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Minkowski's Inequality for Variational Fractional Integrals

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Abstract. Minkowski's inequality is one of the most famous inequalities in mathematics, and has many applications. In this paper, we give Minkowski's inequality for generalized variational integrals that are based on a supermultiplicative function. Our results include previous results about fractional integral inequalities of Minkowski's type.

1. Introduction

Minkowski's inequality is unequivocally one of the most famous inequalities in mathematics. The well-known Minkowski integral inequality is given as follows:

Theorem 1.1.([1]) *Let $s \geq 1$ and $\int_a^b f^s(x)\,dx$ and $\int_a^b g^s(x)\,dx$ be finite. Then*

$$\left(\int_a^b \left(f^s(x) + g^s(x)\right) dx\right)^{\frac{1}{s}} \leq \left(\int_a^b f^s(x)\,dx\right)^{\frac{1}{s}} + \left(\int_a^b g^s(x)dx\right)^{\frac{1}{s}}.$$

The following reverse Minkowski integral inequality was obtained by Bougoffa [3] in 2006.

Theorem 1.2.([3]) *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M,$$

then

$$\left(\int_a^b f^s(x)\,dx\right)^{\frac{1}{s}} + \left(\int_a^b g^s(x)dx\right)^{\frac{1}{s}} \leq c\left(\int_a^b \left(f^s(x) + g^s(x)\right) dx\right)^{\frac{1}{s}},$$

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where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Variational integrals, which generalize fractional integrals, play an important role in all fields of mathematics [2, 5]. They are versatile, and have wide application in applied mathematics.

In 2010, Agrawal [2] introduced a generalized variational integral which generalizes the Riemann-Liouville fractional integral.

Definition 1.3.([2]) A generalized variational integral of order α of a real function f is defined as

$$\mathbb{S}^{\alpha}_{\langle a,t,b,p,q\rangle} f(t) = p\int_a^t k_\alpha(t,s)f(s)ds + q\int_t^b k_\alpha(s,t)f(s)ds = \mathbb{S}^{\alpha}_{P} f(t),$$

where $t \in (a,b),\ p,q \in \mathbb{R},\ P = \langle a,t,b,p,q\rangle$ and $k_\alpha(t,s)$ is a kernel which is nonnegative and depends upon a parameter $\alpha > 0$.

Remark 1.4. Let $k_\alpha(t,s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$ and $P = \langle a,t,b,p,q\rangle$.

(i) If $P = P_1 = \langle a,t,b,1,0\rangle$, then the left Riemann-Liouville fractional integral yields i.e,

$$\mathbb{S}^{\alpha}_{P_1} f(t) = \int_a^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s)ds = \mathbb{J}^{\alpha}_{a+} f(t).$$

(ii) If $P = P_2 = \langle a,t,b,0,1\rangle$, then the right Riemann-Liouville fractional integral is concluded as

$$\mathbb{S}^{\alpha}_{P_2} f(t) = \int_t^b \frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1} f(s)ds = \mathbb{J}^{\alpha}_{b-} f(t).$$

(iii) If $P = P_3 = \langle a,t,b,\frac{1}{2},\frac{1}{2}\rangle$, then we have the Riesz fractional integral as follows

$$\mathbb{S}^{\alpha}_{P_3} f(t) = \frac{1}{2}\mathbb{S}^{\alpha}_{P_1} f(t) + \frac{1}{2}\mathbb{S}^{\alpha}_{P_2} f(t) = \frac{1}{2}\mathbb{J}^{\alpha}_{a+} f(t) + \frac{1}{2}\mathbb{J}^{\alpha}_{b-} f(t).$$

In 2010, Dahmani [4] proved the following inequalities related to Minkowski's inequality for Riemann-Liouville fractional integrals; these generalize the results in [3].

