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Rings which satisfy the Property of Inserting Regular Elements at Zero Products

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Abstract. This article concerns the class of rings which satisfy the property of inserting regular elements at zero products, and rings with such property are called *regular-IFP*. We study the structure of regular-IFP rings in relation to various ring properties that play roles in noncommutative ring theory. We investigate conditions under which the regular-IFPness pass to polynomial rings, and equivalent conditions to the regular-IFPness.

1. Introduction

Throughout this article every ring is an associative ring with identity. Let R be a ring. An element u of R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. A *left regular* element is defined similarly. An element is *regular* if it is both left and right regular (and hence not a zero divisor). We use $C(R)$ and $U(R)$ to de-

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note the monoid of regular elements and the group of units in R, respectively. The Wedderburn radical (i.e., sum of all nilpotent ideals), the upper nilradical (i.e., the sum of all nil ideals), the lower nilradical (i.e., the intersection of all prime ideals), the Jacobson radical, and the set of all nilpotent elements in R are denoted by $N_0(R), N^*(R)$, $N_*(R)$, $J(R)$, and $N(R)$, respectively. It is well-known that $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$ and $C_{f(x)}$ denotes the set of all coefficients of $f(x)$ for $f(x) \in R[x]$. Denote the n by n full (resp., upper triangular) matrix ring over R by $M_n(R)$ (resp., $T_n(R)$). $D_n(R)$ denotes the subring $\{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ of $T_n(R)$. Let I_n and E_{ij} be the identity matrix and the matrix, with (i,j)-entry 1 and elsewhere 0, in $M_n(R)$, respectively. Let $\mathbb{Z}$ (resp., $\mathbb{Z}_n$) denote the ring of integers (modulo n), and $\mathbb{R}$ denote the field of real numbers.

2. Regular-IFP Rings

In this section we study the properties of regular-IFP rings as well as the relations between regular-IFP rings and ring properties that play important roles in noncommutative ring theory. Due to Bell [4], a ring is called *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. Following Kim et al. [15], a ring R is called *unit-IFP* if $ab = 0$ for $a, b \in R$ implies $aU(R)b = 0$. IFP rings are clearly unit-IFP, but not conversely by [15, Example 1.1]. A ring R is usually called *reduced* if $N(R) = 0$. Commutative rings are clearly IFP; and reduced rings are easily shown to be IFP, but not conversely because there exist many non-reduced commutative ring. A ring is usually called *Abelian* if every idempotent is central. Unit-IFP rings are Abelian by [15, Lemma 1.2(2)].

Definition 2.1. A ring R is called *regular-IFP* if $ab = 0$ for $a, b \in R$ implies $aC(R)b = 0$.

Regular-IFP rings are clearly unit-IFP (hence Abelian), but not conversely by the following example.

Example 2.2. There exists a unit-IFP ring that is not regular-IFP. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Identify a, b with their images in R for simplicity. Then R is unit-IFP by [15, Example 1.1]. But R is not a regular-IFP ring because $b^2 = 0$ and $bab \neq 0$ where $a \in C(R)$.

Remark 2.3.

(1) The following conditions are equivalent, which can be proved by applying the regular-IFPness iteratively:

   (i) A ring R is regular-IFP;

   (ii) $a_1C(R)a_2C(R)a_3 \cdots a_{n-1}C(R)a_n = 0$ whenever $a_1a_2 \cdots a_n = 0$ for $a_1, a_2, \ldots, a_n \in R$.

(2) $D_3(R)$ is (regular-)IFP over a reduced ring R by [9, Proposition 2.1]. However $M_n(R)$ and $T_n(R)$, over any ring R for $n \geq 2$, cannot be regular-IFP since they are not Abelian, noting that unit-IFP (or regular-IFP) rings are Abelian.

(3) There exists an Abelian ring that is not regular-IFP. Set $R = D_n(S)$ for $n \geq 4$ over an Abelian ring S. Then R is Abelian by [10, Lemma 2]. Let $A = E_{12}, B = E_{34} \in R$. Then $AB = 0$. Consider $C = I_n + E_{23} \in R$. Then $C \in C(R)$ clearly. But $ACB = E_{14} \neq 0$, so that R is not regular-IFP.

