_a (and E_b, respectively) denote the bottom edge (and the left edge) of the region of $P(j)$, see Subsection 3.1 for the precise definition. For an integer j with $j \neq 0, -1$, depending on whether $\mathcal{X} = \mathrm{IX}$ or X, we construct a region of $P_{\mathcal{X}}(j)$ as follows, see Figure 3.

> [TypeIX] We add a square along the bottom edge E_a first, and add another square along the lengthened left edge $\overline{E_b}$.
>
> [TypeX] We add a square along the left edge E_b first, and add another square along the lengthened bottom edge $\overline{E_a}$.

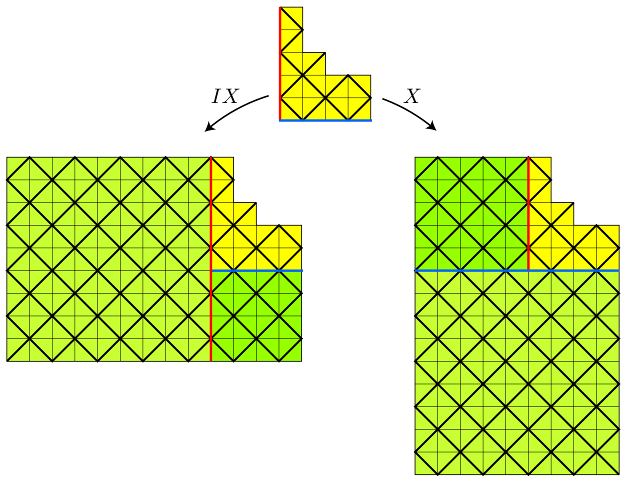


Figure 3: Add squares twice to get $P_{\mathfrak{X}}(j)$ (ex. $P_{IX}(2)$ and $P_X(2)$)

We remark that, by the first square addition along an edge E_a (or E_b, respectively), the other edge E_b (or E_a) is lengthened as $\overline{E_b}$ (or $\overline{E_a}$). The second square is added along the lengthened one. For an edge E, we let $l(E)$ denote the length of E. We have $l(E_a) = |2j|,\ l(E_b) = |2j+1|$ and $l(\overline{E_b}) = l(\overline{E_a}) = l(E_a) + l(E_b)$.

The concrete version of the main theorem (Theorem 1.3) is:

Theorem 1.5. *The plane curve $P_{\mathfrak{X}}(j)$ presents the knot $k_{\mathfrak{X}}(j)$ or its mirror image, as a divide knot.*

We will also show the following lemmas.

Lemma 1.6. *The plane curve $P(j)$ presents the torus knot $T(j, 2j+1)$ as a divide knot.*

Lemma 1.7.([13]) *We let $P_m(j)$ denote a plane curve obtained by a square addition along E_a from $P(j)$, which appears in the process [TypeIX] to construct $P_{IX}(j)$ in Definition 1.4: $P(j) \to P_m(j) \to P_{IX}(j)$.*

The plane curve $P_m(j)$ presents the cable knot $C(T(2,3); j, 6j+1)$, regarded as $C(T(2,3); |j|, 6|j|-1)$ if $j < -1$, of the trefoil as a divide knot:

$$T(j, 2j+1) \ \to \ C(T(2,3); j, 6j+1) \ \to \ k_{IX}(j).$$

The proof of Theorem 1.5 is divided into two parts: In the first half, starting with Baker's Dehn surgery description in [6, 8], we study the knots by usual diagrams. In the second half, we will use divide presentations. We will introduce a convenient method, which we call *Couture move*, to deform L-shaped curves. It was pointed out in the private communication of the author and Olivier Couture. With Couture moves, the proof get much geometric, intuitive and shorter. The author's old proof of Theorem 1.5 was troublesome braid calculus.

Next, we remark on the relation between the surgery coefficients and the area of the regions (of the curves). For an L-shaped region $\mathcal{L}$, we let Area($\mathcal{L}$) denote the area of $\mathcal{L}$, and Conc($\mathcal{L}$) the number of concave points of $\mathcal{L}$, respectively, see Definition 2.4 for the precise definition of concave points. In the simpler case (Conc($\mathcal{L}$) $= 1$) for the knots in the major subfamilies of lens space surgeries, the formula was "Area($\mathcal{L}$) $-$ Surgery coefficient $= 0$ or 1"(Theorem 1.4 in [27], see also [25]). It is modified to:

Lemma 1.8. *On the divide presentation of $k_{\mathfrak{X}}(j)$ by the L-shaped curve $P_{\mathfrak{X}}(j)$ in Theorem 1.5, the area, the number of concave points (of the region $\mathcal{L}$) of $P_{\mathfrak{X}}(j)$ and the coefficient of the lens space surgery along $k_{\mathfrak{X}}(j)$ satisfy*

$$[Area(\mathcal{L}) - Conc(\mathcal{L})] - Surgery\ coefficient = 0 \ \ or\ 1.$$

Lemma 1.8 will be verified by Table 2 in Subsection 3.1.

Question 1.9. *Study divide knots $L(P)$ presented by (generalized) L-shaped curves $P = X \cap \mathcal{L}$. By Lemma 1.8, $c(\mathcal{L}) = Area(\mathcal{L}) - Conc(\mathcal{L})$ or $c(\mathcal{L}) - 1$ could be expected coefficients for exceptional Dehn surgeries of $L(P)$. Study $c(\mathcal{L})$ and $c(\mathcal{L})-1$ surgeries along $L(P)$.*

In the next section, we review theory of A'Campo's divide knots and links briefly and generalize L-shaped plane curve and decide the parametrizing notation. We will introduce a convenient method, which we call *Couture move* on divide presentations, to deform L-shaped curves. It was pointed out in the private communication of the author and Olivier Couture. In Section 3, we review the construction of the knots $k_{\mathfrak{X}}(j)$, give a precise definition of the plane curves $P_{\mathfrak{X}}(j)$ and prove Theorem 1.3, its more precise version Theorem 1.5 and the lemmas.

2. Divide Knots and Plane Curves

We review theory of A'Campo's divide knots and links briefly. We are interested in plane curves constructed as intersection of the $\pi/4$-lattice X and a region. We generalize L-shaped plane curve and decide the parametrizing notation.

2.1. Torus Knots

We start with a presentation of a (positive) torus knot as a divide knot. Let (a, b) be a pair of positive integers and $B(a, b)$ a curve defined as an intersection of the $\pi/4$-lattice X and an $a \times b$ rectangle $\mathcal{R}(a, b)$ whose every vertex is placed at a lattice point ($\in \mathbb{Z}^2$), see Figure 4. If (a, b) is coprime, $B(a, b)$ is a billiard curve in $\mathcal{R}(a, b)$ with slope ± 1.

