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Torsion in Homology of Dihedral Quandles of Even Order

Seung Yeop Yang
Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea
e-mail : seungyeop.yang@knu.ac.kr

Abstract. Niebrzydowski and Przytycki conjectured that the torsion of rack and quandle homology of a dihedral quandle of order $2k$ is annihilated by k, unless $k = 2^t$ for $t > 1$. We partially prove this conjecture.

1. Introduction

Rack [7] and quandle [4] homology are (co)homology theories for self-distributive structures with axioms obtained diagrammatically from Reidemeister moves in classical knot theory. Various homological and homotopical knot invariants have been developed based on these (co)homology theories. More precisely, quandle cocycle invariants [4] are constructed with cocycles of quandle homology and quandle homotopy invariants [13, 17] are obtained using homotopy classes of maps from spheres to the geometric realizations of quandle homology. It is significant to calculate quandle (co)homology and determine explicit quandle cocycles for the study of these invariants.

The free parts of rack and quandle homology groups of finite quandles have been completely computed in [6, 9], but little is known about the torsion parts. The orders of torsion elements of rack and quandle homology were approximated in [9], and their relationship with the inner automorphism group of the quandle was discussed in [6]. Niebrzydowski and Przytycki [11, 12] have developed methods and conjectures about the higher dimensional homology of dihedral quandles. They conjectured that for a dihedral quandle R_p of order odd prime p, $\mathrm{Tor}H_n^Q(R_p) = \mathbb{Z}_p^{f_n}$, where $\{f_n\}$ is a "delayed" Fibonacci sequence, i.e., $f_n = f_{n-1} + f_{n-3}$ and $f_1 = f_2 = 0, f_3 = 1$. It was proved independently by Clauwens [5] and Nosaka [14]. In analogy to the result of group homology, it was conjectured [12] that for a finite quasigroup quandle, the torsion subgroups of its rack and quandle homology are annihilated by the order of the quandle. This was proved in [15] and generalized

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[16, 19] for some connected quandles.

In this paper, we study the torsion subgroups of rack and quandle homology of non-connected quandles. To start with, we partially solve the following conjecture:

Conjecture 1.1.([12]) *The number k annihilates* $\mathrm{Tor}H_n^W(R_{2k})$, *unless* $k = 2^t, t > 1$ *and the number* $2k$ *is the smallest number annihilating* $\mathrm{Tor}H_n^W(R_{2k})$ *for* $k = 2^t, t > 1$, *where* $W = R, Q$.

1.1. Preliminaries

Definition 1.2.([8, 10]) A *quandle* $(X, *)$ is an algebraic structure with a set X and a binary operation $* : X \times X \to X$ satisfying the following axioms:

(1) (Right self-distributivity) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$;

(2) (Invertibility) For each $b \in X$, the right translation $r_b : X \to X$ given by $r_b(x) = x * b$ is invertible;

(3) (Idempotency) $a * a = a$ for any $a \in X$.

If the binary operation satisfies right self-distributivity and invertibility, then $(X, *)$ is call a *rack*. Note that the three axioms above are motivated by Reidemeister moves in knot theory, and racks and quandles can be used to construct (framed) knot invariants.

Example 1.3. Basic quandles can be obtained from groups and modules as follows:

(1) An abelian group A equipped with the binary operation $* : A \times A \to A$ defined by $g * h = 2h - g$ is called a *Takasaki quandle* or *kei*. Specially, when $A = \mathbb{Z}_n$, it is called a *dihedral quandle* and denoted by R_n. See Table 1 for example.

(2) A group G with the operation $g * h = hg^{-1}h$ is called a *core quandle*.

(3) A group G with the conjugate operation $g * h = h^{-1}gh$ is called a *conjugate quandle*.

(4) Let M be a module over the Laurent polynomial ring $\mathbb{Z}[T, T^{-1}]$. A quandle M with the operation $a * b = Ta + (1 - T)b$ is called an *Alexander quandle*.

A *trivial quandle*, a set X with the binary operation $a * b = a$ (i.e., the operation does not depend on the choice of b), is the most elementary quandle. However, it plays an important role in the proof of Theorem 2.2 because its rack and quandle homology groups do not contain torsion subgroups.