(4) Let R be a regular-IFP ring such that $R = C(R) \cup N(R)$. Let $ab = 0$ for $a, b \in R$. Then $aC(R)b = 0$ since R is regular-IFP. Moreover $aN(R)b = 0$ by [15, Lemma 1.2(2)]. Thus R is IFP.

Based on Armendariz [3, Lemma 1], a ring R is called *Armendariz* if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, by Rege and Chhawchharia [19]. Reduced rings are Armendariz by [3, Lemma 1]. The concepts of Armendariz rings and commutative rings are independent of each other by Example 2.2 and [19, Example 3.2], noting that the ring R in Example 2.2 is Armendariz by [2, Example 4.8].

By Goodearl [7], a ring R is called (*von Neumann*) *regular* if for every $a \in R$ there exists $b \in R$ such that $a = aba$, and a ring R is called *strongly regular* if $a \in a^2R$ for every $a \in R$. It is easily checked that $J(R) = 0$ for every regular ring R, and note that a ring is strongly regular if and only if it is Abelian regular, by [7, Theorems 3.2 and 3.5]. Recall that unit-IFP rings are Abelian, and Armendariz rings are also Abelian by [12, Corollary 8]. So for a regular ring R, we have that R is reduced if and only if R is Armendariz if and only if R is IFP if and only if R is regular-IFP if and only if unit-IFP if and only if R is Abelian, by [7, Theorem 3.2].

Following [11], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. Finite rings are clearly locally finite, but not conversely by the existence of algebraic closures of finite fields. It is shown that a ring is locally finite if and only if every finite subset generates a finite subring, in [11, Theorem 2.2(1)].

Proposition 2.4

(1) *Let R be a locally finite ring.*

  (i) *If R is an Armendariz ring, then it is regular-IFP.*

  (ii) *If R is a regular-IFP ring, then $R/J(R)$ is strongly regular with $J(R) = N(R)$.*

(2) *Let R be a right or left Artinian ring. If R is regular-IFP, then $R/J(R)$ is a strongly regular ring with $J(R) = N(R)$.*

Proof. (1)–(i) Let R be an Armendariz ring and suppose $ab = 0$ for $a, b \in R$. Let $c \in C(R)$. Since R is locally finite, $c^n \in I(R)$ for some $n \geq 1$ by the proof of [12,

Proposition 16]. But $c^n \in C(R)$, forcing $c^n = 1$. This yields $ac^nb = 0$, and so $acb = 0$ by [12, Lemma 7]. This implies $aC(R)b = 0$, and hence R is regular-IFP.
(1)–(ii) Since regular-IFP rings are Abelian, we obtain the result by [11, Proposition 2.5].
(2) It is well-known that $J(R)$ is nilpotent for the right (or left) Artinian ring R. Since $R/J(R)$ is semisimple Artinian and Abelian, $R/J(R)$ is a finite direct product of division rings. This completes the proof. □

The class of regular-IFP rings is not closed under homomorphic images as can be seen by the ring R in Example 2.2. But the following constructions preserve the regular-IFPness. We use $\oplus$ and $\prod$ to denote the direct sum and the direct product of rings, respectively.

Proposition 2.5.

(1) *Let R_λ ($\lambda \in \Lambda$) be Abelian rings. Then R_λ is regular-IFP for each $\lambda \in \Lambda$ if and only if $\prod_{\lambda\in\Lambda} R_\lambda$ is regular-IFP if and only if the subring of $\prod_{\lambda\in\Lambda} R_\lambda$ generated by $\oplus_{\lambda\in\Lambda} R_\lambda$ and $1_{\prod_{\lambda\in\Lambda} R_\lambda}$ is regular-IFP.*

(2) *Let R be an Abelian ring and $e^2 = e \in R$. Then R is regular-IFP if and only if both eR and $(1-e)R$ are regular-IFP.*