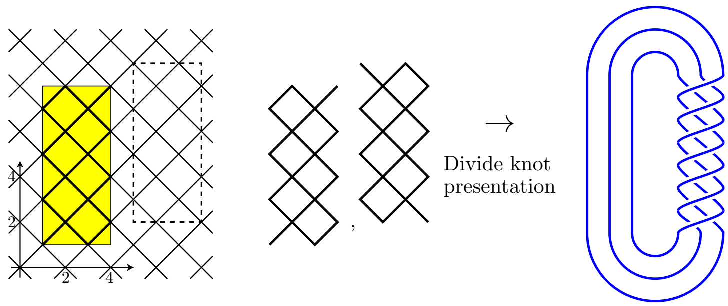


Figure 4: A billiard curve $B(a, b)$ presents a torus knot $T(a, b)$ (ex. $T(3, 7)$)

Lemma 2.1.([1, 2, 3, 4, 5, 12, 16]) *The curve $B(a, b)$ presents the torus link $T(a, b)$ as a divide link.*

Strictly, the curve $B(a, b)$ depends on the placement of the rectangle, whether the left bottom corner of the region is a terminal of the curve or not, see Figure 4 again. Even if (a, b) is not coprime (i.e., case of a torus link), the curve $B(a, b)$ presents $T(a, b)$ for either choice ([16]).

2.2. Basic Facts on Divide Knots

The theory of A'Campo's *divide knots and links* comes from the singularity theory of complex curves. A *divide* P is (originally) a relative, generic immersion of a 1-manifold in the unit disk D in $\mathbb{R}^2$. A'Campo [1, 2, 3, 4] formulated the way to associate to each divide P a link $L(P)$ in S^3. We regard S^3 as

$$S^3 = \{(u, v) \in D \times T_u D \mid |u|^2 + |v|^2 = 1\}$$

and the original construction is

$$L(P) = \{(u, v) \in D \times T_u D \mid u \in P, v \in T_u P, |u|^2 + |v|^2 = 1\} \subset S^3,$$

where $T_u P$ is the subset consisting of vectors tangent to P in the tangent space $T_u D$ of D at u. In the present paper, we regard a PL (piecewise linear) plane curve as a divide by smoothing corners and controlling the size.

Some characterizations of (general) divide knots and links are known, and some topological invariants of $L(P)$ can be gotten from the divide P directly. Here, we list some of them.

Lemma 2.2.([2, 20, 23])

(1) *$L(P)$ is a knot (i.e., connected) if and only if P is an immersed arc.*

(2) *If $L(P)$ is a knot, the unknotting number, the Seifert genus and the 4-genus of $L(P)$ are all equal to the number $d(P)$ of the double points of P.*

(3) *If $P = P_1 \cup P_2$ is the image of an immersion of two arcs, then the linking number of the two component link $L(P) = L(P_1) \cup L(P_2)$ is equal to the number of the intersection points between P_1 and P_2.*

(4) *If P is connected, then $L(P)$ is fibered.*

(5) *Any divide link $L(P)$ is strongly invertible.*

(6) *A divide P and its mirror image $P!$ present the same knot or link: $L(P!) = L(P)$.*

(7) *If P_1 and P_2 are related by some Δ-moves, then the links $L(P_1)$ and $L(P_2)$ are isotopic: If $P_1 \sim_\Delta P_2$ then $L(P_1) = L(P_2)$, see Figure 5.*

(8) *Any divide knot is a closure of a strongly quasi-positive braid, i.e., a product of some σ_{ij} in Figure 5.*

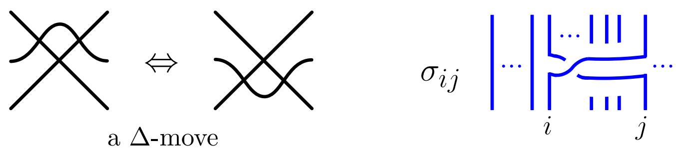


Figure 5: Basics on divide knots

For theory of divide knots, see also [11, 21] and "transverse $\mathbb{C}$-links" defined by Rudolph [23]. In [12] Couture and Perron pointed out a method to get the braid presentation from the divide in a restricted cases, called "ordered Morse" divides. We can apply their method, see also Hirasawa's method [20].

Finally we recall an operation "adding a square" on divides P and its contribution to the divide links $L(P)$.

Lemma 2.3.([27, Lemma 4.2]) *Adding a square on an L-shaped curve P along an edge l (of the region) corresponds to a right handed full-twist on the divide knot $L(P)$ along the unknotted component defined by l, see Figure 7.*

Adding a square is related to "blow-down". Here, blow-down along y-axis ($x = 0$) is the coordinate transformation from (x, y) to (x', y') by $x' = x, y' = yx$ (ex. $y^2 = x + \epsilon$ become ${y'}^2 = {x'}^2(x' + \epsilon)$)

2.3. Curves defined by Regions

In xy-plane, a lattice point or an integer vector $(k, l) \in \mathbb{Z}^2$ is called *even* or *odd*, if $k + l$ is even or odd, respectively. Double points of the $\pi/4$-lattice $X =$

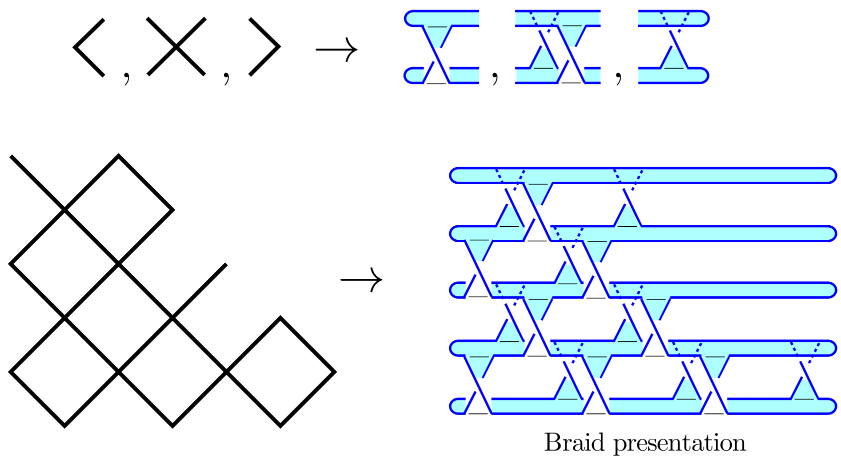


Figure 6: Couture–Perron's method

$\{(x,y)|\cos\pi x = \cos\pi y\}$ are at even points. We are interested in curves constructed as intersection of X and a region, as a generalization of Lemma 2.1.