A *quandle homomorphism* is a map $h : X \to Y$ between two quandles $(X, *)$ and $(Y, *')$ such that $h(a * b) = h(a) *' h(b)$ for all $a, b \in X$. A bijective quandle homomorphism is called a *quandle isomorphism*, and a quandle isomorphism from a quandle X onto itself is called a *quandle automorphism*. Let X be a quandle. The *quandle automorphism group* $\mathrm{Aut}(X)$ of X is the group consisting of quandle

Table 1: Dihedral quandle of order 6

$*$	0	2	4	3	5	1
0	0	4	2	0	4	2
2	4	2	0	4	2	0
4	2	0	4	2	0	4
3	3	1	5	3	1	5
5	1	5	3	1	5	3
1	5	3	1	5	3	1

automorphisms of X. Note that every right translation of a quandle is a quandle automorphism. The subgroup of $\mathrm{Aut}(X)$ generated by all the right translations of X is called the *quandle inner automorphism group*, denoted by $\mathrm{Inn}(X)$.

A quandle X is said to be *quasigroup* or *Latin* if every left translation, $l_b : X \to X$ defined by $l_b(x) = b * x$ is invertible. A quandle X is *connected* if the canonical action of $\mathrm{Inn}(X)$ on X is transitive. Otherwise, X is said to be *non-connected.* Note that every quasigroup quandle is connected, but the converse does not hold in general.

Example 1.4.

(1) The dihedral quandle R_n of order n is quasigroup (i.e., connected) if n is odd. Otherwise, it is non-connected.

(2) The set of all 4-cycles in the symmetric group S_4 with the conjugate operation is a connected quandle, but it is not quasigroup.

(3) An Alexander quandle is quasigroup if and only if $1 - T$ is invertible.

We next review the rack and quandle homology theories.

Definition 1.5.([4, 7])

(1) For a given rack X, let $C_n^R(X)$ be the free abelian group generated by n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ of elements of X. We define the boundary homomorphism $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by for $n \geq 2$,

$$\partial_n(\mathbf{x}) = \sum_{i=2}^{n}(-1)^i\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

and for $n < 2$, $\partial_n = 0$. $(C_n^R(X), \partial_n)$ is called the *rack chain complex* of X.

(2) For a quandle X, define the subgroup $C_n^D(X)$ of $C_n^R(X)$ for $n \geq 2$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some i. We let $C_n^D(X) = 0$ if

$n < 2$. Then $(C_n^D(X), \partial_n)$ forms a sub-chain complex of $(C_n^R(X), \partial_n)$, called the *degenerate chain complex* of X.
The quotient chain complex $(C_n^Q(X) = C_n^R(X)/C_n^D(X), \partial_n')$, where ∂_n' is the induced homomorphism, is called the *quandle chain complex* of X. Hereafter, we denote all boundary maps by ∂_n.

(3) Let A be an abelian group. Define the chain and cochain complexes

$$C_*^W(X;A) = C_*^W(X) \otimes A,\ \partial = \partial \otimes \mathrm{Id},$$
$$C_W^*(X;A) = \mathrm{Hom}(C_*^W(X), A),\ \delta = \mathrm{Hom}(\partial, \mathrm{Id})$$

for W=R, D, and Q. The yielded homology groups

$$H_n^W(X;A) = H_n(C_*^W(X;A)) \text{ and } H_W^n(X;A) = H^n(C_W^*(X;A))$$

for W=R, D, and Q are called the nth *rack, degenerate, and quandle homology groups* and the nth *rack, degenerate, and quandle cohomology groups* of a rack/quandle X with coefficient group A.

The free parts of the rack and quandle homology groups of a finite quandle X, denoted by $\mathrm{Free}H_n^R(X)$ and $\mathrm{Free}H_n^Q(X)$, respectively, were completely computed [6, 9]. In particular, for the dihedral quandle R_m of order m we have:

$$\mathrm{Free}H_n^R(R_m) = \begin{cases} \mathbb{Z}, & \text{if } m \text{ is odd;} \\ \mathbb{Z}^{2^n}, & \text{if } m \text{ is even,} \end{cases}$$
$$\mathrm{Free}H_n^Q(R_m) = \begin{cases} \mathbb{Z}, & \text{if } m \text{ is odd and } n = 1; \\ 0, & \text{if } m \text{ is odd and } n > 1; \\ \mathbb{Z}^2, & \text{if } m \text{ is even.} \end{cases}$$

However, it is a bit difficult to compute the torsion parts because there are fewer methods to calculate them than the group homology theory. As for the torsion parts of the rack and quandle homology of dihedral quandles, if m is odd prime, then

$$\mathrm{Tor}H_n^Q(R_m) = \mathbb{Z}_m^{f_n},$$

where $f_n = f_{n-1} + f_{n-3}$ and $f_1 = f_2 = 0, f_3 = 1$ [5, 14]. Moreover,

$$|R_m|\mathrm{Tor}H_n^R(R_m) = 0 \text{ and } |R_m|\mathrm{Tor}H_n^Q(R_m) = 0$$

if m is odd [14, 15]. However, little is known when m is even.