Proof. (1) Suppose that the subring of $\prod_{\lambda\in\Lambda} R_\lambda$ generated by $\oplus_{\lambda\in\Lambda} R_\lambda$ and $1_{\prod_{\lambda\in\Lambda} R_\lambda}$, say S, is regular-IFP. Let $ab = 0$ for $a, b \in R_\lambda$, and $c \in C(R_\lambda)$. Let $\alpha = (x_i)$ and $\beta = (y_j)$ be sequences in S such that $x_\lambda = a$, $x_i = 0$ for all $i \neq \lambda$, and $y_\lambda = b$, $y_j = 0$ for all $j \neq \lambda$. Then $\alpha\beta = 0$. Consider a sequence $\delta = (z_k) \in S$ in which $z_\lambda = c$ and $z_k = 1_{R_k}$ for all $k \neq \lambda$. Then $\delta \in C(S)$. Since S is regular-IFP, we have $\alpha\delta\beta = 0$. This yields $acb = 0$, and so R_λ is regular-IFP. The remainder of the proof is routine.
(2) The proof is obtained from (1) since $R = eR \oplus (1-e)R$ for $e^2 = e \in R$. □

For a given ring R, recall that R is called *local* if $R/J(R)$ is a division ring; R is called *semilocal* if $R/J(R)$ is semisimple Artinian; and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are clearly Abelian and semilocal.

Corollary 2.6. *A ring R is semiperfect regular-IFP if and only if R is a finite direct product of local regular-IFP rings.*

Proof. Suppose that R is regular-IFP and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [18, Proposition 3.7.2], i.e., each e_iRe_i is a local ring. Since R is regular-IFP, R is Abelian and so $e_iR = e_iRe_i$ for each i. This implies $R = \sum_{i=1}^n e_iR$. Then each e_iR is also a regular-IFP ring by Proposition 2.5.

Conversely assume that R is a finite direct product of local regular-IFP rings. Then R is semiperfect since local rings are semiperfect by [18, Corollary 3.7.1], and moreover R is regular-IFP by Proposition 2.5. □

Recall that homomorphic images of regular-IFP rings need not be regular-IFP. Considering this fact, one may ask whether a ring R is regular-IFP when every homomorphic image of R is regular-IFP. But the following provides a negative answer.

Example 2.7. There exists a non-regular-IFP ring R whose factor rings are regular-IFP. Consider $R = T_2(D)$ over a division ring D. Then every non-trivial factor ring is one of $R/J(R) \cong D \oplus D$, $R/I \cong D$ and $R/K \cong D$, where $J(R) = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}, I = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & D \\ 0 & D \end{pmatrix}$. These factor rings are reduced and so (regular-)IFP. But R cannot be regular-IFP because R is non-Abelian.

3. Extensions of Regular-IFP Rings

In this section we examine the regular-IFP property of ring extensions that play roles in noncommutative ring theory.

Regarding Remark 2.3(3), we have the following.

Proposition 3.1. *For a ring R the following conditions are equivalent:*

(1) *R is a reduced ring;*

(2) *$D_3(R)$ is an IFP ring;*

(3) *$D_3(R)$ is a regular-IFP ting;*

(4) *$D_3(R)$ is a unit-IFP ring;*

(5) *$AN(D_3(R))B = 0$ whenever $AB = 0$ for $A, B \in D_3(R)$.*

Proof. The equivalences of the conditions (1), (2), and (4) are proved by [15, Proposition 2.1], and so they are equivalent to (3).

(4) $\Rightarrow$ (5): Suppose that (4) holds and let $C \in N(D_3(R))$. Then $I_3 - C \in U(D_3(R))$, where I_3 denotes the identity matrix in $D_3(R)$. If $AB = 0$ for $A, B \in D_3(R)$, then $A(I_3 - C)B = 0$ by assumption since $I_3 - C \in U(D_3(R))$, implying that $ACB = 0$.

(5) $\Rightarrow$ (1): Suppose that (5) holds. Assume on the contrary that there exists $0 \neq a \in R$ with $a^2 = 0$. We refer to the argument in the proof of [14, Proposition 2.8]. Consider two matrices

$$A = \begin{pmatrix} a & a & -1 \\ 0 & a & -1 \\ 0 & 0 & a \end{pmatrix} \text{ and } B = \begin{pmatrix} a & 0 & a \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

in $D_3(R)$. Then $AB = 0$, but $AE_{12}B = aE_{13} \neq 0$ for $E_{12} \in N(D_3(R))$, which contradicts (5). Thus R is reduced. $\square$

Following Cohn [6], a ring R is called *reversible* if $ab = 0$ for $a, b \in R$ implies $ba = 0$. It is easily checked that reduced rings are reversible and reversible rings are IFP. The condition “R is a reduced ring” in Proposition 3.1 cannot be weaken by the condition “R is a reversible ring” by next example.