The lattice X has some symmetries: We let r_x denote the reflection along y-axis, $r_x(x,y) = (-x,y)$, $R_{\pi/2}$ the $\pi/2$-rotation (along the origin), and $+\vec{v}_{\rm ev}$ a parallel translation by an <u>even</u> vector. Symmetry of the lattice X is generated by $r_x, R_{\pi/2}$ and some $\vec{v}_{\rm ev}$. A curve constructed as intersection of X and a region $\mathcal{R}$ does not change by the action (on $\mathcal{R}$) of the symmetry of X. We also use a parallel translation $+\vec{v}_{\rm od}$ by an odd vector to place a region $\mathcal{R}$ well.

Definition 2.4.(Condition of regions) We are interested in curves constructed as intersection of X and regions $\mathcal{R}$. We formulate the conditions on regions:

(i) A region $\mathcal{R}$ is a union of a finite number of rectangles.

(ii) Each edge of the rectangles in $\mathcal{R}$ is horizontal or vertical.

(iii) Each vertex of $\mathcal{R}$ is at a lattice point.

(iv) Difference vectors of any pair of concave points of the region $\mathcal{R}$ are even.

Here, a concave point in (iv) is defined as follows: A boundary point p of a region $\mathcal{R}$ is called a *concave point* (of the region) if a neighborhood of $\mathcal{R}$ at p is locally homeomorphic to that of $\{x \le 0\} \cup \{y \le 0\}$ at $(0,0)$ by the symmetry of xy-plane, generated by r_x, $R_{\pi/2}$ and $+\vec{v}$ (by an even or an odd integer vector), see Figure 8.

If a concave point p of $\mathcal{R}$ is at an even point, then the curve $X \cap \mathcal{R}$ is not generic at p, i.e., a terminal point overlaps with an interior point of the curve. We are concerned only with a generic immersed curves. By the condition (iv), all concave

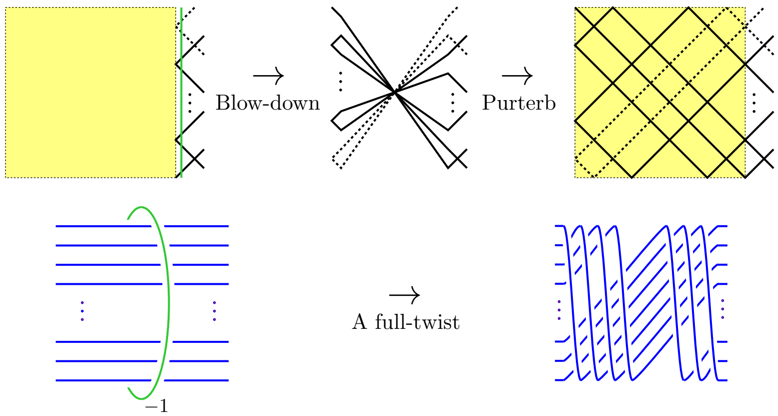


Figure 7: Adding a square

points of either $X \cap \mathcal{R}$ or $X \cap (\mathcal{R} + \vec{v}_{\mathrm{od}})$ are placed at even points, and it defines a generic immersed curve.

Definition 2.5. For a region $\mathcal{R}$ satisfying the condition (i),(ii),(iii) and (iv), either $X \cap \mathcal{R}$ or $X \cap (\mathcal{R} + \vec{v}_{\mathrm{od}})$ is a generic immersed curve. We choose the generic one and define it as *a curve defined by the region* $\mathcal{R}$, see an example ($X \cap (\mathcal{R} + \vec{v}_{\mathrm{od}})$ is chosen) in Figure 8.

We describe a curve $X \cap \mathcal{R}$ by describing (and parametrizing) the region $\mathcal{R}$ by using xy-coordinates.

2.4. L-shaped Curves

We generalize *L-shaped curves* from the old version. It is an extension of "L-shaped curves" in [27, Section 3.2], but the notation (parametrization) is changed.

Definition 2.6.(L-shaped region at the origin) See Figure 8. Let n be a positive integer with $n > 1$. We let

$$[\{(a_i, b_i)\}] := [(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)] \tag{2.1}$$

denote a sequence of lattice points ($\in \mathbb{Z}^2$) in xy-plane satisfying

$$0 < a_1 < a_2 < \cdots < a_n \text{ and } b_1 > b_2 > \cdots > b_n > 0.$$

We define a region $R[\{(a_i, b_i)\}]$ in xy-plane by

$$R[\{(a_i, b_i)\}] = \bigcup_{i=1}^{n} \{(x, y) \,|\, 0 \le x \le a_i, 0 \le y \le b_i\}. \tag{2.2}$$

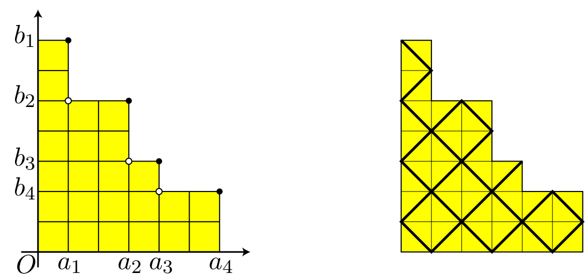


Figure 8: Type $R[(1,7),(3,5),(4,3),(6,2)]$ $(= P(3))$

We will call this region *an L-shaped region of type* $[\{(a_i, b_i)\}]$ *of length* n (at the origin).

If such a region defines a generic immersed curve in the sense of Definition 2.5, we call the curve *L-shaped curve of type* $[\{(a_i, b_i)\}]$.

An L-shaped region of type $[\{(a_i, b_i)\}]$ of length n has $n-1$ concave points at the coordinate (a_i, b_{i+1}) with $1 \le i \le n-1$. Note that the L-shaped region $L[a_1, a_2; b_1, b_2]$ defined in [27, Definition 3.3] is redefined $R[(a_1, b_2), (a_2, b_1)]$ of length 2 here.

It is easy to see:

Lemma 2.7. *The area of an L-shaped region of type* $[\{(a_i, b_i)\}]$ *of length* n *is*

$$\mathit{Area}\ (R[\{(a_i, b_i)\}]) = \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n-1} a_i b_{i+1}.$$

Question 2.8. *Find a formula on the numbers of circle and arc components of (generic) L-shaped curves. When is an L-shaped curve a generic immersed arc?*

2.5. Deformation of Curves, Couture Move

We show a lemma on a deformation of divide presentations of torus knots of type $(n, 2n-1)$ geometrically. We call the method Couture move for short. The move consists of some Δ-moves in Lemma 2.2(7). It was pointed out in the private communication of the author and Olivier Couture in the opportunity of a conference "Singularities, knots, and mapping class groups in memory of Bernard Perron" held in Sept. 2010. By the move, we can deform the divide presentations of $T(n, 2n-1)$ from the curve $B(n, 2n-1)$ in Lemma 2.1 to a generalized L-shaped curve of special type. An example is shown in Figure 9.