2. Annihilation Theorems for Quandle Extensions

Quandle cocycles can be used to construct extensions of quandles in a similar way to obtain extensions of groups using group cocycles. An abelian extension theory for quandles was introduced by Carter, Elhamdadi, Nikiforou, and Saito

[2], and a generalization to extensions with a dynamical cocycle was defined by Andruskiewitsch and Graña [1].

Definition 2.1.([1, 3]) Let X be a quandle and S be a non-empty set. Let $\alpha : X \times X \to \mathrm{Fun}(S \times S, S) = S^{S\times S}$ be a function, so that for $a, b \in X$ and $s, t \in S$ we have $\alpha_{a,b}(s,t) \in S$. Then $S \times X$ is a quandle by the operation $(s,a) * (t,b) = (\alpha_{a,b}(s,t), a*b)$, where $a*b$ denotes the quandle operation in X, if and only if α satisfies the following conditions:

(1) $\alpha_{a,a}(s,s) = s$ for all $a \in X$ and $s \in S$,

(2) $\alpha_{a,b}(-,t) : S \to S$ is a bijection for each $a, b \in X$ and for each $t \in S$,

(3) $\alpha_{a*b,c}(\alpha_{a,b}(s,t),u) = \alpha_{a*c,b*c}(\alpha_{a,c}(s,u), \alpha_{b,c}(t,u))$ for all $a,b,c \in X$ and $s,t,u \in S$.

Such a function α is called a *dynamical quandle cocycle*. The quandle constructed above is denoted by $S \times_\alpha X$, and is called the *extension of X by a dynamical cocycle* α.

We first discuss annihilation of rack and quandle homology groups of quandle extensions using certain dynamic cocycles.

Theorem 2.2. *Suppose that X is a finite quasigroup quandle and S is a non-empty set. Let α be the dynamical cocycle defined by $\alpha_{a,b}(-,t) = \mathrm{Id}_S$ for all $a, b \in X$ and for all $t \in S$. Then the torsion subgroups of $H_n^R(S \times_\alpha X)$ and $H_n^Q(S \times_\alpha X)$ are annihilated by $|X|$.*

Proof. Denote an element (s,x) of $S \times_\alpha X$ by x^s. Let $\mathbf{x} = (x_1^{s_1}, \cdots, x_n^{s_n}) \in C_n^R(S \times_\alpha X)$. We define two chain maps $f_r^j, f_s^j : C_n^R(S \times_\alpha X) \to C_n^R(S \times_\alpha X)$ by

$$f_r^j(\mathbf{x}) = |X|(x_j^{s_1}, \dots, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) \text{ for } 1 \le j \le n,$$

$$f_s^j(\mathbf{x}) = \sum_{y \in X} (y^{s_1}, \dots, y^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) \text{ for } 1 \le j \le n.$$

Using the following chain homotopies D_n^j and F_n^j, we show that $D_n^j : f_r^j \simeq f_s^j$ for each $1 \le j \le n$ and $F_n^j : f_s^{j-1} \simeq f_r^j$ for each $2 \le j \le n$:

$$D_n^j(\mathbf{x}) = \sum_{y \in X} (x_j^{s_1}, \dots, x_j^{s_j}, y^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) \text{ for } 1 \le j \le n,$$
$$F_n^j(\mathbf{x}) = \sum_{y \in X} (x_j^{s_1}, \dots, x_j^{s_{j-1}}, y^{s_j}, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) \text{ for } 2 \le j \le n.$$

Note that $(s,a) * (t,b) = (s, a*b)$, i.e., $a^s * b^t = (a*b)^s$ since $\alpha_{a,b}(-,t) = \mathrm{Id}_S$ for any $a, b \in X$ and for any $s, t \in S$.

Define face maps $d_i^{(*_0)}, d_i^{(*)} : C_n^R(X) \to C_{n-1}^R(X)$ of the boundary homomorphism ∂_n by

$$d_i^{(*_0)}(\mathbf{x}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \text{ and}$$
$$d_i^{(*)}(\mathbf{x}) = (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n),$$

i.e., $\partial_n = \sum_{i=2}^{n} (-1)^i (d_i^{(*_0)} - d_i^{(*)})$.