Example 3.2. We refer to the construction and argument in [16, Example 2.1]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Next, let I be the ideal of A generated by

$$\begin{aligned}&a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\ &b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\ &(a_0+a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{and } r_1r_2r_3r_4,\end{aligned}$$

where the constant terms of $r, r_1, r_2, r_3, r_4 \in A$ are zero. Now set $R = A/I$. We identity

$$a_0, a_1, a_2, b_0, b_1, b_2, c$$

with their images in R for simplicity. Then R is reversible by [16, Example 2.1] but not reduced clearly.

Now, consider

$$A = \begin{pmatrix} a_0 & a_1 & 0\\ 0 & a_0 & 0\\ 0 & 0 & a_0\end{pmatrix}, B = \begin{pmatrix} b_0 & b_1 & 0\\ 0 & b_0 & 0\\ 0 & 0 & b_0\end{pmatrix} \in D_3(R).$$

Then $AB = 0$. But

$$ACB = \begin{pmatrix} a_0 & a_1 & 0\\ 0 & a_0 & 0\\ 0 & 0 & a_0\end{pmatrix}\begin{pmatrix} c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c\end{pmatrix}\begin{pmatrix} b_0 & b_1 & 0\\ 0 & b_0 & 0\\ 0 & 0 & b_0\end{pmatrix} = \begin{pmatrix} 0 & a_0cb_1 + a_1cb_0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \neq 0$$

because $a_0cb_1 + a_1cb_0 \notin I$, noting $C = \begin{pmatrix} c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c\end{pmatrix} \in N(D_3(R))$. Thus $D_3(R)$ does not satisfy the condition (5) of Proposition 3.1.

Remark 3.3.

(1) Note that $D_2(R)$ over a reduced ring R is IFP by [16, Proposition 1.6] and so it is regular-IFP. Moreover, there exists a non-reduced non-commutative reversible ring R over which $D_2(R)$ is regular-IFP by [15, Example 2.2].

However, the ring S is always regular-IFP when $D_2(S)$ is regular-IFP. For, suppose that $D_2(S)$ is regular-IFP and let $ab = 0$ for $a, b \in S$. For $A = \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}, B = \begin{pmatrix} b & 0\\ 0 & b\end{pmatrix} \in D_2(S)$, we have $AB = 0$ and so $AC(D_2(S))B = 0$ by assumption. Set $C = \begin{pmatrix} c & 0\\ 0 & c\end{pmatrix}$ for any $c \in C(S)$. Then $C \in C(D_2(S))$ and $ACB = 0$, entailing $acb = 0$. Thus S is regular-IFP.

(2) Related to (1) above, there exists a reversible ring R such that $D_2(R)$ is not regular-IFP. Let $\mathbb{H}$ be the Hamilton quaternions over $\mathbb{R}$ and $R = D_2(\mathbb{H})$. Then R is reversible [16, Proposition 1.6]. We refer to the argument in [16, Example 1.7]. Consider

$$A = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

in $D_2(R)$. Then $AB = 0$.

Note that

$$C = \begin{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \end{pmatrix} \in C(D_2(R))$$

by [13, Lemma 2.1] because $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \in C(D_2(\mathbb{H}))$. But $ACB \neq 0$, hence $D_2(R)$ is not regular-IFP.

(3) For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$. If R is a reduced ring, then $V_n(R)$ is (regular)-IFP by [17, Lemma 2.3 and Proposition 3.3], but the converse does not hold in general as can be seen by the commutative ring $V_n(R)$ over a non-reduced commutative ring (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$) R for $n \geq 2$.

Proposition 3.4.

(1) *Let M be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is regular-IFP if and only if $M^{-1}R$ is regular-IFP.*

(2) *Let R be a ring. Then $R[x]$ is regular-IFP if and only if $R[x, x^{-1}]$ is regular-IFP.*

Proof. (1) It comes from the fact that $C(M^{-1}R) = M^{-1}C(R)$.
(2) Recall the ring of *Laurent polynomials* in x, written by $R[x, x^{-1}]$. Letting $M = \{1, x, x^2, x^3, \ldots\}$, M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$. By (1), the proof is completed. □

In [15, Example 2.7], we see an IFP ring R over which $R[x]$ is not unit-IFP, where the ring R is constructed in [12, Example 2]. So the regular-IFPness does not pass to polynomial rings since regular-IFP rings are unit-IFP. In the following we see a condition under which the regular-IFPness is preserved by polynomial rings.