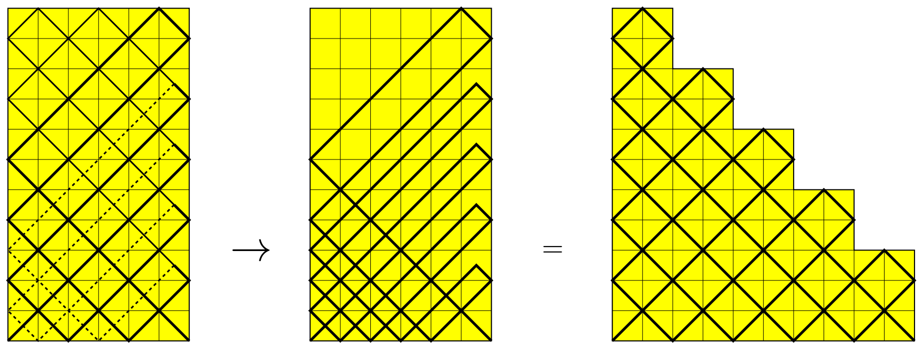


Figure 9: Couture move (ex. $T(6,11)$)

Lemma 2.9. *Let n, j be integers with $1 < 2j \leq n+1$. We define a sequence $[\{(a_i, b_i)^{(n;j)}\}]$ of lattice points of length $2j-1$ as follows:*
Case n is even:

$$(a_i, b_i)^{(n;j)} = \begin{cases} (n+2i-2j, 2n+1-2i) & (1 \leq i \leq j) \\ (n+2i-2j, 4j+1-2i) & (j+1 \leq i \leq 2j-1) \end{cases}.$$

Case n is odd:

$$(a_i, b_i)^{(n;j)} = \begin{cases} (n+2-2j, 2n-1) & (i=1) \\ (n+2i-2j, 2n+2-2i) & (2 \leq i \leq j) \\ (n+2i-2j, 4j-2i) & (j+1 \leq i \leq 2j-1) \end{cases}.$$

The L-shaped curve of type $[\{(a_i, b_i)^{(n;j)}\}]$ for any $j = 1, 2, \dots \lfloor n/2 \rfloor$ presents the torus knot $T(n, 2n-1)$ as a divide knot. Here, $\lfloor x \rfloor$ means the largest integer less than or equal to x.

Note that $a_i + b_i = 2n+1$ for the final (the largest) $j = n/2$ in the case n is even, and also for $j = (n+1)/2$ except $i = 1$, in the case n is odd.

Example 2.10. For $n = 6$, we have

$[\{(a_i, b_i)^{(6;1)}\}] = [(6, 11)]$,
$[\{(a_i, b_i)^{(6;2)}\}] = [(4, 11), (6, 9), (8, 3)]$,
$[\{(a_i, b_i)^{(6;3)}\}] = [(2, 11), (4, 9), (6, 7), (8, 5), (10, 3)]$.

For $n = 7$, we have

$[\{(a_i, b_i)^{(7;1)}\}] = [(7, 13)]$,
$[\{(a_i, b_i)^{(7;2)}\}] = [(5, 13), (7, 12), (9, 2)]$,
$[\{(a_i, b_i)^{(7;3)}\}] = [(3, 13), (5, 12), (7, 10), (9, 4), (11, 2)]$,
$[\{(a_i, b_i)^{(7;4)}\}] = [(1, 13), (3, 12), (5, 10), (7, 8), (9, 6), (11, 4), (13, 2)]$.

Proof. Note that the case $j = 1$ is included in Lemma 2.1, since the L-shaped curve of type $[\{(a_i, b_i)^{(n;1)}\}] = [(n, 2n-1)]$ is the curve $B(n, 2n-1)$, the original divide presentation of $T(n, 2n-1)$. The proof for the case $j > 1$ is by induction with respect to j.

First, we give a proof in the case n is even. The first step from $j = 1$ to 2 (with $n = 6$) is shown in Figure 10, where the thin part in the curve moves to the thin dotted curve. It is easy to see that the move consists of some Δ-moves in Lemma 2.2(7), thus it does not change the knot as a divide presentation. In the final picture in Figure 10, the second step from $j = 2$ to 3 is shown. For larger n and the j-th step, we can check that the deformation of the curves corresponds to the change of types of the L-shaped curves, where the length get longer by two.

For the case n is odd, the original curve $B(n, 2n-1)$ has a terminal point at the right top corner, but the deformations are similar. The first step (with $n = 7$) is shown in Figure 11. □

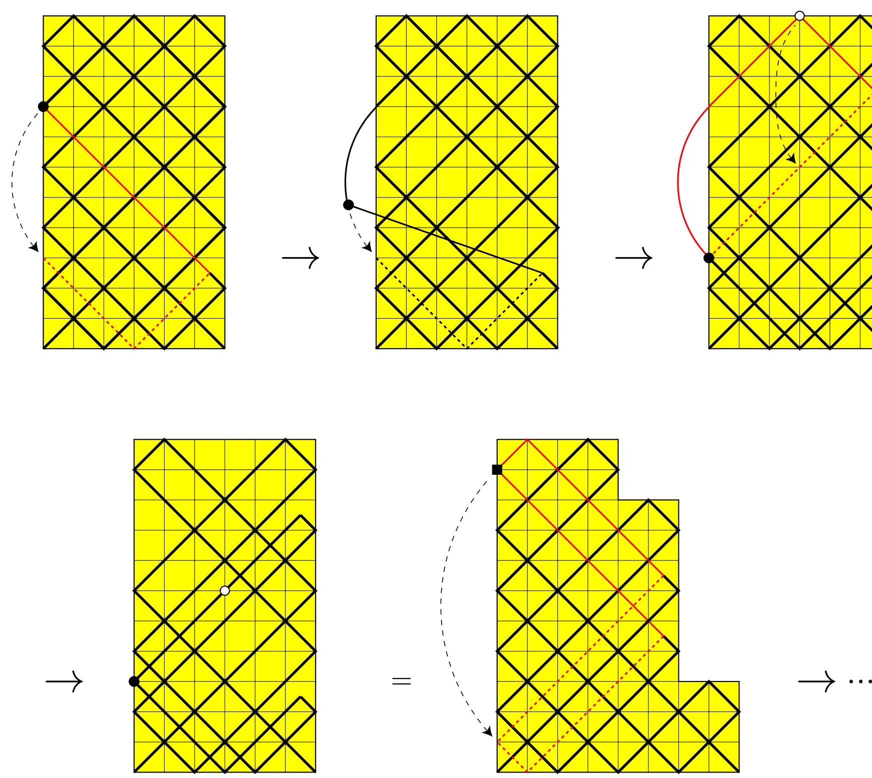


Figure 10: Couture move from $j = 1$ to 2 (n is even, ex. $n = 6$)