Let us first consider the chain homotopy $D_n^j : C_n^R(S \times_\alpha X) \to C_{n+1}^R(S \times_\alpha X)$.

(1) Assume that $i \le j$. The idempotent condition of a quandle implies that

$$d_i^{(*)} D_n^j(\mathbf{x}) = \sum_{y \in X} (x_j^{s_1}, \dots, x_j^{s_{i-1}}, x_j^{s_{i+1}}, \dots, x_j^{s_j}, y^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}).$$

Note that the formula above does not depend on $*$, in particular $(d_i^{(*_0)} - d_i^{(*)}) D_n^j = 0$. Moreover,

$$D_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{y \in X} (x_{j+1}^{s_1}, \dots, x_{j+1}^{s_{i-1}}, x_{j+1}^{s_{i+1}}, \dots, x_{j+1}^{s_{j+1}}, y^{s_{j+1}}, x_{j+2}^{s_{j+2}}, \dots, x_n^{s_n})$$

which is the same as $D_{n-1}^j d_i^{(*_0)}(\mathbf{x})$, hence $D_{n-1}^j (d_i^{(*_0)} - d_i^{(*)}) = 0$.

(2) When $j + 2 \le i \le n + 1$, $\sum_{y \in X} (y^{s_j} * x_{i-1}^{s_{i-1}}) = \sum_{y \in X} (y * x_{i-1})^{s_j} = \sum_{y \in X} (y^{s_j})$ by the invertibility condition of a quandle, and therefore

$$d_i^{(*)} D_n^j(\mathbf{x}) = \sum_{y \in X} ((x_j * x_{i-1})^{s_1}, \dots, (x_j * x_{i-1})^{s_j}, y^{s_j},$$
$$(x_{j+1} * x_{i-1})^{s_{j+1}}, \dots, (x_{i-2} * x_{i-1})^{s_{i-2}}, x_i^{s_i}, \cdots, x_n^{s_n}).$$

On the other hand, if $j + 1 \le i$, then we have

$$D_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{y \in X} ((x_j * x_i)^{s_1}, \dots, (x_j * x_i)^{s_j}, y^{s_j},$$
$$(x_{j+1} * x_i)^{s_{j+1}}, \dots, (x_{i-1} * x_i)^{s_{i-1}}, x_{i+1}^{s_{i+1}}, \cdots, x_n^{s_n}),$$

i.e., $d_{i+1}^{(*_0)} D_n^j = D_{n-1}^j d_i^{(*_0)}$ and $d_{i+1}^{(*)} D_n^j = D_{n-1}^j d_i^{(*)}$ for $j + 1 \le i \le n$.

(3) Suppose that $i = j + 1$. Then we have

$$d_i^{(*_0)} D_n^j(\mathbf{x}) = |X| (x_j^{s_1}, \dots, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) = f_r^j(\mathbf{x}).$$

Note that $\sum_{y \in X} (x_j^{s_k} * y^{s_j}) = \sum_{y \in X} (x_j * y)^{s_k} = \sum_{y \in X} (y^{s_k})$ as X is a quasigroup quandle. Therefore, we obtain the following equality:

$$d_i^{(*)} D_n^j(\mathbf{x}) = \sum_{y \in X} (y^{s_1}, \dots, y^{s_j}, x_{j+1}^{s_{j+1}}, \dots, x_n^{s_n}) = f_s^j(\mathbf{x}).$$

By (1), (2), and (3),

$$\partial_{n+1}D_n^j(\mathbf{x}) + D_{n-1}^j\partial_n(\mathbf{x}) = (-1)^{j+1}(f_r^j(\mathbf{x}) - f_s^j(\mathbf{x})),$$

hence, $D_n^j : f_r^j \simeq f_s^j$ for each $1 \leq j \leq n$.

We next consider the chain homotopy $F_n^j : C_n^R(S \times_\alpha X) \to C_{n+1}^R(S \times_\alpha X)$.