Proposition 3.5. *Let R be a ring.*

(1) $\{c+c_1x+\ldots+c_tx^t \in R[x] \mid c \in C(R)$ *for* $t \geq 1\} \subseteq C(R[x])$ *and* $\{d_0+d_1x+\ldots+d_{s-1}x^{s-1}+dx^s \in R[x] \mid d \in C(R)$ *for* $s \geq 1\} \subseteq C(R[x])$.

(2) *If $R[x]$ is regular-IFP, then so is R.*

(3) *Let R be a regular-IFP ring such that $C(R[x]) = \{c + xN(R)[x] \mid c \in C(R)\}$. If R is Armendariz then $R[x]$ is regular-IFP.*

Proof. (1) Consider $h(x) = c + c_1x + \ldots + c_tx^t \in R[x]$ with $c \in C(R)$. Suppose that $h(x)g(x) = 0$ for any $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$. Then $h(x)g(x) = 0$ implies $cb_0 = 0$ and so $b_0 = 0$ since $c \in C(R)$. From $0 = h(x)g(x) = (c + c_1x + \cdots + c_tx^t)(b_1x + \cdots + b_nx^n)$, we have $cb_1 = 0$ and hence $b_1 = 0$. Continuing this process, we get $b_2 = 0, \ldots, b_n = 0$, showing that $g(x) = 0$. Thus $h(x) \in C(R[x])$ and so $\{c + c_1x_1 + \ldots + c_tx^t \in R[x] \mid c \in C(R)$ for $t \geq 1\} \subseteq C(R[x])$. The proof of the latter part is similar.

(2) It is routine.

(3) Suppose that R is Armendariz. Let $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. Then $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ since R is Armendariz. Hence $aC(R)b = 0$ by hypothesis. Moreover, $aN(R)b = 0$ by help of [15, Lemma 1.2(1)]. This implies that $f(x)C(R[x])g(x) = 0$, showing that $R[x]$ is regular-IFP. □

The next example shows that the condition "R is an Armendariz ring" cannot be dropped in Proposition 3.5(3).

Example 3.6. We use the ring and argument in [12, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Let B be the set of all polynomials with zero constant terms in A. Next, consider the ideal I of A generated by

$$a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2,$$
$$a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), r_1r_2r_3r_4$$

where $r \in A$ and $r_1, r_2, r_3, r_4 \in B$. Set $R = A/I$, and identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Then R is (regular-)IFP but not Armendariz, by [12, Example 2] and [19, Proposition 4.6].

Notice that $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = 0$. But

$$(a_0 + a_1x + a_2x^2)(1 + c)(b_0 + b_1x + b_2x^2) = (a_0 + a_1x + a_2x^2)c(b_0 + b_1x + b_2x^2) \neq 0$$

because $a_0cb_1 + a_1cb_0 \notin I$, noting $1 + c \in C(R[x])$ as in [15, Example 2.7]. Thus $R[x]$ is not regular-IFP.

Considering Proposition 3.5, it is natural to ask whether $C_{f(x)} \cap C(R) \neq \emptyset$ when $f(x) \in C(R[x])$, where R is a given ring. But the answer is negative by the following.

Example 3.7. Let A be any ring and $R = A \times A$. Consider a polynomial $f(x) = (1,0) + (0,1)x$ in $R[x]$. Suppose $f(x)g(x) = 0$ for $g(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ with $a_i = (b_i, c_i)$. Then from $f(x)g(x) = 0$, we obtain $f_1(x)g_1(x) = 0$ and $f_2(x)g_2(x) = 0$, where

$$f_1(x) = 1 + 0x, f_2(x) = 0 + x, \text{ and } g_1(x) = \sum_{i=0}^{m} b_i x^i, g_2(x) = \sum_{i=0}^{m} c_i x^i.$$

This implies $\sum_{i=0}^{m} b_i x^i = 0$ and $\sum_{i=0}^{m} c_i x^{i+1} = 0$, so that $b_i = 0$ and $c_i = 0$ for all i. Thus $a_i = 0$ for all i, and hence $f(x) \in C(R[x])$. But $(1,0), (0,1) \notin C(R)$.