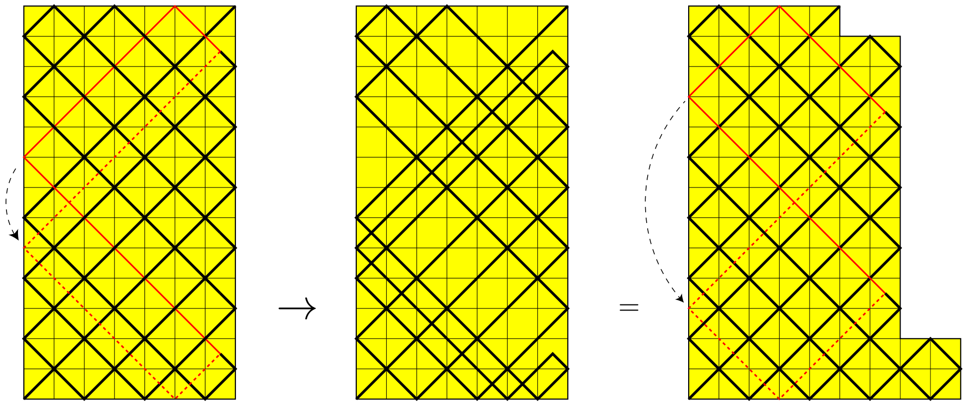


Figure 11: Couture move from $j = 1$ to 2 (n is odd, ex. $n = 7$)

Definition 2.11.(Couture move) The deformations of the curves in the proof of Lemma 2.9, especially the first and the second halves in Figure 10, and their generalizations for larger n, are useful for general L-shaped curves. In fact, we can use such deformations if the regions used in the moves are locally homeomorphic. We call them *Couture moves*.

In the present paper, we will use Couture moves for generalized L-shaped curves in Figures 16, 17 and 20.

3. Details on Sporadic Knots and Proof

We give a precise definition of the plane curves $P(j)$, verify the formula in Lemma 1.8 and prove Theorem 1.5 and the lemmas. We start the proof with Baker's description in [6, 8] of sporadic knots, and we use Couture moves on divides in the second half of the proof.

3.1. Precise Definition of Curves

We define divides $P(j)$ by the method introduced in the last section.

Definition 3.1.(Precise definition of $P(j)$) For an integer j ($j \neq 0, -1$), we define a sequence $[\{(a_i, b_i)^{[j]}\}]$ of lattice points $(a_i, b_i)^{[j]}$ as follows.

(Case $j > 0$) Starting with $[\{(a_i, b_i)^{[1]}\}] = [(1, 3), (2, 1)]$, we define $[\{(a_i, b_i)^{[j]}\}]$ inductively with respective to j, by

$$\begin{cases} (a_1, b_1)^{[j]} = (1, 2j+1) & (i = 1) \\ (a_i, b_i)^{[j]} = (b_{j+2-i}, a_{j+2-i})^{[j-1]} + (1, 1) & (1 < i \leq j+1) \end{cases}.$$

(Case $j < -1$) We define $[\{(a_i, b_i)^{[j]}\}]$ by

$$(a_i, b_i)^{[j]} = (2i, -2j + 1 - 2i) \quad (1 \leq i \leq -j).$$

We define a plane curve $P(j)$ as an L-shaped curve of type $[(a_i, b_i)^{[j]}]$, whose length is $j + 1$ (if $j > 0$) or $-j$ (if $j < -1$). See and verify the examples in Figure 2.

By Lemma 2.7, it is easy to see

Lemma 3.2.

$$Area(R[\{(a_i, b_i)^{[j]}\}]) = \begin{cases} 2j^2 + 2j & (j > 0) \\ 2j^2 & (j < -1) \end{cases}.$$

The plane curve $P_{\mathfrak{X}}(j)$ is constructed by adding a square twice as in Definition 1.4. Since a square addition along an edge E of length $l(E)$ increases the area by $l(E)^2$, the area of the region of $P_{IX}(j)$ with $j > 0$ is calculated as

$$\begin{aligned} Area(R[\{(a_i, b_i)^{[j]}\}]) + l(E_a)^2 + l(\overline{E_b})^2 &= (2j^2 + 2j) + (2j)^2 + (4j + 1)^2 \\ &= 22j^2 + 10j + 1 \end{aligned}$$

On the other hand, the area of the region of $P_X(j)$ with $j > 0$ is calculated as

$$\begin{aligned} Area(R[\{(a_i, b_i)^{[j]}\}]) + l(E_b)^2 + l(\overline{E_a})^2 &= (2j^2 + 2j) + (2j + 1)^2 + (4j + 1)^2 \\ &= 22j^2 + 14j + 2 \end{aligned}$$

We calculate them also in the cases $j < -1$ and list them in Table 2. Lemma 1.8 is proved by Table 2.

Curve	Coeff. p of $k_{\mathfrak{X}}(j)$	Area (Case $j > 0$)	Area (Case $j < -1$)
$P_{IX}(j)$	$22j^2 + 9j + 1$	$22j^2 + 10j + 1$	$22j^2 + 8j + 1$
$P_X(j)$	$22j^2 + 13j + 2$	$22j^2 + 14j + 2$	$22j^2 + 12j + 2$

Table 2: Area of the plane curve $P_{\mathfrak{X}}(j)$

Next, we calculate and verify that the numbers of double points of $P(j)$ and $P_{\mathfrak{X}}(j)$. By Lemma 2.2(2), they are equal to the genus of the presented knots $L(P(j))$ and $L(P_{\mathfrak{X}}(j)) = k_{\mathfrak{X}}(j)$, respectively. Since $L(P(j))$ is $T(j, 2j + 1)$ by Lemma 1.6, we have:

Lemma 3.3. *The number $d(P(j))$ of double points of the plane curve $P(j)$ is equal to the genus of the torus knot $T(j, 2j + 1)$:*

$$d(P(j)) = \frac{(|j| - 1)(|2j + 1| - 1)}{2} = \begin{cases} j(j - 1) & (j > 0) \\ (j + 1)^2 & (j < -1) \end{cases}.$$