Theorem 1.5.([4]) *Let* $\alpha > 0, s \geq 1$ *and* f, g *be positive on* $[0,\infty)$ *such that* $t > 0$, $\mathbb{J}^{\alpha}_{0+} f^s(t) < \infty$ *and* $\mathbb{J}^{\alpha}_{0+} g^s(t) < \infty$. *If* $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0,t]$, *then*

$$\left[\mathbb{J}^{\alpha}_{0+} f^s(t)\right]^{\frac{1}{s}} + \left[\mathbb{J}^{\alpha}_{0+} g^s(t)\right]^{\frac{1}{s}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}\left[\mathbb{J}^{\alpha}_{0+} (f+g)^s(t)\right]^{\frac{1}{s}}.$$

Theorem 1.6.([4]) *Suppose that* $\alpha > 0, s \geq 1$ *and* f, g *are two positive functions on* $(0,\infty)$ *such that* $t > 0, \mathbb{J}^\alpha f^s(t) < \infty$ *and* $\mathbb{J}^\alpha g^s(t) < \infty$. *If* $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0,t]$, *then*

$$\Big[\mathbb{J}^\alpha f^s(t)\Big]^{\frac{2}{s}} + \mathbb{J}^\alpha g^s(t)\Big]^{\frac{2}{s}} \geq (\frac{(M+1)(m+1)}{M} - 2)\Big[\mathbb{J}^\alpha f^s(t)\Big]^{\frac{1}{s}}\Big[\mathbb{J}^\alpha g^s(t)\Big]^{\frac{1}{s}}.$$

In this article, we are going to extend these theorems for the generalized variational integral.

2. Main Results

In this section, we give a generalized Minkowski type inequality for the generalized variational integral. Before we begin our results we need the following definition and lemma.

Definition 2.1.([6, 7, 8]) A function $\varphi : (0,\infty) \to (0,\infty)$ is called *C-submultiplicative* with $C > 0$ if

$$\varphi(xy) \leq C\varphi(x)\varphi(y), \tag{2.1}$$

for all $x, y \in (0,\infty)$. If inequality (2.1) is reversed, then φ will be called *C-supermultiplicative.*

In the following theorem, we give a more general version of Theorem 1.5 based on a supermultiplicative function.

Theorem 2.2. *Let* f *and* g *be positive functions. Let two functions* $\varphi_i : (0,\infty) \to (0,\infty), i = 1,2$ *be increasing such that* φ_1 *is* M_1*-supermultiplicative and* φ_2 *is* M_2*-submultiplicative and* $\mathbb{S}_P^\alpha\varphi_i(f)(t)$ *and* $\mathbb{S}_P^\alpha\varphi_i(g)(t)$ *are finite. If there exist* $C_1, C_2 \in (0,\infty)$ *such that*

$$(f+g)(t) \geq \max\{C_1 f(t), C_2 g(t)\},$$

then

$$\begin{aligned}&\varphi_2\big[\mathbb{S}_P^\alpha\varphi_1(f)(t)\big] + \varphi_2\big[\mathbb{S}_P^\alpha\varphi_1(g)(t)\big]\\ &\leqslant M_2\left(\varphi_2(\frac{1}{M_1\varphi_1(C_1)}) + \varphi_2(\frac{1}{M_1\varphi_1(C_2)})\right)\varphi_2\big[\mathbb{S}_P^\alpha\varphi(f+g)(t)\big]\end{aligned}$$

Proof. Since $(f+g)(t) \geq \max\{C_1 f(t), C_2 g(t)\}$ and φ_1 is increasing, then

$$\varphi_1(C_1 f(t)) \leq \varphi_1(f+g)(t), \tag{2.2}$$
$$\varphi_1(C_2 g(t)) \leq \varphi_1(f+g)(t), \tag{2.3}$$

If φ_1 is M_1-supermultiplicative, then by Definition 2.1, (2.2) and (2.3), we obtain

$$M_1\varphi_1(C_1)\varphi_1(f(t)) \le \varphi_1(f+g)(t), \tag{2.4}$$

$$M_1\varphi_1(C_2)\varphi_1(g(t)) \le \varphi_1(f+g)(t). \tag{2.5}$$

Multiplying both sides of (2.4) and (2.5) by $pk_\alpha(t,\tau)$ and integrating respect to τ on $[a,t]$, we have

$$pM_1\varphi_1(C_1)\int_a^t k_\alpha(t,\tau)\varphi_1(f)(\tau)\,d\tau \le p\int_a^t k_\alpha(t,\tau)\varphi_1(f+g)(\tau)d\tau, \tag{2.6}$$

$$pM_1\varphi_1(C_2)\int_a^t k_\alpha(t,\tau)\varphi_1(g)(\tau)\,d\tau \le p\int_a^t k_\alpha(t,\tau)\varphi_1(f+g)(\tau)d\tau. \tag{2.7}$$