We consider next some equivalent conditions to the regular-IFP property in relation to the sum of coefficients of polynomials which satisfy some property of inserting regular polynomials. For $f(x) \in R[x]$, let $f(1)$ denote the sum of all coefficients of $f(x)$.

Proposition 3.8. *For a ring R the following conditions are equivalent:*

(1) *R is regular-IFP;*

(2) *If $f_1(x)f_2(x)\cdots f_n(x) = 0$ for $f_1(x), f_2(x), \ldots, f_n(x) \in R[x]$, then the sum of all coefficients of every polynomial in*

$$f_1(x)C(R)[x]f_2(x)C(R)[x]\cdots f_{n-1}(x)C(R)[x]f_n(x)$$

is zero;

(3) *If $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, then the sum of all coefficients of every polynomial in $f(x)C(R)[x]g(x)$ is zero;*

(4) *If $f(x)g(x) = 0$ for linear polynomials $f(x), g(x)$ in $R[x]$, then the sum of all coefficients of every polynomial in $f(x)C(R)[x]g(x)$ is zero;*

(5) *$f(x)g(x) = 0$ implies $f(x)C(R)[x]g(x) = 0$ for linear polynomials $f(x), g(x)$ in $R[x]$.*

Proof. The procedure of the proof is almost similar to one of [15, Proposition 2.8], but we write it here for completeness. (1) $\Rightarrow$ (2): Assume that the condition (1) holds. Let $f_1(x)f_2(x)\cdots f_n(x) = 0$ for $f_1(x), f_2(x), \ldots, f_n(x) \in R[x]$. Then we have

$$f_1(1)f_2(1)\cdots f_{n-1}(1)f_n(1) = 0.$$

By Remark 2.3(1), we have $f_1(1)C(R)f_2(1)C(R)\cdots f_{n-1}(1)C(R)f_n(1) = 0$. This yields that the sum of all coefficients of every polynomial in

$$f_1(x)C(R)[x]f_2(x)C(R)[x]\cdots f_{n-1}(x)C(R)[x]f_n(x)$$

is zero.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (5): Assume that the condition (4) holds. Let $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ such that $f(x)g(x) = 0$. Then $a_0b_0 = 0, a_0b_1 + a_1b_0 = 0$ and $a_1b_1 = 0$. From $a_0b_0 = 0$ and $a_1b_1 = 0$, we get $(a_0x)(b_0x) = 0$ and $(a_1x)(b_1x) = 0$; hence, by (4), we have

$$a_0C(R)b_0 = 0 \text{ and } a_1C(R)b_1 = 0. \tag{3.1}$$

From $f(x)g(x) = 0$, we have

$$0 = f(1)cg(1) = (a_0 + a_1)c(b_0 + b_1) = a_0cb_0 + a_0cb_1 + a_1cb_0 + a_1cb_1 = a_0cb_1 + a_1cb_0$$

for all $c \in C(R)$ by (4) and the equalities (3.1). Therefore $f(x)C(R)[x]g(x) = 0$. $\square$

4. Related Topic

Based on Proposition 3.1(5), a ring R is called *nilpotent-IFP* [8] if $aN(R)b = 0$ whenever $ab = 0$ for $a, b \in R$. Every unit-IFP ring is nilpotent-IFP by the proof of Proposition 3.1 and this direction is irreversible by [15, Example 2.5]. For a unit-IFP (or regular-IFP) ring R, we have $N_0(R) = N_*(R) = N^*(R)$ by [15, Theorem 1.3(1)]. But there exists a unit-IFP (hence nilpotent-IFP) ring R such that $N_0(R) \subsetneq N(R)$ and $N(R) \nsubseteq J(R)$, by [15, Example 1.1]. The following partially extends the result of [15, Theorem 1.3].

Theorem 4.1. *For a nilpotent-IFP ring R, we have the following.*

(1) $N_0(R) = N_*(R) = N^*(R)$.