Since the number of double points increases by $l(l-1)/2$ by adding a square along an edge of length l, $d(P_{IX}(j))$ is calculated as
(Case $j > 0$)

$$\begin{aligned} d(P_{IX}(j)) &= d(P(j)) + \frac{l(E_a)\cdot(l(E_a)-1)}{2} + \frac{l(\overline{E_b})\cdot(l(\overline{E_b})-1)}{2} \\ &= j(j-1) + \frac{2j\cdot(2j-1)}{2} + \frac{(4j+1)\cdot 4j}{2} \\ &= 11j^2. \end{aligned}$$

(Case $j < -1$)

$$\begin{aligned} d(P_{IX}(j)) &= (j+1)^2 + \frac{(-2j)\cdot(-2j-1)}{2} + \frac{(-4j-1)\cdot(-4j-2)}{2} \\ &= 11j^2 + 9j + 2. \end{aligned}$$

They are equal to the genus of the knots $k_{IX}(j)$, see Table 1. We leave the case of TypeX to the readers.

3.2. Proof of the Main Theorem

In the first half of the proof, we study the knots in the usual diagram and Dehn surgery description. In the second half, we will use divide presentation.

We start with Baker's Dehn surgery description of the knots $k_{\mathfrak{X}}(j)$ in Figure 12 from [6]. Throughout the paper, we fix

$$(\alpha, \beta) = \begin{cases} (-2,-3) & \text{for TypeIX} \\ (-3,-2) & \text{for TypeX} \end{cases}.$$

It is easy to see that the framed sublink of thick seven components presents S^3, by usual weighted tree diagram in the right half of Figure 12. We call the sublink the non-trivial diagram of S^3. In the resulting S^3, the thin component is the sporadic knot $k_{\mathfrak{X}}(j)$ as a knot.

[Case $j > 0$] First, we assume that $j > 0$ until the final paragraph. To get a usual diagram of the knot $k_{\mathfrak{X}}(j)$, we have to chase the thin component (with framing) during the deformation from the non-trivial diagram to the empty diagram of S^3. It is not easy but straight forward. In the middle of the process, we reach a diagram in Figure 13, where boxed $+1$ in the diagram means a right handed full-twist. The thin component $k(j)$ is a torus knot $T(j, j+1)$ with framing $j(j+1)$, which is a coefficient of reducible Dehn surgery. We name the four component link $L(j) = k(j) \cup u_{-1} \cup u_\alpha \cup u_\beta$, where u_x the x-framed unknotted component for $x = -1, \alpha, \beta$.

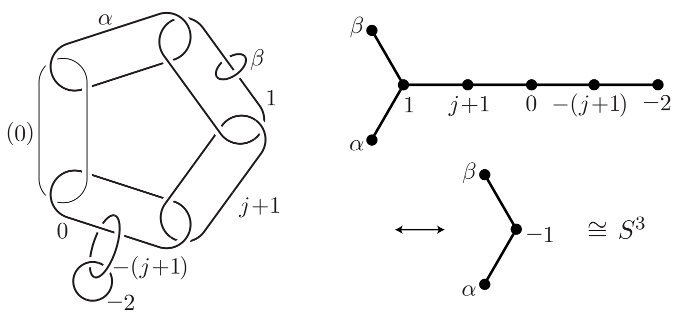


Figure 12: Baker's description

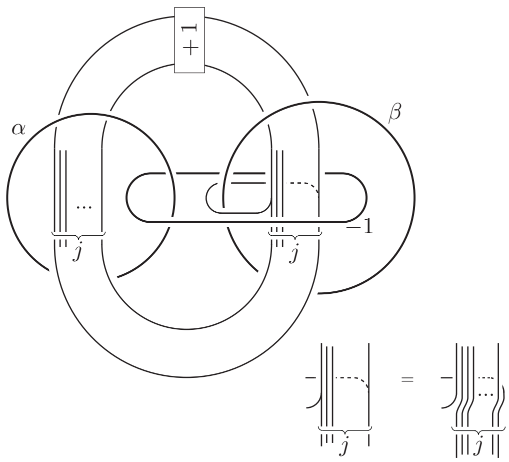


Figure 13: Construction of $k_{\mathfrak{X}}(j)$: Link $L(j)$

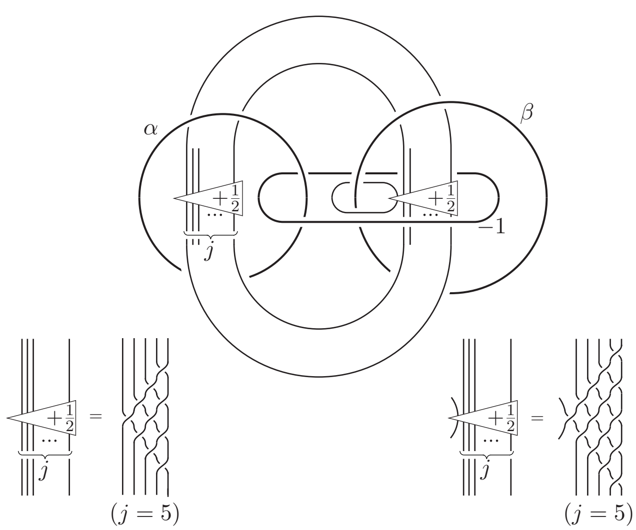


Figure 14: Link $L(j)$ with a strong involution

Second, we decompose the full-twist at the boxed $+1$ as two half-twists, denoted by $+\frac{1}{2}$ in triangles, and deform the diagram by isotopy as in Figure 14. Note that the half-twist in the right is of $j+1$ strings. We can see that $L(j)$ is strongly invertible with respect to the horizontal axis, see Lemma 2.2(5). As a quotient of the involution, ignoring the crossing data (over or under), we have a plane curve properly immersed in the half plane. The curve can be modified as in Figure 15. This is a divide presentation of $L(j)$. It can be checked by Couture–Perron's method in Figure 6. This process is related to the original construction of divide knots: For a link of singularity of a complex plane curve, the divide is a real part of a "good" perturbation (called real Morsification) of the equation of the singularity.