Similarly,

$$qM_1\varphi_1(C_1)\int_t^b k_\alpha(\tau,t)\varphi_1(f)d\tau \le q\int_t^b k_\alpha(\tau,t)\varphi_1(f+g)(\tau)d\tau. \tag{2.8}$$

$$qM_1\varphi_1(C_2)\int_t^b k_\alpha(\tau,t)\varphi_1(g)d\tau \le q\int_t^b k_\alpha(\tau,t)\varphi_1(f+g)(\tau)d\tau. \tag{2.9}$$

Now by adding (2.6) and (2.8), we have

$$\begin{aligned}&pM_1\varphi_1(C_1)\int_a^t k_\alpha(t,\tau)\varphi_1(f)(\tau)\,d\tau + qM_1\varphi_1(C_1)\int_t^b k_\alpha(\tau,t)\varphi_1(f)d\tau\\ &\le p\int_a^t k_\alpha(t,\tau)\varphi_1(f+g)(\tau)d\tau + q\int_t^b k_\alpha(\tau,t)\varphi_1(f+g)(\tau)d\tau,\end{aligned}$$

which is equivalent to

$$M_1\varphi_1(C_1)[\mathbb{S}_P^\alpha\varphi_1(f)(t)] \le \mathbb{S}_P^\alpha\varphi_1(f+g)(t)$$

$$\mathbb{S}_P^\alpha\varphi_1(f)(t) \le \frac{1}{M_1\varphi_1(C_1)}[\mathbb{S}_P^\alpha\varphi_1(f+g)(t)]$$

Since φ_2 is M_2-submultiplicative, then

$$\varphi_2\big[\mathbb{S}_P^\alpha\varphi_1(f)(t)\big] \le M_2\varphi_2\Big(\frac{1}{M_1\varphi_1(C_1)}\Big)\varphi_2\big[\mathbb{S}_P^\alpha\varphi_1(f+g)(t)\big]. \tag{2.10}$$

Now by adding (2.7) and (2.9), we have

$$\begin{aligned}&pM_1\varphi_1(C_2)\int_a^t k_\alpha(t,\tau)\varphi_1(g)(\tau)\,d\tau + qM_1\varphi_1(C_2)\int_t^b k_\alpha(\tau,t)\varphi_1(g)d\tau\\ &\le p\int_a^t k_\alpha(t,\tau)\varphi_1(f+g)(\tau)d\tau + q\int_t^b k_\alpha(\tau,t)\varphi_1(f+g)(\tau)d\tau,\end{aligned}$$

which is equivalent to

$$M_1\varphi_1(C_2)[\mathbb{S}_P^{\alpha}\varphi_1(g)\,(t)] \leq \mathbb{S}_P^{\alpha}\varphi_1(f+g)\,(t)\,.$$

Then

$$\mathbb{S}_P^{\alpha}\varphi_1(g)(t) \leq \frac{1}{M_1\varphi_1(C_2)}[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)]$$

Hence,

$$\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi(g)(t)\big] \leq M_2\varphi_2(\frac{1}{M_1\varphi_1(C_2)})\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)\big]. \tag{2.11}$$

By adding (2.10) and (2.11),

$$\begin{aligned}&\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f)(t)\big] + \varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(g)(t)\big]\\ &\leqslant M_2\left(\varphi_2(\frac{1}{M_1\varphi_1(C_1)}) + \varphi_2(\frac{1}{M_1\varphi_1(C_2)})\right)\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)\big]\end{aligned}$$

we obtain the desired result. □

If $\varphi_1(x) = x^p, \varphi_2(x) = x^{\frac{1}{p}}, p \geq 1, M_1 = M_2 = 1,\ C_1 = \frac{M+1}{M}$ and $C_2 = m+1$ in Theorem 2.2, then the following corollary is achieved.

Corollary 2.3. *Let $\alpha > 0, p \geq 1$ and f, g be two positive functions on $[0,\infty)$ such that $\mathbb{S}_P^{\alpha}f^p(t)$ and $\mathbb{S}_P^{\alpha}g^p(t)$ are finite. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0,t]$, then*

$$\Big[\mathbb{S}_P^{\alpha}f^p(t)\Big]^{\frac{1}{p}} + \Big[\mathbb{S}_P^{\alpha}g^p(t)\Big]^{\frac{1}{p}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}\Big[\mathbb{S}_P^{\alpha}(f+g)^p(t)\Big]^{\frac{1}{p}}.$$

If $P = P_1 = \langle a, t, b, 1, 0\rangle$ in Corollary 2.3, then the following result holds.