(2) $J(R[x]) = N_0(R[x]) = N_*(R[x]) = N^*(R[x]) = N_0(R)[x] = N_*(R)[x] = N^*(R)[x] = N_0(R)[x] \subseteq J(R)[x]$. *Moreover,* $J(R[x]) = J(R)[x]$ *when* $J(R)$ *is nil.*

Proof. (1) Let $a \in N^*(R)$. Then $a^n = 0$ for some $n \geq 1$ and $RaR \subseteq N^*(R) \subseteq N(R)$. Since R is nilpotent-IFP,

$$a(RaR)a(RaR)a \cdots a(RaR)a(RaR)a = 0,$$

and hence $(RaR)^{2n-1} = 0$ and so $a \in N_0(R)$. Thus we have $N_0(R) = N_*(R) = N^*(R)$.

(2) By help of [1, Theorem 1] and [5, Corollary 4], we have $J(R[x]) \subseteq N^*(R)[x]$ and $N_0(R)[x] = N_0(R[x])$, respectively. By (1) and the facts that $N_*(R[x]) \subseteq N^*(R[x])$, $N^*(R[x]) \subseteq J(R[x])$, and $N^*(R)[x] \subseteq J(R)[x]$. Thus we get

$$\begin{aligned} J(R[x]) \subseteq N^*(R)[x] &= N_*(R)[x] = N_0(R)[x] \\ &= N_0(R[x]) \subseteq N_*(R[x]) \subseteq N^*(R[x]) \subseteq J(R[x]), \end{aligned}$$

and so

$$\begin{aligned} J(R[x]) = N^*(R)[x] &= N_*(R[x]) = N_0(R[x]) \\ &= N^*(R[x]) = N_*(R)[x] = N_0(R)[x] \subseteq J(R)[x]. \end{aligned}$$

Moreover, $J(R[x]) = J(R)[x]$ since $J(R) = N^*(R)$ when $J(R)$ is nil. □

Notice that $J(R[x])$ is always nil in any nilpotent-IFP ring R by Theorem 4.1(2).

On the other hand, Hong et al. [9] consider the duo property on the monoid of regular elements as follows. They call a ring R *right* (resp., *left*) *duo on regularity* (simply, *right* (resp., *left*) *DR*) if $C(R)a \subseteq aC(R)$ (resp., $aC(R) \subseteq C(R)a$) for all $a \in R$; and a ring is called *DR* if it is both left and right DR. Thus it is clear that a ring R is DR if and only if $C(R)a = aC(R)$ for all $a \in R$. A ring R is clearly DR when $C(R)$ is contained in the center of R. Division rings are clearly DR.

Proposition 4.2. *Every one-sided DR ring is regular-IFP.*

Proof. Suppose that R is a right DR ring and let $ab = 0$ for $a, b \in R$. Then $a(C(R)b) \subseteq a(bC(R)) = 0$ since R is right DR. The proof is done as desired. The proof for the case of left DR is similar to the above. □

Notice that the converse of Proposition 4.2 does not hold in general by the next example.

Example 4.3. Let R be the Hamilton quaternions over $\mathbb{Z}$. Then R is a domain and so regular-IFP. But R is not right DR by [9, Example 2.5(1)]. Moreover R is also not left DR by a similar argument to [9, Example 2.5(1)].

Consider the group ring KQ_8, where K is a field and Q_8 denotes the quaternion group.

Corollary 4.4. *For a field K of characteristic 0 and $R = KQ_8$, the following conditions are equivalent:*

(1) *R is DR;*

(2) *R is regular-IFP;*

(3) *R is unit-IFP;*

(4) *R is nilpotent-IFP;*

(5) *R is Abelian;*

(6) *The equation $1 + x^2 + y^2 = 0$ has no solutions in K;*

(7) *R is isomorphic to a finite direct product of division rings;*

(8) *R is reduced.*

Proof. The equivalences of (1), (5), (6), (7) and (8) are shown in [9, Proposition 2.13]. (1) $\Rightarrow$ (2) comes from Proposition 4.2. (2)$\Rightarrow$ (3) is obvious, and (3) $\Rightarrow$ (4) is noted above. (4) $\Rightarrow$ (5) is proved by [8, Proposition 1.5(1)]. □

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