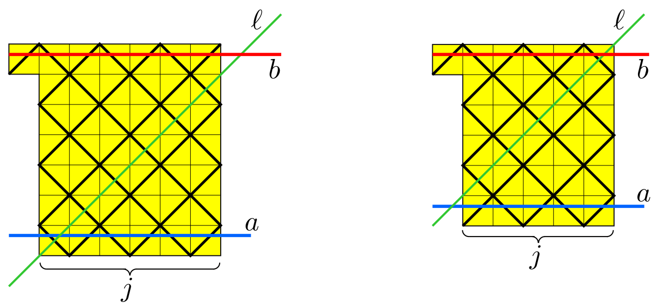


Figure 15: Divide presentation $C(j)$ of $L(j)$ (ex. $j=6$ and $j=5$)

In the divide presentation of $L(j)$ in Figure 15, a line segment ℓ is placed slightly different whether j is even or odd. We name the plane curve $C(j) = c(j) \cup \ell \cup a \cup b$, where $c(j)$ presents $k(j) = T(j, j+1)$ by Lemma 2.1, since $c(j)$ is an L-shaped curve of type $[(j, j+1), (j+1, 1)]$ and is isotopic to $B(j, j+1)$. The line segments ℓ, a, b present $u_{-1}, u_\alpha, u_\beta$, respectively, as a divide presentation.

The linking matrix of $L(j)$ with a suitable orientation of the link

$$\begin{pmatrix} j(j+1) & j & j & j+1 \\ j & -1 & 1 & 1 \\ j & 1 & \alpha & 1 \\ j+1 & 1 & 1 & \beta \end{pmatrix},$$

is equal to the matrix of the number of intersection points of the components of the divide by Lemma 2.2(3).

From now on, we go into the second half of the proof, and study the knots and links by divide presentation. We use Δ-moves on divides freely, see Lemma 2.2(7). We have two (or three) steps: (i) Blow-down along ℓ (take a full-twist along u_{-1}, (ii) (Only if j is odd) Modify the curve by some Δ-moves, and (iii) Deform the curve by Couture moves. Our goal is the divide $P(j)$ with edges $\overline{a}, \overline{b}$, where $\overline{a}$ (and $\overline{b}$) is a small parallel push-off of the bottom edge E_a (the left edge E_b) of the region of $P(j)$ into the interior.

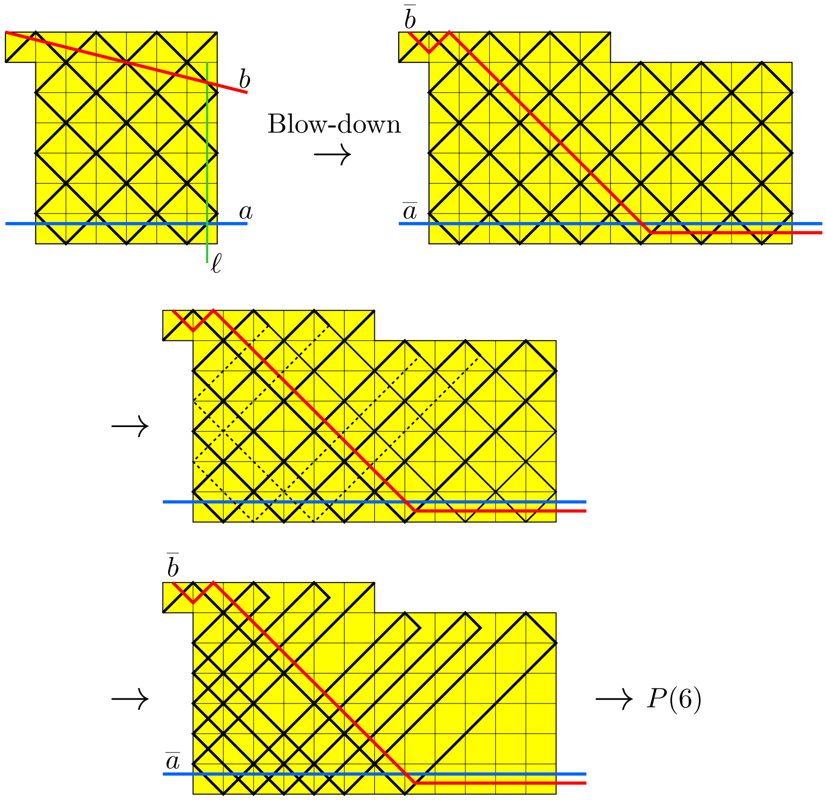


Figure 16: Deformation: Case j is even (ex. $j = 6$)

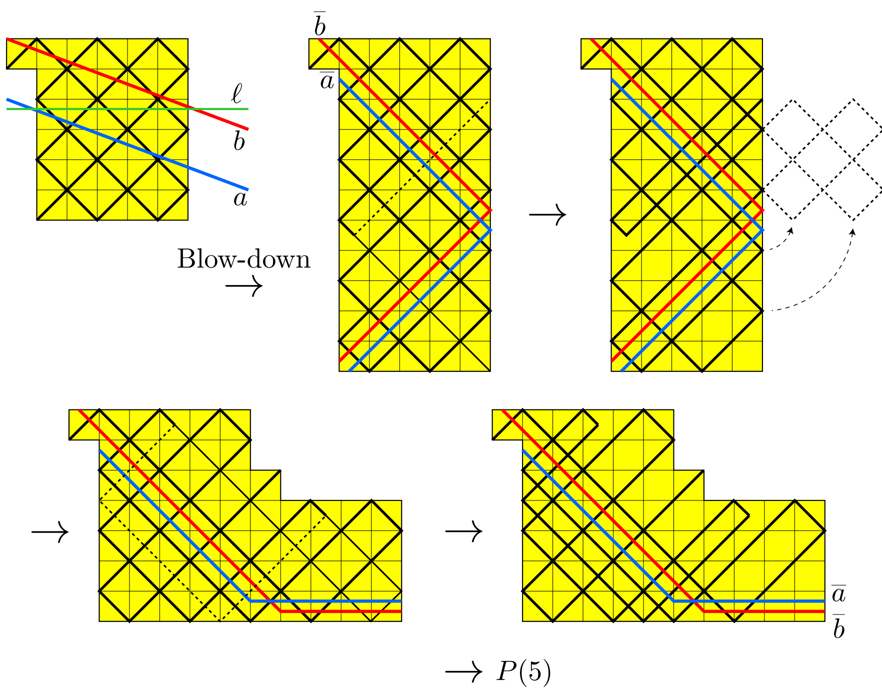


Figure 17: Deformation: Case j is odd (ex. $j = 5$)

(Step (i)) We take a right handed full-twist of $k(j) \cup u_\alpha \cup u_\beta$ along the unknot u_{-1}. We name the resulting link

$$\overline{L(j)} = \overline{k(j)} \cup \overline{u_\alpha} \cup \overline{u_\beta}.$$

This full-twist is done as a blow-down along the line ℓ, i.e., by adding a square by Lemma 2.3. In the case where j is even, we slide ℓ and b by Δ-moves as the first picture in Figure 16 and add a square. Otherwise, we add a square along the bottom edge as in Figure 17.