(4) If $i \leq j-1$, then

$$d_i^{(*)}F_n^j(\mathbf{x}) = \sum_{y\in X}(x_j^{s_1}, \ldots, x_j^{s_{i-1}}, x_j^{s_{i+1}}, \ldots, x_j^{s_{j-1}}, y^{s_j}, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \ldots, x_n^{s_n}),$$

so this formula does not depend on $*$, in particular $(d_i^{(*_0)} - d_i^{(*)})F_n^j = 0$.
Moreover, if $i \leq j$, then

$$F_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{y\in X}(x_{j+1}^{s_1}, \ldots, x_{j+1}^{s_{i-1}}, x_{j+1}^{s_{i+1}}, \ldots, x_{j+1}^{s_j}, y^{s_{j+1}}, x_{j+1}^{s_{j+1}}, x_{j+2}^{s_{j+2}}, \ldots, x_n^{s_n})$$

which is the same as $F_{n-1}^j d_i^{(*_0)}(\mathbf{x})$, hence $F_{n-1}^j(d_i^{(*_0)} - d_i^{(*)}) = 0$.

(5) Note that $\sum_{y\in X}(y^{s_j} * x_{i-1}^{s_{i-1}}) = \sum_{y\in X}(y * x_{i-1})^{s_j} = \sum_{y\in X}(y^{s_j})$ by the invertibility condition of a quandle.
If $i = j+1$, then $(d_i^{(*_0)} - d_i^{(*)})F_n^j = 0$. Assume that $j+2 \leq i \leq n+1$.
Then

$$d_i^{(*)}F_n^j(\mathbf{x}) = \sum_{y\in X}((x_j * x_{i-1})^{s_1}, \ldots, (x_j * x_{i-1})^{s_{j-1}}, y^{s_j}, (x_j * x_{i-1})^{s_j}, \ldots, (x_{i-2} * x_{i-1})^{s_{i-2}}, x_i^{s_i}, \cdots, x_n^{s_n}).$$

On the other hand, if $j+1 \leq i$, then

$$F_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{y\in X}((x_j * x_i)^{s_1}, \ldots, (x_j * x_i)^{s_{j-1}}, y^{s_j}, (x_j * x_i)^{s_j}, \ldots, (x_{i-1} * x_i)^{s_{i-1}}, x_{i+1}^{s_{i+1}}, \cdots, x_n^{s_n}).$$

Thus, $d_{i+1}^{(*_0)}F_n^j = F_{n-1}^j d_i^{(*_0)}$ and $d_{i+1}^{(*)}F_n^j = F_{n-1}^j d_i^{(*)}$ for $j+1 \leq i \leq n$.

(6) Assume that $i = j$. Then we have

$$d_i^{(*_0)}F_n^j(\mathbf{x}) = |X|(x_j^{s_1}, \ldots, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \ldots, x_n^{s_n}) = f_r^j(\mathbf{x}).$$

Since X is a quasigroup quandle, $\sum_{y\in X}(x_j^{s_k} * y^{s_j}) = \sum_{y\in X}(x_j * y)^{s_k} = \sum_{y\in X}(y^{s_k})$. Hence, we have

$$d_i^{(*)}F_n^j(\mathbf{x}) = \sum_{y\in X}(y^{s_1}, \ldots, y^{s_{j-1}}, x_j^{s_j}, \ldots, x_n^{s_n}) = f_s^{j-1}(\mathbf{x}).$$

By (4), (5), and (6),

$$\partial_{n+1}F_n^j(\mathbf{x}) + F_{n-1}^j\partial_n(\mathbf{x}) = (-1)^j(f_r^j(\mathbf{x}) - f_s^{j-1}(\mathbf{x})),$$

thus, $F_n^j : f_s^{j-1} \simeq f_r^j$ for each $2 \leq j \leq n$.

Finally, we obtain the following sequence of chain homotopic chain maps:

$$|X|\mathrm{Id}_{C_n^R(S\times_\alpha X)} = f_r^1 \simeq f_s^1 \simeq f_r^2 \simeq \cdots \simeq f_r^{n-1} \simeq f_s^{n-1} \simeq f_r^n \simeq f_s^n.$$

Let $\tau(S)$ denote the trivial quandle with the set S, i.e., $s * t = s$ for any $s, t \in \tau(S)$. Consider the chain maps $p : C_n^R(S \times_\alpha X) \to C_n^R(\tau(S))$ and $\phi : C_n^R(\tau(S)) \to C_n^R(S \times_\alpha X)$ given by

$$p(x_1^{s_1}, \cdots, x_n^{s_n}) = (s_1, \cdots, s_n) \text{ and } \phi(s_1, \cdots, s_n) = \sum_{y\in X}(y^{s_1}, \cdots, y^{s_n}).$$

Clearly $\phi \circ p = f_s^n$, so that we have the same induced homomorphisms

$$|X|\mathrm{Id}_{H_n^R(S\times_\alpha X)} = (f_r^1)_* = (f_s^n)_* = \phi_* \circ p_*,$$

where $p_* : H_n^R(S \times_\alpha X) \to H_n^R(\tau(S))$ and $\phi_* : H_n^R(\tau(S)) \to H_n^R(S \times_\alpha X)$. Since $\tau(S)$ is a trivial quandle, $H_n^R(\tau(S))$ has no torsion. Therefore,

$$|X|\mathbf{z} = |X|\mathrm{Id}_{H_n^R(S\times_\alpha X)}(\mathbf{z}) = \phi_*(p_*(\mathbf{z})) = \phi_*(0) = 0$$

for every $\mathbf{z} \in \mathrm{Tor}(H_n^R(S \times_\alpha X))$ as desired.