Corollary 2.4.([4]) *Let $\alpha > 0, s \geq 1$ and f, g be positive on $[0,\infty)$ such that $t > 0,\ \mathbb{J}_{0+}^{\alpha}f^s(t) < \infty$ and $\mathbb{J}_{0+}^{\alpha}g^s(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M,\ \tau \in [0,t]$, then*

$$\Big[\mathbb{J}_{0+}^{\alpha}f^s(t)\Big]^{\frac{1}{s}} + \Big[\mathbb{J}_{0+}^{\alpha}g^s(t)\Big]^{\frac{1}{s}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}\Big[\mathbb{J}_{0+}^{\alpha}(f+g)^s(t)\Big]^{\frac{1}{s}}.$$

Theorem 2.5. *Let $\alpha > 0$, f and g be two nonnegative functions on $[0,\infty)$. Let two functions $\varphi_i : (0,\infty) \to (0,\infty), i = 1,2$ be increasing such that φ_1 is M_1-*supermultiplicative and φ_2 is M_2-submultiplicative and $\mathbb{S}_P^{\alpha}\varphi_i(f)(t)$ and $\mathbb{S}_P^{\alpha}\varphi_i(g)(t)$ are finite. If there exist $C_1, C_2 \in (0,\infty)$ such that

$$(f+g)\,(t) \geq \max\{C_1f(t), C_2g(t)\},$$

then

$$\begin{aligned}&\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f)(t)\big]\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(g)(t)\big]\\ &\leqslant M_2\varphi_2(\frac{1}{M_1\varphi_1(C_1)})\varphi_2(\frac{1}{M_1\varphi_1(C_2)})\left(\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)\big]\right)^2.\end{aligned}$$

Proof. Multiplying (2.10) and (2.11), we have

$$\begin{aligned}&\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f)(t)\big]\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(g)(t)\big]\\ &\leqslant M_2\varphi_2(\frac{1}{M_1\varphi_1(C_1)})\varphi_2(\frac{1}{M_1\varphi_1(C_2)})\left(\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)\big]\right)^2.\end{aligned}$$

Then

$$\frac{\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f)(t)\big]\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(g)(t)\big]}{M_2\varphi_2(\frac{1}{M_1\varphi_1(C_1)})\varphi_2(\frac{1}{M_1\varphi_1(C_2)})}\leq\Big(\varphi_2\big[\mathbb{S}_P^{\alpha}\varphi_1(f+g)(t)\big]\Big)^2. \qquad \square$$

If $P=P_1=\langle a,t,b,1,0\rangle,\varphi_1(x)=x^p,\varphi_2(x)=x^{\frac{1}{p}}$, $p\geq 1, M_1=M_2=1, C_1=\frac{M+1}{M}$ and $C_2=m+1$ in Theorem 2.5, then following result holds by using Minkowski inequality.

Corollary 2.6.([4]) *Suppose that $\alpha>0, s\geq 1$ and f,g are two positive functions on $(0,\infty)$ such that $t>0, \mathbb{J}^{\alpha}f^s(t)<\infty$ and $\mathbb{J}^{\alpha}g^s(t)<\infty$. If $0<m\leq\frac{f(\tau)}{g(\tau)}\leq M$, $\tau\in[0,t]$, then*

$$\Big[\mathbb{J}^{\alpha}f^s(t)\Big]^{\frac{2}{s}}+\mathbb{J}^{\alpha}g^s(t)\Big]^{\frac{2}{s}}\geq(\frac{(M+1)(m+1)}{M}-2)\Big[\mathbb{J}^{\alpha}f^s(t)\Big]^{\frac{1}{s}}\Big[\mathbb{J}^{\alpha}g^s(t)\Big]^{\frac{1}{s}}.$$

3. Conclusions

In this paper, we have proven a Minkowski type inequality for the generalized variational integral. We have also observed that the results obtained in this paper are generalizations of some earlier results. An interesting problem would be to study the methods in this paper to establish the Hermite-Hadamard inequalities for convex functions via the generalized variational integral.

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