(Step (ii)) If j is odd, the curve has a terminal point at the right bottom corner. We move the terminal point (and its segment) up along the right edge, as the second deformation of in Figure 17. We also slide the other added part at the bottom to the right by some Δ-moves.

(Step (iii)) The resulting curve is near L-shaped, but the lines are not in the required position. Here we use Couture move, see the second halves of Figure 16 in the case j is even, and Figure 17 in the case j is odd. By some obvious Δ-moves, we have the required curve $P(j) \cup \overline{a} \cup \overline{b}$, which presents $\overline{L(j)}$ as a divide link.

The sublink $\overline{u_\alpha} \cup \overline{u_\beta}$ is a Hopf link and the framings are $(\alpha+1, \beta+1) = (-1, -2)$ for TypeIX, or $= (-2, -1)$ for TypeX. By the construction of $P_{\mathfrak{X}}(j)$ in Definition 1.4, and by the correspondence between adding a square to a divide and a full-twist of the divide link in Lemma 2.3, we have the required divide presentation of $k_{\mathfrak{X}}(j)$ with $j > 0$.

ww [Case $j < -1$] Finally, we study the case $j < -1$. The method is similar. We define an integer j' by $j = -(j'+1)$ for figures. Starting with Baker's description in Figure 12, we have a link $L(j)$ in Figure 18.

Divide description of $L(j)$ is in Figure 19, contrast to Figure 15. Especially, the non-trivial component $c(j)$ and $c(j')$ (ex. $c(-6)$ and $c(5)$) are the same curves presenting $T(j, j+1) = T(j'+1, j')$ but the other segment components are placed differently.

In Figure 20, we show some deformations. Figures 18, 19, 20 (Case $j < -1$) are contrast to Figures 13, 15, 16 and 17 (Case $j > 0$), respectively. □

Proof of Lemma 1.6. We study a sublink $k(j) \cup u_{-1}$ of $L(j)$ in Figure 13 (or Figure 18, respectively), presented by $c(j) \cup \ell$ in $C(j)$ in Figure 15 (or Figure 19) as a divide link. The component $k(j)$ is a torus knot $T(j, j+1)$ (or $T(j'+1, j')$). By the diagrams of $L(j)$ in Figure 15 and Figure 19, we can see that $\overline{k(j)}$ is a torus knot $T(j, 2j+1)$ (or $T(j'+1, 2j'+1)$), presented by $P(j)$ as a divide knot. The case $j < -1$ is a little harder, see Lemma 2.9. We have the lemma. □

Proof of Lemma 1.7. In [26], a divide knot presentation of cable knots (under some conditions) is studied. Here we use Δ-moves on divides freely.

First, assume $j > 0$. The plane curve $P_m(j)$ is obtained by adding a square twice from $c(j)$ isotopic to $B(j, j+1)$: we add a square along ℓ to $c(j)$ first (then we

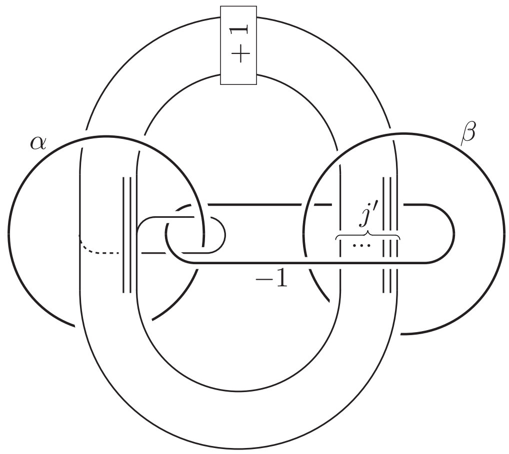


Figure 18: Link $L(j)$ in the case $j < -1$ $(j = -(j'+1))$

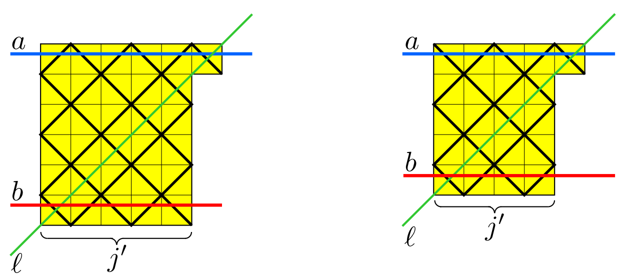
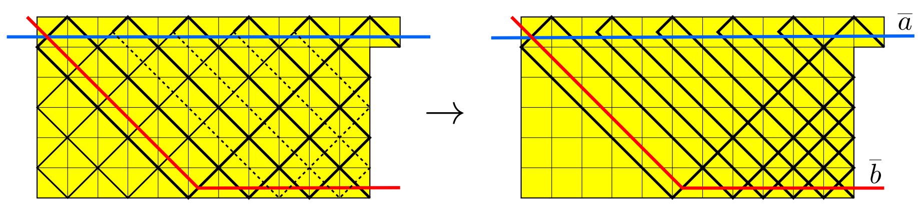
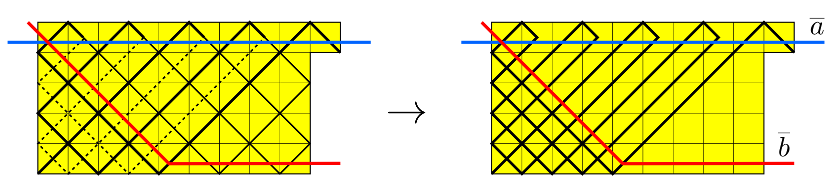


Figure 19: Divide presentation of $L(j)$ with $j < -1$ (ex. $j = -6$ and $j = -5$)



$\longrightarrow P(-6)$

Case j is even (ex. $j = -6$)



$\longrightarrow P(-5)$

Case j is odd (ex. $j = -5$)

Figure 20: Deformation $(j < -1)$

have $P(j)$), and add another square along E_a or $\overline{a}$ second. We see the plane curve obtained by the first square addition (blow-down) in Figure 16 or Figure 17. Since the line $\overline{a}$ can be moved to an edge of the L-shaped region by Δ-moves, we can add the second square. The curve becomes an L-shaped curve of type $[(3j, 2j), (3j+1, j)]$. By [26], it presents the required cable knot $C(T(2,3); j, 6j+1)$.

The proof in the case $j < -1$ is similar. From the first curves in Figure 20, we have an L-shaped curve of type $[(|j|+1, 3|j|), (2|j|, 3|j|-1)]$. By [26], we have the lemma. □

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