Furthermore, since the rack homology of a quandle splits into the quandle homology and the degenerate homology [9], i.e., $H_n^R(S \times_\alpha X) = H_n^Q(S \times_\alpha X) \oplus H_n^D(S \times_\alpha X)$, the torsion of $H_n^Q(S \times_\alpha X)$ is also annihilated by $|X|$. □

Using Theorem 2.2, we partially prove Conjecture 1.1.

Corollary 2.3. *Let R_{2k} be the dihedral quandle of order $2k$. The number k annihilates both $\mathrm{Tor}H_n^R(R_{2k})$ and $\mathrm{Tor}H_n^Q(R_{2k})$, if k is odd.*

Proof. If $k = 1$, we are done because R_2 is a trivial quandle and therefore $H_n^R(R_2)$ and $H_n^Q(R_2)$ have no torsion by definition.

Suppose that $k > 1$. Let $S = \{e, o\}$ be the set with two elements. We define the dynamical cocycle $\alpha : R_k \times R_k \to S^{S\times S}$ by $\alpha_{[a],[b]}(-, t) = \mathrm{Id}_S$ for all $[a], [b] \in R_k$ and for all $t \in S$. Note that if k is odd, then $S \times_\alpha R_k \cong R_{2k}$ via the quandle isomorphism $h : S\times_\alpha R_k \to R_{2k}$ defined by $h(e, [m]) = [2m]$ and $h(o, [m]) = [2m+k]$ for each $[m] \in R_k$. Furthermore, since k is odd, the dihedral quandle R_k is a quasigroup quandle. Therefore, Theorem 2.2 implies that the torsion subgroups of $H_n^R(R_{2k}) = H_n^R(S \times_\alpha R_k)$ and $H_n^Q(R_{2k}) = H_n^Q(S \times_\alpha R_k)$ are annihilated by k. □

Table 2 contains some computational results on homology groups of dihedral quandles of small even order.

Table 2: Homology of dihedral quandles of order $2k$ when k is odd

n	1	2	3
$H_n^R(R_2)$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$
$H_n^Q(R_2)$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$H_n^D(R_2)$	0	$\mathbb{Z}^2$	$\mathbb{Z}^6$
$H_n^R(R_6)$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8 \oplus \mathbb{Z}_3^2$
$H_n^Q(R_6)$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3^2$
$H_n^D(R_6)$	0	$\mathbb{Z}^2$	$\mathbb{Z}^6$
$H_n^R(R_{10})$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8 \oplus \mathbb{Z}_5^2$
$H_n^Q(R_{10})$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_5^2$
$H_n^D(R_{10})$	0	$\mathbb{Z}^2$	$\mathbb{Z}^6$

Remark 2.4. Corollary 2.3 does not hold when we replace the condition "k is odd" with "k is even" in the corollary. For example, $H_3^Q(R_8) = \mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^2$.

3. Future Research

Corollary 2.3 can be used to compute rack and quandle homology groups of dihedral quandles of even order. In 2016, Takefumi Nosaka suggested the following open problem during the Knots in Hellas conference:

Problem 3.1. *$H_3^Q(R_{2p}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ if p is odd prime.*

An open problem of whether $|\mathrm{Inn}(X)|$ of a finite quandle X annihilates $\mathrm{Tor}H_n^R(X)$ and $\mathrm{Tor}H_n^Q(X)$ for every dimension n was suggested in [16]. It is known that $\mathrm{Inn}(R_m)$ is isomorphic to the dihedral group of order $2m$ if m is odd and the dihedral group of order m if m is even. One can prove the following open problem by generalizing Corollary 2.3 in case of even k:

Problem 3.2. *$\mathrm{Tor}H_n^R(R_m)$ and $\mathrm{Tor}H_n^Q(R_m)$ are annihilated by $|\mathrm{Inn}(R_m)|$ for all n.*

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